Miscellaneous Reverse Order Laws for Generalized Inverses of Matrix Products with Applications

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Abstract. One of the fundamental research problems in the theory of generalized inverses of matrices is to establish reverse order laws for generalized inverses of matrix products. Under the assumption that A, B, and C singular matrices of the appropriate sizes, two reverse order laws for generalized inverses of the matrix products AB and ABC can be written as $(AB)^{(i,\dots,j)} = B^{(i_2,\dots,j_2)}A^{(i_1,\dots,j_1)}$ and $(ABC)^{(i,\dots,j)} = C^{(i_3,\dots,j_3)}B^{(i_2,\dots,j_2)}A^{(i_1,\dots,j_1)}$, or other mixed reverse order laws. These equalities do not necessarily hold for different choices of generalized inverses of the matrices. Thus it is a tremendous work to classify and derive necessary and sufficient conditions for the reverse order law to hold because there are all 15 types of $\{i,\dots,j\}$-generalized inverse for a given matrix according to the combinatoric choice of the four Penrose equations. In this paper, we shall establish several decades of mixed reverse order laws for $\{1\}$- and $\{1,2\}$-generalized inverses of AB and ABC, and give a classified investigation to a family of reverse order laws $(ABC)^{(i,\dots,j)} = C^{-1}B^{(k,\dots,l)}A^{-1}$ for the eight commonly-used types of generalized inverses by means of the block matrix representation method (BMRM) and the matrix rank method (MRM). A variety of consequences and applications these reverse order laws are presented.

Keywords: matrix product; generalized inverse; reverse order law; BMRM; MRM.

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1 Introduction

Throughout this article, we denote by $\mathbb{C}^{m\times n}$ the set of all $m\times n$ complex matrices; by A^*, $r(A)$, and $\mathscr{R}(A)$ the conjugate transpose, the rank, and the range (column space) of a matrix $A\in\mathbb{C}^{m\times n}$, respectively; by I_m the identity matrix of order m; and $[A,\, B]$ be a row block matrix consisting of A and B. A matrix $A\in\mathbb{C}^{m\times m}$ is said to be EP (or range Hermitian) if $\mathscr{R}(A^*)=\mathscr{R}(A)$ holds. We next introduce the definition and notation of generalized inverses of a matrix. The Moore–Penrose inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying the four Penrose equations

$$\text{(i) } AXA=A,\quad \text{(ii) } XAX=X,\quad \text{(iii) } (AX)^*=AX,\quad \text{(iv) } (XA)^*=XA. \tag{1.1}$$

A matrix X is called an $\{i,\dots,j\}$-generalized inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the ith, …, jth equations in (1.1). The collection of all $\{i,\dots,j\}$-generalized inverses of A is denoted by $\{(A^{(i,\dots,j)}\}$. There are all 15 types of $\{i,\dots,j\}$-generalized inverses for a given matrix A by definition, but people are mainly interested in the types that involve the first equation:

$$A^{\dagger},\ \ A^{(1,3,4)},\ \ A^{(1,2,4)},\ \ A^{(1,2,3)},\ \ A^{(1,4)},\ \ A^{(1,3)},\ \ A^{(1,2)},\ \ A^{(1)}, \tag{1.2}$$

which are usually called the eight commonly-used types of generalized inverses of A in the literature; see e.g., [3, 4, 19]. In addition, we denote by $P_A = AA^{\dagger}$, $E_A = I_m - AA^{\dagger}$, and $F_A = I_n - A^{\dagger}A$, the three orthogonal projectors (Hermitian idempotent matrices) induced from A.

Matrix expressions that involve a family of generalized inverses $A_1^{(i_1,\dots,j_1)}, A_2^{(i_2,\dots,j_2)},\dots, A_k^{(i_k,\dots,j_k)}$ can generally be expressed as $f(A_1^{(i_1,\dots,j_1)}, A_2^{(i_2,\dots,j_2)},\dots, A_k^{(i_k,\dots,j_k)})$, where $f(\cdot)$ denotes certain algebraic operations of matrices, while matrix equalities that involve generalized inverses can be written as

$$f_1(A_1^{(i_1,\dots,j_1)}, A_2^{(i_2,\dots,j_2)},\dots, A_k^{(i_k,\dots,j_k)}) = f_2(B_1^{(s_1,\dots,t_1)}, B_2^{(s_2,\dots,t_2)},\dots, B_l^{(s_l,\dots,t_l)}). \tag{1.3}$$

Since generalized inverses of a singular matrix are not unique, the matrix expressions may vary with respect to the choices of the generalized inverses. Hence (1.3) can also be described by the following matrix set equality

$$\left\{f_1(A_1^{(i_1,\dots,j_1)}, A_2^{(i_2,\dots,j_2)},\dots, A_k^{(i_k,\dots,j_k)})\right\} = \left\{f_2(B_1^{(s_1,\dots,t_1)}, B_2^{(s_2,\dots,t_2)},\dots, B_l^{(s_l,\dots,t_l)})\right\}. \tag{1.4}$$

We next describe some examples of (1.3) for the generalized inverses of matrix products. Recall a fundamental facts in linear algebra that for any three nonsingular matrices A, B, and C of the same size, the products AB and ABC are nonsingular as well, and the reverse order laws $(AB)^{-1} = B^{-1}A^{-1}$ and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, as well as the cancellation law $C(ABC)^{-1}A = B^{-1}$ always hold. These identities can be used to simplify matrix expressions that involve inverse operations of products of nonsingular matrices. If some or all of A, B, and C are singular, generalized inverses of AB and ABC can be written as certain expressions composed by A, B,

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and C and their generalized inverses. The special matrix equalities for generalized inverses which people are interested are the following reverse order laws

$$(AB)^{(i,\dots,j)} = B^{(i_2,\dots,j_2)}A^{(i_1,\dots,j_1)}, \quad (ABC)^{(i,\dots,j)} = C^{(i_3,\dots,j_3)}B^{(i_2,\dots,j_2)}A^{(i_1,\dots,j_1)} \tag{1.5}$$

for the products AB and ABC. In addition to (1.5), generalized inverses of the matrix products AB and ABC can be written as various mixed reverse order laws, such as,

$$(AB)^{(i,\dots,j)} = B^{(i_2,\dots,j_2)}XA^{(i_1,\dots,j_1)}, \tag{1.6}$$

$$(AB)^{(i,\dots,j)} = B^{(i_2,\dots,j_2)}A^{(i_1,\dots,j_1)} + Y, \tag{1.7}$$

$$(ABC)^{(i,\dots,j)} = C^{(i_3,\dots,j_3)}YB^{(i_2,\dots,j_2)}XA^{(i_1,\dots,j_1)}, \tag{1.8}$$

$$(ABC)^{(i,\dots,j)} = C^{(i_3,\dots,j_3)}B^{(i_2,\dots,j_2)}A^{(i_1,\dots,j_1)} + Z, \tag{1.9}$$

etc., for certain matrices X, Y, and Z composed by A, B, C, and their generalized inverses. Eq. (1.5)–(1.9) do not necessarily hold for different choices of generalized inverses of the matrices. Thus people wish to find identifying conditions for (1.5)–(1.9) to hold under various assumptions. This is really a tremendous work because of different possible choices of $\{i,\dots,j\}$-generalized inverses for a given matrix. Approaches on reverse order laws were started in the 1960s and have been one of the attractive and fruitful research fields in matrix algebra and operator theory, but only a small part of reverse order laws were considered; for instance, $(AB)^{(1)} = B^{(1)}A^{(1)}$ and $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ were approached in [2, 9, 11, 23–25, 32, 40, 41] among others; the special case of the second equality in (1.5) for the Moore–Penrose inverse is given by $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$, which was considered in [5, 8, 12, 14, 26, 36], while mixed reverse order laws for Moore–Penrose inverses of AB and ABC were formulated and approached in [6, 7, 10, 13, 17, 29, 30, 32, 34, 38] among others. It is worth to point out that the present author initiated the use of the matrix rank method in the study of reverse order laws, which can manipulate various complicated calculations associated with generalized inverses. In spite of many efforts, most of (1.5) remain unresolved because there are no analytical methods in mathematics to establish general algebraic identities or solve general algebraic equations.

