Energy conservation for inhomogeneous Navier-Stokes equations

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Abstract

In this paper, we focus on the energy conservation for the weak solutions of inhomogeneous Navier-Stokes equations. It is proved that if the function of density belongs to $L^\infty(0,T;L^\infty(\mathbb{T}^N))\cap L^p(0,T;W^{1,p}(\mathbb{T}^N))$, and the function of velocity belongs to $L^s(0,T;L^r(\mathbb{T}^N))$ with $\frac{2}{s}+\frac{2}{r}=1$, then the energy equality holds. This result can be seen as a inhomogeneous version for Shinbrot's criterion.

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Keywords: Inhomogeneous Navier-Stokes equations; energy equality; Shinbrot's criterion.

1 Introduction

In this paper we are concerned with the problem of energy conservation for the weak solutions of the following inhomogeneous Navier-Stokes equations in the periodic domain.

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0\,, & \\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)-\Delta u+\nabla\pi=0\,, & \text{in } \mathbb{T}^N\times(0,T)\,, \\ \operatorname{div} u=0\,, & \text{in } \mathbb{T}^N\times(0,T)\,, \\ \rho(\cdot,\,0)=\rho_0\,,\rho u(\cdot,\,0)=\rho_0u_0\,, & \text{in } \mathbb{T}^N\,, \end{cases} \tag{1.1}$$

where u is the velocity, π denotes the pressure, and $\rho\geq 0$ is the density of fluid, and we define $u_0=0$ on the set $\{x:\rho_0(x)=0\}$.

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When ρ is a positive constant, the system (1.1) reduces to the classical Navier-Stokes. It is well known that the weak solutions (Leray-Hopf weak solutions)of Navier-Stokes exist globally in any given interval, and the weak solutions enjoy the following energy inequality, see [9, 7].

$$\|u(t_0)\|_2^2 + 2\int_0^{t_0} \|\nabla u(\tau)\|_2^2 \, d\tau \leq \|u_0\|_2^2 \, ,$$

for any $t_0 \in [0,T)$. It is natural to ask if it is possible to replace the the sign "less or equal" by "equal"? In other words, does the energy equality hold? A first results are due to Lions [10] and Prodi [12], they proved that if $u \in L^4(0,T;L^4(\mathbb{T}^N))$, then the energy equality holds. Later on, in 1974, Shinbrot [14] extended Lions and Prodi's results,he showed that if $u \in L^s(0,T;L^r(\mathbb{T}^N))$ with $\frac{2}{s}+\frac{2}{r}=1$, then the energy equality holds. We also recall that the Serrin's regularity criterion: if $u \in L^s(0,T;L^r(\mathbb{T}^N))$ with $\frac{2}{s}+\frac{N}{r}=1$, then u is regular, and hence u satisfies the energy equality, see [13]. Recently, compared with Shinbrot's results, Berselli and Chiodaroli [2] looked for conditions involving the gradient of the velocity, instead of the velocity itself, see also [3] for Newtonian fluids and non-Newtonian fluids.

The inhomogeneous Navier-Stokes equations are very important in geophysical fluid dynamics. It is called upon to describe situations in which a fluid is inhomogeneous with respect to density. DiPerna and Lions [6, 11] proved the global existence of weak solutions to (1.1) in any space dimension even if the initial data permits regions of vacuum. In the sixties and seventies, the Russian school studied the system 1.1 for the initial density with positive lower bound, see [1, 8]. It was proved that a unique strong solution exists locally for arbitrary initial data. Moreover, these papers also establish global well-posedness results for small solutions in dimension $N \geq 3$, while for the two dimensional case they establish the existence of large strong solutions. However, as far as we know, there is no result on the results for the energy conservation of system 1.1.

Recently, Chen and Yu [4] studied the energy conservation for inhomogeneous Euler equations. They extended the classical result of Constantin-E-Titi [5] to the inhomogeneous Euler equations, and proved the following theorem.