The paper is organized as follows. In Section 2, we give an introduction to the theory of generalized inverses of matrices and present various formulas for calculating ranks of matrices that we shall use in the sequel. In Section 3, we formulate a variety of mixed reverse order laws for generalized inverses of AB and ABC and give their proofs using definitions of generalized inverses and matrix rank formulas. In Section 4, we consider a special product ABC, where A and C are nonsingular, and derive necessary and sufficient conditions for the following four matrix set relations

$$\{(ABC)^{(i,\dots,j)}\} \cap \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \neq \emptyset \qquad (8^2 = 64 \text{ situations}), \tag{1.10}$$

$$\{(ABC)^{(i,\dots,j)}\} \supseteq \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \qquad (8^2 = 64 \text{ situations}), \tag{1.11}$$

$$\{(ABC)^{(i,\dots,j)}\} \subseteq \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \qquad (8^2 = 64 \text{ situations}), \tag{1.12}$$

$$\{(ABC)^{(i,\dots,j)}\} = \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \qquad (8^2 = 64 \text{ situations}) \tag{1.13}$$

to hold for the eight common-used generalized inverses of ABC and B by means of the block matrix representation method (BMRM) and the matrix rank method (MRM). We also present a variety of consequences of the these reverse order laws in Sections 4 and 5.

2 Preliminaries

In this section, we present an introduction on the theory on generalized inverses of matrices and describe the matrix rank methods by which we establish and simplify various matrix equalities that involve generalized inverses.

Lemma 2.1 ([3, 4, 19]). *Let $A \in \mathbb{C}^{m\times n}$. Then the following results hold.*

(a) $A^{\dagger}$ *satisfies the following equalities*

$$(A^{\dagger})^* = (A^*)^{\dagger}, \qquad (A^{\dagger})^{\dagger} = A, \tag{2.1}$$

$$(A^*)^{\dagger}A^* = (AA^{\dagger})^* = AA^{\dagger}, \quad A^*(A^*)^{\dagger} = (A^{\dagger}A)^* = A^{\dagger}A, \tag{2.2}$$

$$\mathscr{R}(A) = \mathscr{R}(AA^*) = \mathscr{R}(AA^*A) = \mathscr{R}(AA^{\dagger}) = \mathscr{R}[(A^{\dagger})^*], \tag{2.3}$$

$$\mathscr{R}(A^*) = \mathscr{R}(A^*A) = \mathscr{R}(A^*AA^*) = \mathscr{R}(A^{\dagger}) = \mathscr{R}(A^{\dagger}A), \tag{2.4}$$

$$r(A) = r(A^*) = r(A^{\dagger}) = r(AA^*) = r(A^*A) = r(AA^{\dagger}) = r(A^{\dagger}A). \tag{2.5}$$

(b) *The general expressions of the seven commonly-used types of generalized inverses* $A^{(1,3,4)}$, $A^{(1,2,4)}$, $A^{(1,2,3)}$, $A^{(1,4)}$, $A^{(1,3)}$, $A^{(1,2)}$, *and* $A^{(1)}$ *of* A *can be written in the following 7 matrix-valued functions*

$$A^{(1)} = A^{\dagger} + F_A U_1 + U_2 E_A, \tag{2.6}$$
$$A^{(1,2)} = (A^{\dagger} + F_A U_1) A (A^{\dagger} + U_2 E_A), \tag{2.7}$$
$$A^{(1,3)} = A^{\dagger} + F_A U, \tag{2.8}$$
$$A^{(1,4)} = A^{\dagger} + U E_A, \tag{2.9}$$
$$A^{(1,2,3)} = A^{\dagger} + F_A U A A^{\dagger}, \tag{2.10}$$
$$A^{(1,2,4)} = A^{\dagger} + A^{\dagger} A U E_A, \tag{2.11}$$
$$A^{(1,3,4)} = A^{\dagger} + F_A U E_A, \tag{2.12}$$

where U, U_1, $U_2 \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(c) *The following matrix equalities hold*

$$AA^{(1)} = AA^{(1,2)} = AA^{(1,4)} = AA^{(1,2,4)} = AA^{\dagger} + AUE_A, \tag{2.13}$$
$$AA^{(1,3)} = AA^{(1,2,3)} = AA^{(1,3,4)} = AA^{\dagger}, \tag{2.14}$$
$$A^{(1)}A = A^{(1,2)}A = A^{(1,3)}A = A^{(1,2,3)}A = A^{\dagger}A + F_A UA, \tag{2.15}$$
$$A^{(1,4)}A = A^{(1,2,4)}A = A^{(1,3,4)}A = A^{\dagger}A, \tag{2.16}$$

where $U \in \mathbb{C}^{n\times m}$ *is arbitrary.*

(d) *The following set inclusions hold*

$$A^{\dagger} \in \{A^{(1,2,3)}\} \subseteq \{A^{(1,2)}\} \subseteq \{A^{(1)}\}, \tag{2.17}$$
$$A^{\dagger} \in \{A^{(1,2,3)}\} \subseteq \{A^{(1,3)}\} \subseteq \{A^{(1)}\}, \tag{2.18}$$
$$A^{\dagger} \in \{A^{(1,2,4)}\} \subseteq \{A^{(1,2)}\} \subseteq \{A^{(1)}\}, \tag{2.19}$$
$$A^{\dagger} \in \{A^{(1,2,4)}\} \subseteq \{A^{(1,4)}\} \subseteq \{A^{(1)}\}, \tag{2.20}$$
$$A^{\dagger} \in \{A^{(1,3,4)}\} \subseteq \{A^{(1,3)}\} \subseteq \{A^{(1)}\}. \tag{2.21}$$
$$A^{\dagger} \in \{A^{(1,3,4)}\} \subseteq \{A^{(1,4)}\} \subseteq \{A^{(1)}\}, \tag{2.22}$$

(e) *The following matrix set equalities hold*

$$\{(A^{(1,3,4)})^*\} = \{(A^*)^{(1,3,4)}\}, \quad \{(A^{(1,2,4)})^*\} = \{(A^*)^{(1,2,3)}\}, \tag{2.23}$$
$$\{(A^{(1,2,3)})^*\} = \{(A^*)^{(1,2,4)}\}, \quad \{(A^{(1,4)})^*\} = \{(A^*)^{(1,3)}\}, \tag{2.24}$$
$$\{(A^{(1,3)})^*\} = \{(A^*)^{(1,4)}\}, \quad \{(A^{(1,2)})^*\} = \{(A^*)^{(1,2)}\}, \tag{2.25}$$
$$\{(A^{(1)})^*\} = \{(A^*)^{(1)}\}. \tag{2.26}$$

(f) *The following rank equalities*

$$r(A^{(1,2,4)}) = r(A^{(1,2,3)}) = r(A^{(1,2)}) = r(A^{\dagger}) = r(A) \tag{2.27}$$

hold for all $A^{(1,2,4)})$, $A^{(1,2,3)}$, *and* $A^{(1,2)})$, *and the following rank equalities hold*

$$\max_{A^{(1)}} r(A^{(1)}) = \max_{A^{(1,3)}} r(A^{(1,3)}) = \max_{A^{(1,4)}} r(A^{(1,4)}) = \max_{A^{(1,3,4)}} r(A^{(1,3,4)}) = \min\{\, m,\, n\}, \tag{2.28}$$
$$\min_{A^{(1)}} r(A^{(1)}) = \min_{A^{(1,3)}} r(A^{(1,3)}) = \min_{A^{(1,4)}} r(A^{(1,4)}) = \min_{A^{(1,3,4)}} r(A^{(1,3,4)}) = r(A). \tag{2.29}$$