Theorem 1.1. *Let (ρ,u) be a weak solution of (1.1) in the sense of distributions. Assume*

$$\rho \in L^\infty(0,T;L^\infty(\mathbb{T}^N)) \cap L^p(0,T;W^{1,p}(\mathbb{T}^N)) \, , \quad u \in L^q(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N)) \, , \tag{1.2}$$

for any $\frac{1}{p}+3q \leq 1$ and $\alpha > \frac{1}{3}$

$$\sqrt{\rho} u \in L^\infty(0,T;L^2(\mathbb{T}^N)) \tag{1.3}$$

and

$$u_0 \in L^2(\mathbb{T}^N).$$

Then the energy

$$E(t) = \int_{\mathbb{T}^N} \rho |u|^2 dx$$

is conserved.

The purpose of the present paper is to extend the Shinbrot's results on Navier-Stokes equations to the inhomogeneous Navier-Stokes equations. Specifically, we will prove the following results.

Theorem 1.2. *Let* (ρ, u) *be a weak solution of* (1.1) *in the sense of distributions, and satisfy*

$$\sqrt{\rho} u \in L^\infty(0,T;L^2(\mathbb{T}^N)) \,, \quad \nabla u \in L^2(0,T;L^2(\mathbb{T}^N)) \,, \tag{1.4}$$

and

$$u_0 \in L^2(\mathbb{T}^N).$$

Assume

$$\rho \in L^\infty(0,T;L^\infty(\mathbb{T}^N)) \cap L^p(0,T;W^{1,p}(\mathbb{T}^N)) \,, \quad u \in L^4(0,T;L^4(\mathbb{T}^N)) \,, \tag{1.5}$$

for any $p \geq 4$*. Then the energy equality holds, that is,*

$$\|(\sqrt{\rho} u)(t_0)\|_2^2 \, + \, 2 \int_0^{t_0} \|\nabla u(\tau)\|_2^2 \, d\tau = \, \|\sqrt{\rho_0} u_0\|_2^2 \,,$$

for any $t_0 \in [0,T)$*.*

Note that $L^\infty(0,T;L^2(\mathbb{T}^N)) \cap L^s(0,T;L^r(\mathbb{T}^N)) \subset L^4(0,T;L^4(\mathbb{T}^N))$ if $\frac{2}{s} + \frac{2}{r} = 1$. Hence, we have the following conclusion.

Corollary 1.3. *Let* (ρ, u) *be a weak solution of* (1.1) *in the sense of distributions, and satisfy*

$$\sqrt{\rho} u \in L^\infty(0,T;L^2(\mathbb{T}^N)) \,, \quad \nabla u \in L^2(0,T;L^2(\mathbb{T}^N)) \,,$$

and

$$u_0 \in L^2(\mathbb{T}^N).$$

Assume

$$\rho \in L^\infty(0,T;L^\infty(\mathbb{T}^N)) \cap L^p(0,T;W^{1,p}(\mathbb{T}^N)) \,, \quad u \in L^s(0,T;L^r(\mathbb{T}^N)) \,,$$

for any $p \geq 4$*, and* $\frac{2}{s} + \frac{2}{r} = 1$*. Then the energy equality holds.*

2 Proof of Theorem 1.2

To obtain Theorem 1.2, the following lemma is very crucial. It was proved by Lions [11], see also [4], Lemma 2.1.

Lemma 2.1. *Let ∂ be a partial derivative in space or time. Let $f\,, \partial f \in L^p(\mathbb{R}^+ \times \mathbb{T}^N)$, $g \in L^q(\mathbb{R}^+ \times \mathbb{T}^N)$ with $1 \le p,\, q \le \infty$, and $\frac{1}{p} + \frac{1}{p} \le 1$. Then, we have*

$$\|[\partial(fg)]^\epsilon - \partial(fg^\epsilon)\|_{L^r(\mathbb{R}^+\times\mathbb{T}^N)} \le C(\|\partial_t f\|_{L^p(\mathbb{R}^+\times\mathbb{T}^N)} + \|\nabla f\|_{L^p(\mathbb{R}^+\times\mathbb{T}^N)})\|g\|_{L^q(\mathbb{R}^+\times\mathbb{T}^N)} \quad (2.1)$$

for a constant $C > 0$ in dependent of ϵ, f and g, and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$[\partial(fg)]^\epsilon - \partial(fg^\epsilon) \to 0 \quad in\ L^r(\mathbb{R}^+ \times \mathbb{T}^N)\,, \quad (2.2)$$

as $\epsilon \to 0$ if $r < \infty$.

Next, we prove Theorem 1.2. Our argument is in the spirit of the argument in [4]. Compared with the Euler equations, Navier-Stokes equations have a additional regularity, $\nabla u \in L^2(0, T; L^2(\mathbb{T}^N))$, this is crucial to obtain Theorem 1.2.

Proof of Theorem 1.2. As [4], we choose $\Phi(t, x) = (\psi(t)u^\epsilon)^\epsilon$ as a test function, where $\psi(t)$ is the class of all smooth compactly supported functions on $(0, +\infty)$. Multiplying Φ on both sides of the second equation in (1.1), one can obtain

$$\int_0^T \int_{\mathbb{T}^N} \Phi[(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \Delta u + \nabla\pi]dxdt = 0\,,$$

which implies that

$$\int_0^T \int_{\mathbb{T}^N} \psi(t)u^\epsilon[(\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla\pi]^\epsilon dxdt + \int_0^T \int_{\mathbb{T}^N} \psi(t)|\nabla u^\epsilon|^2 dxdt = 0\,. \quad (2.3)$$

It is easy to get that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^N} \psi(t)u^\epsilon((\rho u)_t)^\epsilon dxdt \\
&= \int_0^T \int_{\mathbb{T}^N} \psi(t)u^\epsilon[((\rho u)_t)^\epsilon - (\rho u^\epsilon)_t]dxdt + \int_0^T \int_{\mathbb{T}^N} \psi(t)u^\epsilon(\rho u^\epsilon)_t dxdt \\
&=: A_\epsilon + \int_T \int_{\mathbb{T}^N} \psi(t)\rho\partial_t \frac{|u^\epsilon|^2}{2} dxdt + \int_T \int_{\mathbb{T}^N} \psi(t)\rho_t|u^\epsilon|^2 dxdt\,,
\end{aligned} \quad (2.4)$$

and

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^N} \psi(t) u^\epsilon (\mathrm{div}(\rho u \otimes u)^\epsilon)^\epsilon dxdt \\
&= \int_0^T \int_{\mathbb{T}^N} \psi(t) u^\epsilon [(\mathrm{div}(\rho u \otimes u)^\epsilon)^\epsilon - \mathrm{div}(\rho u \otimes u^\epsilon)] dxdt + \int_0^T \int_{\mathbb{T}^N} \psi(t) \mathrm{div}(\rho u \otimes u^\epsilon) dxdt \\
&=: B_\epsilon + \int_T \int_{\mathbb{T}^N} \psi(t) \rho u \cdot \nabla \frac{|u^\epsilon|^2}{2} dxdt + \int_T \int_{\mathbb{T}^N} \psi(t) \mathrm{div}(\rho u) |u^\epsilon|^2 dxdt \\
&= B_\epsilon - \int_T \int_{\mathbb{T}^N} \psi(t) \rho_t |u^\epsilon|^2 dxdt \,.
\end{aligned}
\tag{2.5}
$$

From (2.3)-(2.5), we have

$$
-\int_T \int_{\mathbb{T}^N} \psi_t \frac{1}{2} \rho |u^\epsilon|^2 dxdt + \int_0^T \int_{\mathbb{T}^N} \psi(t) |\nabla u^\epsilon|^2 dxdt + A_\epsilon + B_\epsilon = 0 \,. \tag{2.6}
$$