(g) *The following equivalent facts hold*

Lemma 2.2 ([28]). *Let $A \in \mathbb{C}^{m\times n}$ and $G \in \mathbb{C}^{n\times m}$. Then*

$$\min_{A^{(1)}} r(A^{(1)} - G) = r(A - AGA), \tag{2.30}$$

$$\min_{A^{(1,2)}} r(A^{(1,2)} - G) = \max\{\, r(A - AGA),\ r(G) + r(A) - r(GA) - r(AG)\,\}, \tag{2.31}$$

$$\min_{A^{(1,3)}} r(A^{(1,3)} - G) = r(A^*AG - A^*), \tag{2.32}$$

$$\min_{A^{(1,4)}} r(A^{(1,4)} - G) = r(GAA^* - A^*), \tag{2.33}$$

$$\min_{A^{(1,2,3)}} r(A^{(1,2,3)} - G) = r(A^*AG - A^*) + r\begin{bmatrix} A^* \\ G \end{bmatrix} - r\begin{bmatrix} A^* \\ AG \end{bmatrix}, \tag{2.34}$$

$$\min_{A^{(1,2,4)}} r(A^{(1,2,4)} - G) = r(GAA^* - A^*) + r[A^*,\, G] - r[A^*,\, GA], \tag{2.35}$$

$$\min_{A^{(1,3,4)}} r(A^{(1,3,4)} - G) = r(A^*AG - A^*) + r(GAA^* - A^*) - r(A - AGA), \tag{2.36}$$

$$r(A^{\dagger} - G) = r\begin{bmatrix} A^*AA^* & A^* \\ A^* & G \end{bmatrix} - r(A). \tag{2.37}$$

In particular,

$$G \in \{A^{(1)}\} \Leftrightarrow AGA = A, \tag{2.38}$$

$$G \in \{A^{(1,2)}\} \Leftrightarrow AGA = A \text{ and } r(G) = r(A), \tag{2.39}$$

$$G \in \{A^{(1,3)}\} \Leftrightarrow A^*AG = A^*, \tag{2.40}$$

$$G \in \{A^{(1,4)}\} \Leftrightarrow GAA^* = A^*, \tag{2.41}$$

$$G \in \{A^{(1,2,3)}\} \Leftrightarrow A^*AG = A^* \text{ and } r(G) = r(A) \Leftrightarrow A^*AG = A^* \text{ and } GE_A = 0, \tag{2.42}$$

$$G \in \{A^{(1,2,4)}\} \Leftrightarrow GAA^* = A^* \text{and } r(G) = r(A) \Leftrightarrow GAA^* = A^* \text{ and } F_AG = 0, \tag{2.43}$$

$$G \in \{A^{(1,3,4)}\} \Leftrightarrow A^*AG = A^* \text{ and } GAA^* = A^*, \tag{2.44}$$

$$\begin{aligned} G = A^{\dagger} &\Leftrightarrow A^*AG = A^*,\ GAA^* = A^*,\ \text{and } r(G) = r(A) \\ &\Leftrightarrow A^*AG = A^*,\ GAA^* = A^*,\ GE_A = 0,\ F_AG = 0. \end{aligned} \tag{2.45}$$

Lemma 2.3 ([15]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and $C \in \mathbb{C}^{l\times n}$. Then*

$$r[A,\, B] = r(A) + r(B - AA^{(1)}B), \quad r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^{(1)}A) \tag{2.46}$$

hold for all $A^{(1)}$.

Lemma 2.4 ([15]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then*

$$r(AB) = r(A) + r(B) - n + r[(I_n - BB^{(1)})(I_p - A^{(1)}A)], \tag{2.47}$$

$$r(ABC) = r(AB) + r(BC) - r(B) + r[(I_n - (BC)(BC)^{(1)})B(I_p - (AB)^{(1)}(AB))] \tag{2.48}$$

hold for all $A^{(1)}$, $B^{(1)}$, $(AB)^{(1)}$, and $(BC)^{(1)}$.

Lemma 2.5 ([28]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then*

$$\max_{A^{(1,2)}} r(D - CA^{(1,2)}B) = \min\left\{ r(A) + r(D),\, r[C,\, D],\, r\begin{bmatrix} B \\ D \end{bmatrix},\ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}, \tag{2.49}$$

$$\min_{A^{(1,2)}} r(D - CA^{(1,2)}B) = r\begin{bmatrix} B \\ D \end{bmatrix} + r[\,C,\ D\,] + r(A) + \max\{\, r_1,\ r_2\}, \tag{2.50}$$

where

$$r_1 = r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}, \quad r_2 = r(D) - r\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

Lemma 2.6 ([35]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times q}$, and $C \in \mathbb{C}^{p\times q}$. Then the following two formulas hold*

$$\max_{A^{(1,2)},C^{(1,2)}} r(A^{(1,2)}BC^{(1,2)}) = \min\{\, r(A),\ r(B),\ r(C)\,\}, \tag{2.51}$$

$$\min_{A^{(1,2)},C^{(1,2)}} r(A^{(1,2)}BC^{(1,2)}) = \max\{\, 0,\ r(A) + r(B) + r(C) - r[A,\, B] - r[B^*,\, C^*]\}. \tag{2.52}$$

We also use the following results to establish matrix equalities that involve generalized inverses.

Lemma 2.7 ([27,28]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then the following results hold.*

(a) *There exists an $A^{(1)}$ such that $CA^{(1)}B = D$ holds if and only if*

$$\mathscr{R}(D) \subseteq \mathscr{R}(C), \quad \mathscr{R}(D^*) \subseteq \mathscr{R}(B^*), \text{ and } r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[\, A,\, B] - r(A); \tag{2.53}$$

$CA^{(1)}B = D$ holds for all $A^{(1)}$ if and only if

$$[C,\, D] = 0 \text{ or } \begin{bmatrix} B \\ D \end{bmatrix} = 0 \text{ or } r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A). \tag{2.54}$$

(b) *$CA^{(1,2)}B = D$ holds for all $A^{(1,2)}$ if and only if*

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = 0 \text{ or } [C,\, D] = 0 \text{ or } \begin{bmatrix} B \\ D \end{bmatrix} = 0 \text{ or } r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A). \tag{2.55}$$

(c) *$CA^{(1,3)}B = D$ holds for all $A^{(1,3)}$ if and only if*

$$\begin{bmatrix} B \\ D \end{bmatrix} = 0 \text{ or } r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} = r(A). \tag{2.56}$$

(d) *$CA^{(1,4)}B = D$ holds for all $A^{(1,4)}$ if and only if*

$$[C,\, D] = 0 \text{ or } r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} = r(A). \tag{2.57}$$

(e) *$CA^{(1,2,3)}B = D$ holds for all $A^{(1,2,3)}$ if and only if*

$$\begin{bmatrix} A^*B \\ D \end{bmatrix} = 0 \text{ or } r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} = r(A). \tag{2.58}$$

(f) *$CA^{(1,2,4)}B = D$ holds for all $A^{(1,2,4)}$ if and only if*

$$[CA^*,\, D] = 0 \text{ or } r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} = r(A). \tag{2.59}$$

(g) *$CA^{(1,3,4)}B = D$ holds for all $A^{(1,3,4)}$ if and only if*

$$r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} = r(A) \text{ or } r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} = r(A). \tag{2.60}$$

(h) *$CA^{\dagger}B = D$ holds if and only if*

$$r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r(A). \tag{2.61}$$

We next give a specified introduction to the MRM. Recall a basic fact about the link between the nullity of matrix and its rank that $A = 0$ holds if and only if $r(A) = 0$ holds. Thus, two matrices A and B of the same size are equal, namely, $A = B$, if and only if $r(A - B) = 0$. Furthermore, assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two sets consisting of matrices of the same size. Then

$$\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset \Leftrightarrow \min_{A\in\mathcal{S}_1,\, B\in\mathcal{S}_2} r(A - B) = 0;$$

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \Leftrightarrow \max_{A\in\mathcal{S}_1} \min_{B\in\mathcal{S}_2} r(A - B) = 0$$

hold obviously in view of the MRM. The basic facts are quite familiar to the reader with common background in linear algebra, but they provide a highly flexible framework for characterizing equalities of matrices under many situations, precisely, if certain formulas for calculating the rank of $X - Y$ are derived, we can use the formulas to characterize relationships between two matrices X and Y and to obtain many valuable results on relationships between two matrix sets. This method, called the matrix rank method, is available for studying various matrix expressions involving generalized inverses of matrices. Perhaps, no methods in linear algebra and matrix theory, as described above, is more elementary than the rank method in characterizing equalities of matrices.

3 Mixed Reverse Order Law for Generalized Inverses

We first give two groups of set inclusions associated with the matrix sets $\{(AB)^{(1)}\}$ and $\{(AB)^{(1,2)}\}$.

Theorem 3.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given. Then,*

(a) *the following set inclusions hold*

$$\{\,(AB)^{(1)}\} \supseteq \{\,(A^{(1)}AB)^{(1)}A^{(1)}\}, \tag{3.1}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,B^{(1)}(ABB^{(1)})^{(1)}\}, \tag{3.2}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,(A^*AB)^{(1)}A^*\}, \tag{3.3}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,B^*(ABB^*)^{(1)}\}, \tag{3.4}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,(AA^*AB)^{(1)}AA^*\}, \tag{3.5}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,B^*B(ABB^*B)^{(1)}\}, \tag{3.6}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,B^{(1)}(A^{(1)}ABB^{(1)})^{(1)}A^{(1)}\}, \tag{3.7}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,B^*(A^*ABB^*)^{(1)}A^*\}, \tag{3.8}$$
$$\{\,(AB)^{(1)}\} \supseteq \{\,B^*B(AA^*ABB^*B)^{(1)}AA^*\}; \tag{3.9}$$

(b) *the following set inclusions hold*

$$\{\,(AB)^{(1,2)}\} \supseteq \{\,(A^{(1,2)}AB)^{(1,2)}A^{(1,2)}\}, \tag{3.10}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,B^{(1,2)}(ABB^{(1,2)})^{(1,2)}\}, \tag{3.11}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,(A^*AB)^{(1,2)}A^*\}, \tag{3.12}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,B^*(ABB^*)^{(1,2)}\}, \tag{3.13}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,(AA^*AB)^{(1,2)}AA^*\}, \tag{3.14}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,B^*B(ABB^*B)^{(1,2)}\}, \tag{3.15}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,B^{(1,2)}(A^{(1,2)}ABB^{(1,2)})^{(1,2)}A^{(1,2)}\}, \tag{3.16}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,B^*(A^*ABB^*)^{(1,2)}A^*\}, \tag{3.17}$$
$$\{\,(AB)^{(1,2)}\} \supseteq \{\,B^*B(AA^*ABB^*B)^{(1,2)}AA^*\}. \tag{3.18}$$

Proof. The whole proofs are based on the definitions of generalized inverses and direct verifications. For any generalized inverses $A^{(1)}$, $B^{(1)}$, $(A^{(1)}AB)^{(1)}$, $(ABB^{(1)})^{(1)}$, $(A^*AB)^{(1)}$, $(ABB^*)^{(1)}$, $(A^{(1)}ABB^{(1)})^{(1)}$, $(A^*ABB^*)^{(1)}$, and $(AA^*ABB^*A)^{(1)}$, it is easy to verify by definition and Lemma 2.1(a) that

$$\begin{aligned}
&AB[(A^{(1)}AB)^{(1)}A^{(1)}]AB = A(A^{(1)}AB)(A^{(1)}AB)^{(1)}(A^{(1)}AB) = A(A^{(1)}AB) = AB,\\
&AB[B^{(1)}(ABB^{(1)})^{(1)}]AB = (ABB^{(1)})(ABB^{(1)})^{(1)}(ABB^{(1)})B = (ABB^{(1)})B = AB,\\
&AB[(A^*AB)^{(1)}A^*]AB = (A^\dagger)^*(A^*AB)(A^*AB)^{(1)}(A^*AB) = (A^\dagger)^*(A^*AB) = AB,\\
&AB[B^*(ABB^*)^{(1)}]AB = (ABB^*)(ABB^*)^{(1)}(ABB^*)(B^\dagger)^* = (ABB^*)(B^\dagger)^* = AB,\\
&AB[AA^*AB)^{(1)}AA^*]AB = [(AA^*)^\dagger]^*(AA^*AB)(AA^*AB)^{(1)}(AA^*AB) = [(AA^*)^\dagger]^*AA^*AB = AB,\\
&ABB^*B(ABB^*B)^{(1)}AB = (ABB^*B)(ABB^*B)^{(1)}(ABB^*B)[(B^*B)^\dagger]^* = ABB^*B[(B^*B)^\dagger]^* = AB,
\end{aligned}$$

and

$$\begin{aligned}
AB[B^{(1)}(A^{(1)}ABB^{(1)})^{(1)}A^{(1)}]AB &= A(A^{(1)}ABB^{(1)})(A^{(1)}ABB^{(1)})^{(1)}(A^{(1)}ABB^{(1)})B\\
&= A(A^{(1)}ABB^{(1)})B = AB,\\
AB[B^*(A^*ABB^*)^{(1)}A^*]AB &= (A^\dagger)^*(A^*ABB^*)(A^*ABB^*)^{(1)}(A^*ABB^*)(B^\dagger)^*(A^\dagger)^*(A^*ABB^*)(B^\dagger)^*\\
&= AB,\\
AB[B^*B(AA^*ABB^*B)^{(1)}AA^*]AB &= [(AA^*)^\dagger]^*(AA^*ABB^*B)(AA^*ABB^*B)^{(1)}(AA^*ABB^*B)[(B^*B)^\dagger]^*\\
&= [(AA^*)^\dagger]^*(AA^*ABB^*B)[(B^*B)^\dagger]^* = AB,
\end{aligned}$$

establishing (3.1)–(3.9).

Since $\{M^{(1)}\} \supseteq \{M^{(1,2)}\}$ for any matrix M, it is easy to see from (3.1)–(3.9) that

$$\{(AB)^{(1)}\} \supseteq \{(A^{(1)}AB)^{(1)}A^{(1)}\} \supseteq \{(A^{(1,2)}AB)^{(1,2)}A^{(1,2)}\}, \tag{3.19}$$
$$\{(AB)^{(1)}\} \supseteq \{B^{(1)}(ABB^{(1)})^{(1)}\} \supseteq \{B^{(1,2)}(ABB^{(1,2)})^{(1,2)}\}, \tag{3.20}$$
$$\{(AB)^{(1)}\} \supseteq \{(A^*AB)^{(1)}A^*\} \supseteq \{(A^*AB)^{(1,2)}A^*\}, \tag{3.21}$$
$$\{(AB)^{(1)}\} \supseteq \{B^*(ABB^*)^{(1)}\} \supseteq \{B^*(ABB^*)^{(1,2)}\}, \tag{3.22}$$
$$\{(AB)^{(1)}\} \supseteq \{(AA^*AB)^{(1)}AA^*\} \supseteq \{(AA^*AB)^{(1,2)}AA^*\}, \tag{3.23}$$
$$\{(AB)^{(1)}\} \supseteq \{B^*B(ABB^*B)^{(1)}\} \supseteq \{B^*B(ABB^*B)^{(1,2)}\}, \tag{3.24}$$
$$\{(AB)^{(1)}\} \supseteq \{B^{(1)}(A^{(1)}ABB^{(1)})^{(1)}A^{(1)}\} \supseteq \{B^{(1,2)}(A^{(1,2)}ABB^{(1,2)})^{(1,2)}A^{(1,2)}\}, \tag{3.25}$$
$$\{(AB)^{(1)}\} \supseteq \{B^*(A^*ABB^*)^{(1)}A^*\} \supseteq \{B^*(A^*ABB^*)^{(1,2)}A^*\}, \tag{3.26}$$
$$\{(AB)^{(1)}\} \supseteq \{B^*B(AA^*ABB^*B)^{(1,2)}AA^*\} \supseteq \{B^*B(AA^*ABB^*B)^{(1,2)}AA^*\} \tag{3.27}$$