For A_ϵ, by Lemma 2.1, we have

$$
\begin{aligned}
|A_\epsilon| \leq & \|\psi(t)\|_{L^\infty(0,T)} \|u^\epsilon\|_{L^4(0,T;L^4(\mathbb{T}^N))} \|((\rho u)_t)^\epsilon - (\rho u^\epsilon)_t\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^N))} \\
\leq & C\|u\|^2_{L^4(0,T;L^4(\mathbb{T}^N))} \|\rho_t\|_{L^2(0,T;L^2(\mathbb{T}^N))} \\
\leq & C\|u\|^2_{L^4(0,T;L^4(\mathbb{T}^N))} \|u \cdot \nabla \rho\|_{L^2(0,T;L^2(\mathbb{T}^N))} \\
\leq & C\|u\|^3_{L^4(0,T;L^4(\mathbb{T}^N))} \|\nabla \rho\|_{L^p(0,T;L^p(\mathbb{T}^N))} \,,
\end{aligned}
$$

and

$$
\|((\rho u)_t)^\epsilon - (\rho u^\epsilon)_t\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^N))} \to 0, \quad \text{as } \epsilon \to 0 \,, \tag{2.7}
$$

i.e. $A_\epsilon \to 0$ as $\epsilon \to 0$. For B_ϵ, similarly, we have

$$
\begin{aligned}
|B_\epsilon| \leq & \|\psi(t)\|_{L^\infty(0,T)} \|u^\epsilon\|_{L^4(0,T;L^4(\mathbb{T}^N))} \|(\mathrm{div}(\rho u \otimes u)^\epsilon)^\epsilon - \mathrm{div}(\rho u \otimes u^\epsilon)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^N))} \\
\leq & C\|u\|^2_{L^4(0,T;L^4(\mathbb{T}^N))} \|\nabla(\rho u)\|_{L^2(0,T;L^2(\mathbb{T}^N))} \\
\leq & C\|u\|^2_{L^4(0,T;L^4(\mathbb{T}^N))} \left(\|u \otimes \nabla \rho\|_{L^2(0,T;L^2(\mathbb{T}^N))} + \|\rho \nabla u\|_{L^2(0,T;L^2(\mathbb{T}^N))}\right) \\
\leq & C\|u\|^2_{L^4(0,T;L^4(\mathbb{T}^N))} \Big(\|u\|_{L^4(0,T;L^4(\mathbb{T}^N))} \|\nabla \rho\|_{L^p(0,T;L^p(\mathbb{T}^N))} \\
& + \|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}^N))} \|\nabla u\|_{L^2(0,T;L^2(\mathbb{T}^N))}\Big) \,,
\end{aligned}
$$

and

$$
\|(\mathrm{div}(\rho u \otimes u)^\epsilon)^\epsilon - \mathrm{div}(\rho u \otimes u^\epsilon)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^N))} \to 0, \quad \text{as } \epsilon \to 0 \,, \tag{2.8}
$$

i.e. $B_\epsilon \to 0$ as $\epsilon \to 0$. Now, letting $\epsilon \to 0$, by (2.7), (2.8) and the dominated convergence theorem, we have

$$
-\int_0^T \int_{\mathbb{T}^N} \psi_t \frac{1}{2} \rho |u|^2 dxdt + \int_0^T \int_{\mathbb{T}^N} \psi |\nabla u|^2 dxdt = 0 \,. \tag{2.9}
$$

Next, by (1.4) and (1.5), following the proof of (3.17) in [4], we have, for any $t \to t_0$

$$\sqrt{\rho}u(t) \to \sqrt{\rho}u(t_0), \quad \text{strongly in } L^2(\mathbb{T}^N) \text{ as } t \to t_0^+. \tag{2.10}$$

Now, for $t_0 > 0$, as [4], choose some positive τ and α such that $\tau + \alpha < t_0$ and define the following test function

$$\psi_\tau(t) = \begin{cases} 0\,, & 0 \le t \le \tau\,, \\ \frac{t-\tau}{\alpha}\,, & \tau \le t \le \tau + \alpha\,, \\ 1\,, & \tau + \alpha \le t \le t_0\,, \\ \frac{t_0 - t}{\alpha}\,, & t_0 \le t \le t_0 + \alpha\,, \\ 0\,, & t_0 + \alpha \le t\,. \end{cases}$$