hold. Also by definition,

$$(A^{(1,2)}AB)^{(1,2)}A^{(1,2)}AB(A^{(1,2)}AB)^{(1,2)}A^{(1,2)} = (A^{(1,2)}AB)^{(1,2)}A^{(1,2)}, \tag{3.28}$$
$$B^{(1,2)}(ABB^{(1,2)})^{(1,2)}ABB^{(1,2)}(ABB^{(1,2)})^{(1,2)} = B^{(1,2)}(ABB^{(1,2)})^{(1,2)}, \tag{3.29}$$
$$(A^*AB)^{(1,2)}A^*AB(A^*AB)^{(1,2)} = (A^*AB)^{(1,2)}, \tag{3.30}$$
$$B^*(ABB^*)^{(1,2)}ABB^*(ABB^*)^{(1,2)} = B^*(ABB^*)^{(1,2)}, \tag{3.31}$$
$$(AA^*AB)^{(1,2)}AA^*AB(AA^*AB)^{(1,2)}AA^* = (AA^*AB)^{(1,2)}AA^*, \tag{3.32}$$
$$B^*B(ABB^*B)^{(1,2)}ABB^*B(ABB^*B)^{(1,2)} = B^*B(ABB^*B)^{(1,2)}, \tag{3.33}$$
$$B^{(1,2)}(A^{(1,2)}ABB^{(1,2)})^{(1,2)}A^{(1,2)}ABB^{(1,2)}(A^{(1,2)}ABB^{(1,2)})^{(1,2)}A^{(1,2)} = B^{(1,2)}(A^{(1,2)}ABB^{(1,2)})^{(1,2)}A^{(1,2)}, \tag{3.34}$$
$$B^*(A^*ABB^*)^{(1,2)}A^*ABB^*(A^*ABB^*)^{(1,2)}A^* = B^*(A^*ABB^*)^{(1,2)}A^*, \tag{3.35}$$
$$B^*B(AA^*ABB^*B)^{(1,2)}AA^*ABB^*B(AA^*ABB^*B)^{(1,2)}AA^* = B^*B(AA^*ABB^*B)^{(1,2)}AA^*. \tag{3.36}$$

Combining (3.19)–(3.27) with (3.28)–(3.36) leads to (3.10)–(3.18). □

Theorem 3.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given. and denote $P = I_n - A^{(1)}A$, $Q = I_n - BB^{(1)}$, $U = I_n - A^{(1,2)}A$, and $V = I_n - BB^{(1,2)}$. Then,*

(a) *the set inclusion $\{(AB)^{(1)}\} \supseteq \{B^{(1)}A^{(1)} - B^{(1)}P(QP)^{(1)}QA^{(1)}\}$ always holds;*

(b) $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1,2)} - B^{(1,2)}U(VU)^{(1,2)}VA^{(1,2)}\} \Leftrightarrow r(AB) = r(A) = r(B) \Leftrightarrow \mathscr{R}(AB) = \mathscr{R}(A)$ *and* $\mathscr{R}[(AB)^*] = \mathscr{R}(B^*)$.

Proof. Noting $QP(QP)^{(1)}QP = QP$ and pre- and post-multiplying with A and B, we obtain

$$AQP(QP)^{(1)}QPB = AQPB, \tag{3.37}$$

where both sides are given by

$$\begin{aligned} AQPB &= A(I_n - BB^{(1)})(I_n - A^{(1)}A)B = A(I_n - A^{(1)}A - BB^{(1)} + BB^{(1)}A^{(1)}A)B \\ &= AB - ABB^{(1)}A^{(1)}AB, \\ AQP(QP)^{(1)}QPB &= A(I_n - BB^{(1)})P(QP)^{(1)}Q(I_n - A^{(1)}A)B \\ &= AP(QP)^{(1)}QB - AP(QP)^{(1)}QA^{(1)}AB - ABB^{(1)}P(QP)^{(1)}QB \\ &\quad + ABB^{(1)}P(QP)^{(1)}QA^{(1)}AB \\ &= ABB^{(1)}P(QP)^{(1)}QA^{(1)}AB. \end{aligned}$$

Substituting these equalities into (3.37) yields $AB[B^{(1)}A^{(1)} - B^{(1)}P(QP)^{(1)}QA^{(1)}]AB = AB$. Thus (a) holds.

Since $\{M^{(1)}\} \supseteq \{M^{(1,2)}\}$ for any matrix M, thus it is easy to see from (3.36) that

$$\{(AB)^{(1)}\} \supseteq \{B^{(1)}A^{(1)} - B^{(1)}U(VU)^{(1)}VA^{(1)}\} \supseteq \{B^{(1,2)}A^{(1,2)} - B^{(1,2)}U(VU)^{(1,2)}VA^{(1,2)}\}. \tag{3.38}$$

We next determine the maximum and minimum ranks of $B^{(1,2)}A^{(1,2)} - B^{(1,2)}U(VU)^{(1,2)}VA^{(1,2)}$. Since $r[(VU)^{(1,2)}] = r(VU)$ and $VU(VU)^{(1,2)}VU = VU$, it follows that $r[U(VU)^{(1,2)}V] = r[VU(VU)^{(1,2)}VU] = r(VU)$. Also note that $(U(VU)^{(1,2)})^2 = U(VU)^{(1,2)}$, namely, $U(VU)^{(1,2)}V$ is idempotent, we obtain

$$r[I_n - U(VU)^{(1,2)}V] = n - r[U(VU)^{(1,2)}V] = n - r[(VU)^{(1,2)}] = n - r(VU) = r(A) + r(B) - r(AB) \tag{3.39}$$

holds for all $(VU)^{(1,2)}$ by (2.47). By (2.46) and elementary block matrix operations,

$$\begin{aligned} r[B,\, I_n - U(VU)^{(1,2)}V] &= r(B) + r[V - (VU)(VU)^{(1,2)}V] = r(B) + r[VU,\, V] - r(VU) \\ &= r(B) + (V) - r(VU) = n - r(VU) = r(A) + r(B) - r(AB), \end{aligned} \tag{3.40}$$

$$\begin{aligned} r\begin{bmatrix} A \\ I_n - U(VU)^{(1,2)}V \end{bmatrix} &= r(A) + r[U - U(VU)^{(1,2)}(VU)] = r(A) + r\begin{bmatrix} U \\ VU \end{bmatrix} - r(VU) \\ &= r(A) + r(U) - r(VU) = n - r(VU) = r(A) + r(B) - r(AB). \end{aligned} \tag{3.41}$$

Next by (2.51), (2.52), (3.39), (3.40), and (3.41),

$$\begin{aligned} &\max_{A^{(1,2)},B^{(1,2)}} r[B^{(1,2)}A^{(1,2)} - B^{(1,2)}U(VU)^{(1,2)}VA^{(1,2)}] = \min_{A^{(1,2)},B^{(1,2)}} r[B^{(1,2)}(I_n - U(VU)^{(1,2)}V)A^{(1,2)}] \\ &= \min\{r(A^{(1,2)}),\ r(B^{(1,2)}),\ r(I_n - U(VU)^{(1,2)}V)\} = \min\{r(A),\ r(B),\ r(A) + r(B) - r(AB)\} \\ &= \min\{r(A),\ r(B)\}, \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} &\min_{A^{(1,2)},B^{(1,2)}} r[B^{(1,2)}A^{(1,2)} - B^{(1,2)}U(VU)^{(1,2)}VA^{(1,2)}] \\ &= \min_{A^{(1,2)},B^{(1,2)}} r[B^{(1,2)}(I_n - U(VU)^{(1,2)}V)A^{(1,2)}] \\ &= \max\left\{0,\ r(A^{(1,2)}) + r(B^{(1,2)}) + r[I_n - U(VU)^{(1,2)}V] - r[B,\, I_n - U(VU)^{(1,2)}V] - r\begin{bmatrix} A \\ I_n - U(VU)^{(1,2)}V \end{bmatrix}\right\} \\ &= \max\{0,\ r(AB)\} = r(AB). \end{aligned} \tag{3.43}$$

Combining (3.42) and (3.43), we see that $r[B^{(1,2)}A^{(1,2)} - B^{(1,2)}U(VU)^{(1,2)}VA^{(1,2)}] = r(AB)$ holds for all $A^{(1,2)}$, $B^{(1,2)}$, and $(VU)^{(1,2)}$ if and only if $r(AB) = r(A) = r(B)$. Combining this fact with (3.38) and applying (2.39), we obtain (b). □

The mixed reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger} - B^{\dagger}[(I_n - BB^{\dagger})(I_n - A^{\dagger}A)]^{\dagger}A^{\dagger}$ for the Moore–Penrose inverses was proposed and approached by the present author in [29] using the matrix rank methodology.

We turn now to the case for generalized inverses of a triple matrix product. There are a large variety of mixed-type reverse-order laws that can be formulated mostly by try and fail method. Here we present such a list as follows.

Theorem 3.3. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$ be given, and denote $M = ABC$. Then,*

(a) *the following set inclusions hold*

$$\begin{aligned}
&\{M^{(1)}\} \supseteq \{(A^{(1)}M)^{(1)}A^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{C^{(1)}(MC^{(1)})^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{(A^{*}M)^{(1)}A^{*}\}, \\
&\{M^{(1)}\} \supseteq \{C^{*}(MC^{*})^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{(AA^{*}M)^{(1)}AA^{*}\}, \\
&\{M^{(1)}\} \supseteq \{C^{*}C(MC^{*}C)^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{C^{(1)}(A^{(1)}MC^{(1)})^{(1)}A^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{C^{*}(A^{*}MC^{*})^{(1)}A^{*}\}, \\
&\{M^{(1)}\} \supseteq \{[(AB)^{(1)}M]^{(1)}(AB)^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{(BC)^{(1)}[M(BC)^{(1)}]^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{[(AB)^{*}M]^{(1)}(AB)^{*}\}, \\
&\{M^{(1)}\} \supseteq \{(BC)^{*}[M(BC)^{*}]^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{[(ABB^{(1)})^{(1)}M]^{(1)}(ABB^{(1)})^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{(B^{(1)}BC)^{(1)}[M(B^{(1)}BC)^{(1)}]^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{[(ABB^{*})^{(1)}M]^{(1)}(ABB^{*})^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{(B^{*}BC)^{(1)}[M(B^{*}BC)^{(1)}]^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{C^{*}C(AA^{*}MC^{*}C)^{(1)}AA^{*}\}, \\
&\{M^{(1)}\} \supseteq \{(BC)^{(1)}[(AB)^{(1)}M(BC)^{(1)}]^{(1)}(AB)^{(1)}\}, \\
&\{M^{(1)}\} \supseteq \{(BC)^{*}[(AB)^{*}M(BC)^{*}]^{(1)}(AB)^{*}\},
\end{aligned}$$

$$
\begin{aligned}
&\{ M^{(1)} \} \supseteq \{ (B^{(1)}BC)^{(1)}[(ABB^{(1)})^{(1)}M(B^{(1)}BC)^{(1)}]^{(1)}(ABB^{(1)})^{(1)} \},\\
&\{ M^{(1)} \} \supseteq \{ (B^*BC)^{(1)}[(ABB^*)^{(1)}M(B^*BC)^{(1)}]^{(1)}(ABB^*)^{(1)} \},\\
&\{ M^{(1)} \} \supseteq \{ (B^{(1)}BC)^*[(ABB^{(1)})^*M(B^{(1)}BC)^*]^{(1)}(ABB^{(1)})^* \},\\
&\{ M^{(1)} \} \supseteq \{ (B^*BC)^*[(ABB^*)^*M(B^*BC)^*]^{(1)}(ABB^*)^* \};
\end{aligned}
$$

(b) *the following set inclusions hold*

$$
\begin{aligned}
&\{ M^{(1,2)} \} \supseteq \{ (A^{(1,2)}M)^{(1,2)}A^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ C^{(1,2)}(MC^{(1,2)})^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ (A^*M)^{(1,2)}A^* \},\\
&\{ M^{(1,2)} \} \supseteq \{ C^*(MC^*)^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ (AA^*M)^{(1,2)}AA^* \},\\
&\{ M^{(1,2)} \} \supseteq \{ C^*C(MC^*C)^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ C^{(1,2)}(A^{(1,2)}MC^{(1,2)})^{(1,2)}A^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ C^*(A^*MC^*)^{(1,2)}A^* \},\\
&\{ M^{(1,2)} \} \supseteq \{ [(AB)^{(1,2)}M]^{(1,2)}(AB)^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ (BC)^{(1,2)}[M(BC)^{(1,2)}]^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ [(AB)^*M]^{(1,2)}(AB)^* \},\\
&\{ M^{(1,2)} \} \supseteq \{ (BC)^*[M(BC)^*]^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ [(ABB^{(1,2)})^{(1,2)}M]^{(1,2)}(ABB^{(1,2)})^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ (B^{(1,2)}BC)^{(1,2)}[M(B^{(1,2)}BC)^{(1,2)}]^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ [(ABB^*)^{(1,2)}M]^{(1,2)}(ABB^*)^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ (B^*BC)^{(1,2)}[M(B^*BC)^{(1,2)}]^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ C^*C(AA^*MC^*C)^{(1,2)}AA^* \},\\
&\{ M^{(1,2)} \} \supseteq \{ (BC)^{(1,2)}[(AB)^{(1,2)}M(BC)^{(1,2)}]^{(1,2)}(AB)^{(1,2)} \},
\end{aligned}
$$

$$
\begin{aligned}
&\{ M^{(1,2)} \} \supseteq \{ (BC)^*[(AB)^*M(BC)^*]^{(1,2)}(AB)^* \},\\
&\{ M^{(1,2)} \} \supseteq \{ (B^{(1,2)}BC)^{(1,2)}[(ABB^{(1,2)})^{(1,2)}M(B^{(1,2)}BC)^{(1,2)}]^{(1,2)}(ABB^{(1,2)})^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ (B^{(1,2)}BC)^*[(ABB^{(1,2)})^*M(B^{(1,2)}BC)^*]^{(1,2)}(ABB^{(1,2)})^* \},\\
&\{ M^{(1,2)} \} \supseteq \{ (B^*BC)^{(1,2)}[(ABB^*)^{(1,2)}M(B^*BC)^{(1,2)}]^{(1,2)}(ABB^*)^{(1,2)} \},\\
&\{ M^{(1,2)} \} \supseteq \{ (B^*BC)^*[(ABB^*)^*M(B^*BC)^*]^{(1,2)}(ABB^*)^* \}.
\end{aligned}
$$

Proof. It follows from the direct verification and the definitions of $\{1\}$- and $\{1,2\}$-generalized inverse of matrices. □

We next prove two results related to the mixed reverse order laws:

$$
\begin{aligned}
(ABC)^{(1)} &= (BC)^{(1)}B(AB)^{(1)} - (BC)^{(1)}BP(QBP)^{(1)}QB(AB)^{(1)},\\
(ABC)^{(1,2)} &= (BC)^{(1,2)}B(AB)^{(1,2)} - (BC)^{(1,2)}BU(VBU)^{(1,2)}VB(AB)^{(1,2)}
\end{aligned}
$$

using definitions and the matrix rank formulas, where $P = I_p - (AB)^{(1)}AB$, $Q = I_n - BC(BC)^{(1)}$, $U = I_p - (AB)^{(1,2)}AB$, and $V = I_n - BC(BC)^{(1,2)}$.