Then we can deduce from (2.9) that

$$\begin{aligned} &-\frac{1}{\alpha}\int_\tau^{\tau+\alpha}\int_{\mathbb{T}^N}\frac{1}{2}\rho|u|^2 dxds + \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha}\int_{\mathbb{T}^N}\frac{1}{2}\rho|u|^2 dxds + \int_{\tau+\alpha}^{t_0}\int_{\mathbb{T}^N}|\nabla u|^2 dxds \\ = &-\frac{1}{\alpha}\int_\tau^{\tau+\alpha}\int_{\mathbb{T}^N}(s-\tau)|\nabla u|^2(s) dxds - \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha}\int_{\mathbb{T}^N}(t_0 - s)|\nabla u|^2(s) dxds\,. \end{aligned}$$

Let $\alpha \to 0$, due to (2.10), and

$$|\frac{1}{\alpha}\int_\tau^{\tau+\alpha}\int_{\mathbb{T}^N}(s-\tau)|\nabla u|^2(s) dxds| \le \int_\tau^{\tau+\alpha}\int_{\mathbb{T}^N}|\nabla u|^2(s) dxds\,,$$

and

$$|\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha}\int_{\mathbb{T}^N}(t_0 - s)|\nabla u|^2(s) dxds| \le \int_{t_0}^{t_0+\alpha}\int_{\mathbb{T}^N}|\nabla u|^2(s) dxds\,,$$

we have

$$-\int_{\mathbb{T}^N}\frac{1}{2}\rho|u|^2(\tau) dx + \int_{\mathbb{T}^N}\frac{1}{2}\rho|u|^2(t_0) dx + \int_\tau^{t_0}\int_{\mathbb{T}^N}|\nabla u|^2 dxds = 0\,.$$

Finally, let $\tau \to 0$, we can obtain Theorem 1.2. □

References

[1] S. A. Antontesv, A. V. Kazhikov and V. N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, North-Holland, Amsterdam, 1990.

[2] L.C. Berselli, E. Chiodaroli, On the energy equality for the 3D Navier-Stokes equations, Nonlinear Analysis 192 (2020) 111704.

[3] H. Beirão da Veiga, J. Yang, On the energy equality for solutions to Newtonian and non-Newtonian fluids, Nonlinear Analysis, 185 (2019) 388-402.

[4] R. M. Chen, C. Yu, Onsager's energy conservation for inhomogeneous Euler equations, Journal de Mathématiques Pures et Appliquées, 131 (2019) 1-16.

[5] P. Constantin, W. E, E.S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equation, Comm. Math. Phys. 165 (1) (1994) 207-209.

[6] R. J. DiPerna and P. L. Lions, Equations différentielles ordinaires et équations de transport avec des coefficients irréguliers, in: Séminaire EDP 1988-1989, Ecole Polytechnique, Palaiseau, 1989.

[7] E. Hopf, Über die Anfangswertaufgabe fur die hydrodynamischen Grundgleichungen, Math. Nachr. 4 (1951) 213-231.

[8] A. V. Kazhikov, Resolution of boundary value problems for nonhomogeneous viscous fluids, Dokl. Akad. Nauk., 216 (1974), 1008-1010.

[9] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1) (1934) 193-248.

[10] J.L. Lions, Sur la régularité et l'unicité des solutions turbulentes des équations de Navier Stokes, Rend. Semin. Mat. Univ. Padova 30 (1960) 16-23.

[11] P. L. Lions, Mathematical Topics in Fluid Mechanics, Vol. I: Incompressible Models, Oxford Lecture Series in Math. and Its Appl., 3, Oxford University Press, New York, 1996.

[12] G. Prodi, Un teorema di unicità per le equazioni di Navier-Stokes, Ann. Mat. Pura Appl. (4) 48 (1959) 173-182.

[13] J. Serrin, The initial value problem for the Navier-Stokes equations, in: Nonlinear Problems (Proc. Sympos. Madison, Wis.), Univ. of Wisconsin Press, Madison, Wis, 1963, pp. 69C98.

[14] M. Shinbrot, The energy equation for the Navier-Stokes system, SIAM J. Math. Anal. 5 (1974) 948-954.