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$ be given, and denote $M = ABC$. Then,*

(a) $\{ M^{(1)} \} \supseteq \{ (BC)^{(1)}B(AB)^{(1)} - (BC)^{(1)}BP(QBP)^{(1)}QB(AB)^{(1)} \}$ *always hold;*

(b) $\{ M^{(1,2)} \} \supseteq \{ (BC)^{(1,2)}B(AB)^{(1,2)} - (BC)^{(1,2)}BU(VBU)^{(1,2)}VB(AB)^{(1,2)} \} \Leftrightarrow r(M) = r(AB) = r(BC) \Leftrightarrow \mathscr{R}(M) = \mathscr{R}(AB)$ *and* $\mathscr{R}(M^*) = \mathscr{R}[(BC)^*]$.

Proof. Noting $VBU = (VBU)(VBU)^{(1)}(VBU)$ and pre- and post-multiplying A and C, we obtain

$$AVBUC = AVBU(VBU)^{(1)}VBUC, \tag{3.44}$$

where

$$
\begin{aligned}
AVBUC &= ABC - ABC(BC)^{(1)}BC - AB(AB)^{(1)}ABC + ABC(BC)^{(1)}B(AB)^{(1)}ABC \\
&= ABC(BC)^{(1)}B(AB)^{(1)}ABC - ABC, \\
AVBU(VBU)^{(1)}VBUC &= (A - ABC(BC)^{(1)})BU(VBU)^{(1)}VB(C - (AB)^{(1)}ABC) \\
&= ABU(VBU)^{(1)}VBC - ABC(BC)^{(1)}BU(VBU)^{(1)}VBC \\
&\quad - ABU(VBU)^{(1)}VB(AB)^{(1)}ABC + ABC(BC)^{(1)}BU(VBU)^{(1)}(AB)^{(1)}ABC \\
&= ABC(BC)^{(1)}BU(VBU)^{(1)}(AB)^{(1)}ABC.
\end{aligned}
$$

Substituting these two equalities into (3.44) yields $M[(BC)^{(1)}B(AB)^{(1)} - (BC)^{(1)}BU(VBU)^{(1)}(AB)^{(1)}]M = M$, establishing (a).

Result (a) obviously implies that

$$
\{\, M^{(1)} \,\} \supseteq \{\, (BC)^{(1,2)}B(AB)^{(1,2)} - (BC)^{(1,2)}BU(VBU)^{(1,2)}VB(AB)^{(1,2)} \,\} \tag{3.45}
$$

We next determine the maximum and minimum rank of $(BC)^{(1,2)}[B - BU(VBU)^{(1,2)}VB](AB)^{(1,2)}$. By (2.46) and elementary block matrix operations,

$$
r(VB) = r[B - (BC)(BC)^{(1,2)}B] = r[BC,\, B] - r(BC) = r(B) - r(BC), \tag{3.46}
$$

$$
r(BU) = r[B - B(AB)^{(1,2)}AB] = r[(AB)^*,\, B^*] - r(AB) = r(B) - r(AB). \tag{3.47}
$$

By (2.46) and elementary block matrix operations,

$$
\begin{aligned}
&\max_{(VBU)^{(1,2)}} r[B - BU(VBU)^{(1,2)}VB] \\
&= \min\left\{ r[BU,\, B],\; r\begin{bmatrix} VB \\ B \end{bmatrix},\; r\begin{bmatrix} VBU & VB \\ BU & B \end{bmatrix} - r(VBU) \right\} \\
&= \min\{ r(B),\; r(B) - r(VBU) \} \\
&= r(B) - r(VBU) = r(AB) + r(AB) - r(ABC) \quad \text{(by (2.48))};
\end{aligned} \tag{3.48}
$$

$$
\begin{aligned}
&\min_{(VBU)^{(1,2)}} r[B - BU(VBU)^{(1,2)}VB] \\
&= r\begin{bmatrix} VB \\ B \end{bmatrix} + r[BU,\, B] + r(VBU) \\
&+ \max\left\{ r\begin{bmatrix} VBU & VB \\ BU & B \end{bmatrix} - r\begin{bmatrix} VBU & 0 & VB \\ 0 & BU & B \end{bmatrix} - r\begin{bmatrix} VBU & 0 \\ 0 & VB \\ BU & B \end{bmatrix},\; r(B) - r\begin{bmatrix} VBU & 0 \\ BU & B \end{bmatrix} - r\begin{bmatrix} VBU & VB \\ 0 & B \end{bmatrix} \right\} \\
&= 2r(B) + r(VBU) + \min\{ -2r(VBU) - r(B),\quad -2r(VBU) - r(B) \} \\
&= r(B) - r(VBU) = r(AB) + r(AB) - r(ABC) \quad \text{(by (2.48))}.
\end{aligned} \tag{3.49}
$$

Combining (3.48) and (3.49), we see that

$$
r[B - BU(VBU)^{(1,2)}VB] = r(AB) + r(AB) - r(ABC) \tag{3.50}
$$

holds for all $(VBU)^{(1,2)}$. By (2.51), (2.52), (3.46), (3.47), and elementary block matrix operations,

$$
\begin{aligned}
&r[BC,\, B - BU(VBU)^{(1,2)}VB] = r(BC) + r[VB - (VBU)(VBU)^{(1,2)}VB] \\
&= r(BC) + r[VBU,\; VB] - r(VBU) = r(B) - r(VBU) = r(AB) + r(AB) - r(ABC),
\end{aligned} \tag{3.51}
$$

$$
r[(AB)^*,\; (B - BU(VBU)^{(1,2)}VB)]^*] = r(AB) + r(AB) - r(ABC). \tag{3.52}
$$

Also by (2.51) and (2.52),

$$
\begin{aligned}
&\max_{(AB)^{(1,2)},(BC)^{(1,2)}} r[(BC)^{(1,2)}B(AB)^{(1,2)} - (BC)^{(1,2)}BU(VBU)^{(1,2)}VB(AB)^{(1,2)}] \\
&= \max_{(AB)^{(1,2)},(BC)^{(1,2)}} r\{(BC)^{(1,2)}[B - BU(VBU)^{(1,2)}VB](AB)^{(1,2)}\} \\
&= \max_{(AB)^{(1,2)},\; (BC)^{(1,2)}} \{ r((AB)^{(1,2)}),\; r((BC)^{(1,2)}),\quad r[B - BU(VBU)^{(1,2)}VB)] \} \\
&= \min\{ r(AB),\quad r(BC),\quad r(AB) + r(ABC) - r(ABC) \} \quad \text{(by (3.50))} \\
&= \min\{ r(AB),\quad r(BC) \},
\end{aligned} \tag{3.53}
$$

and

$$
\begin{aligned}
&\min_{(AB)^{(1,2)},(BC)^{(1,2)}} r[(BC)^{(1,2)}B(AB)^{(1,2)} - (BC)^{(1,2)}BU(VBU)^{(1,2)}VB(AB)^{(1,2)}] \\
&= \min_{(AB)^{(1,2)},(BC)^{(1,2)}} r\{(BC)^{(1,2)}[B - BU(VBU)^{(1,2)}VB](AB)^{(1,2)}\} \\
&= \max\{0, \ \ r(AB) + r((BC) + r[B - BU(VBU)^{(1,2)}VB] \\
&\qquad - r[BC,\, B - BU(VBU)^{(1,2)}VB] - r[(AB)^*,\, (B - BU(VBU)^{(1,2)}VB)]^*\} \\
&= \max\{0, \ \ r(ABC)\} \ \ \text{(by (3.50), (3.51), and (3.52))} \\
&= r(ABC).
\end{aligned} \tag{3.54}
$$

Combining (3.53) and (3.54), we see that $r[(BC)^{(1,2)}B(AB)^{(1,2)} - (BC)^{(1,2)}BU(VBU)^{(1,2)}VB(AB)^{(1,2)}] = r(ABC)$ holds for all $(AB)^{(1,2)}$, $(BC)^{(1,2)}$, and $(VU)^{(1,2)}$ if and only if $r(ABC) = r(AB) = r(BC)$. Combining this fact with (3.45) and applying (2.39) lead to (b). □

4 Reverse Order Laws for Generalized Inverses of a Triple Matrix Product

We first prepare some formulas associated with various fundamental matrix calculations in (1.10)–(1.13).

Lemma 4.1. *Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and $C \in \mathbb{C}^{n\times n}$ be given and assume that A and C are nonsingular. Also denote $M = ABC$. Then,*

(a) *the following rank equalities*

$$
r(M) = r(B), \quad r(CM^{(i,\dots,j)}A) = r(M^{(i,\dots,j)}), \quad r(C^{-1}B^{(k,\dots,l)}A^{-1}) = r(B^{(k,\dots,l)}) \tag{4.1}
$$

hold for all $M^{(i,\dots,j)}$ and $B^{(k,\dots,l)}$;

(b) *the following set inclusions hold*

$$
C^{-1}B^{\dagger}A^{-1} \in \{C^{-1}B^{(1,3,4)}A^{-1}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}, \tag{4.2}
$$

$$
C^{-1}B^{\dagger}A^{-1} \in \{C^{-1}B^{(1,3,4)}A^{-1}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}, \tag{4.3}
$$

$$
C^{-1}B^{\dagger}A^{-1} \in \{C^{-1}B^{(1,2,4)}A^{-1}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}, \tag{4.4}
$$

$$
C^{-1}B^{\dagger}A^{-1} \in \{C^{-1}B^{(1,2,4)}A^{-1}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}, \tag{4.5}
$$

$$
C^{-1}B^{\dagger}A^{-1} \in \{C^{-1}B^{(1,2,3)}A^{-1}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}, \tag{4.6}
$$

$$
C^{-1}B^{\dagger}A^{-1} \in \{C^{-1}B^{(1,2,3)}A^{-1}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}; \tag{4.7}
$$

(c) *the following equivalent facts hold*

$$
\{M^{(i,\dots,j)}\} \cap \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \neq \emptyset \Leftrightarrow \{CM^{(i,\dots,j)}A\} \cap \{B^{(k,\dots,l)}\} \neq \emptyset, \tag{4.8}
$$

$$
\{M^{(i,\dots,j)}\} \supseteq \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \Leftrightarrow \{CM^{(i,\dots,j)}A\} \supseteq \{B^{(k,\dots,l)}\}, \tag{4.9}
$$

$$
\{M^{(i,\dots,j)}\} \subseteq \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \Leftrightarrow \{CM^{(i,\dots,j)}A\} \subseteq \{B^{(k,\dots,l)}\}, \tag{4.10}
$$

$$
\{M^{(i,\dots,j)}\} = \{C^{-1}B^{(k,\dots,l)}A^{-1}\} \Leftrightarrow \{CM^{(i,\dots,j)}A\} = \{B^{(k,\dots,l)}\} \tag{4.11}
$$

hold for the eight common-used generalized inverses of B and M;

(d) *the following equalities*

$$
\begin{aligned}
MC^{-1}B^{(1)}A^{-1}M &= MC^{-1}B^{(1,2)}A^{-1}M = MC^{-1}B^{(1,3)}A^{-1}M \\
&= MC^{-1}B^{(1,4)}A^{-1}M = MC^{-1}B^{(1,2,3)}A^{-1}M \\
&= MC^{-1}B^{(1,2,4)}A^{-1}M = MC^{-1}B^{(1,3,4)}A^{-1}M \\
&= MC^{-1}B^{\dagger}A^{-1}M = M
\end{aligned} \tag{4.12}
$$

hold for all the eight commonly-used types of generalized inverses of B, and

$$
\begin{aligned}
BCM^{(1)}AB &= BCM^{(1,2)}AB = BCM^{(1,3)}AB = BCM^{(1,4)}AB \\
&= BCM^{(1,2,3)}AB = BCM^{(1,2,4)}AB = BCM^{(1,3,4)}AB \\
&= BCM^{\dagger}AB = B
\end{aligned} \tag{4.13}
$$

hold for all the eight commonly-used types of generalized inverses of M;

(e) $M^*MC^{-1}B^{(1)}A^{-1} = M^*MC^{-1}B^{(1,2)}A^{-1} = M^*MC^{-1}B^{(1,4)}A^{-1} = M^*MC^{-1}B^{(1,2,4)}A^{-1} = M^*$ *hold for some* $B^{(1)}$, $B^{(1,2)}$, $B^{(1,4)}$, *and* $B^{(1,2,4)}$;

(f) $M^*MC^{-1}B^{(1)}A^{-1} = M^*MC^{-1}B^{(1,2)}A^{-1} = M^*MC^{-1}B^{(1,4)}A^{-1} = M^*MC^{-1}B^{(1,2,4)}A^{-1} = M^*$ *hold for all* $B^{(1)}$, $B^{(1,2)}$, $B^{(1,4)}$, *and* $B^{(1,2,4)} \Leftrightarrow B = 0$ *or* $r(B) = m$;

(g) $M^*MC^{-1}B^{(1,3)}A^{-1} = M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^*MC^{-1}B^{(1,3,4)}A^{-1} = M^*MC^{-1}B^{\dagger}A^{-1} = M^* \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$;

(h) $B^*BCM^{(1)}A = B^*BCM^{(1,2)}A = B^*BCM^{(1,4)}A = B^*BCM^{(1,2,4)}A = B^*$ *hold for all* $M^{(1)}$, $M^{(1,2)}$, $M^{(1,4)}$, *and* $M^{(1,2,4)} \Leftrightarrow B = 0$ *or* $r(B) = m$;

(i) $B^*BCM^{(1,3)}A = B^*BCM^{(1,2,3)}A = B^*BCM^{(1,3,4)}A = B^*BCM^{\dagger}A = B^* \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

Proof. Result (a) follows from the nonsingularity of A and C. Pre- and post-multiplying (2.22)–(2.17) with C^{-1} and A^{-1} respectively yield the set inclusions in Result (b). Pre- and post-multiplying (1.10)–(1.13) with C and A respectively yield the following equivalent facts in Result (c). Result (d) follows from direct verification.

The following results

$$\begin{aligned} &M^*MC^{-1}B^{(1)}A^{-1} = M^*MC^{-1}B^{(1,2)}A^{-1} = M^*MC^{-1}B^{(1,4)}A^{-1} = M^*MC^{-1}B^{(1,2,4)}A^{-1} = M^* \\ &\Leftrightarrow (AB)^*ABB^{(1)} = (AB)^*ABB^{(1,2)} = (AB)^*ABB^{(1,4)} = (AB)^*ABB^{(1,2,4)} = (AB)^*A, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &M^*MC^{-1}B^{(1,3)}A^{-1} = M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^*MC^{-1}B^{(1,3,4)}A^{-1} = M^*MC^{-1}B^{\dagger}A^{-1} = M^* \\ &\Leftrightarrow (AB)^*ABB^{\dagger} = (AB)^*A \end{aligned} \tag{4.15}$$

follow from (2.13)–(2.16) and the nonsingularity of A and C. Furthermore, it is easy to derive from Lemma 2.7 that

$$\begin{aligned} &(AB)^*ABB^{(1)} = (AB)^*ABB^{(1,2)} = (AB)^*ABB^{(1,4)} = (AB)^*ABB^{(1,2,4)} = (AB)^*A \\ &\text{are solvable for some } B^{(1)},\ B^{(1,2)},\ B^{(1,4)},\ \text{and } B^{(1,2,4)}, \end{aligned} \tag{4.16}$$

combining this fact with (4.14) leads to Result (e);

$$\begin{aligned} &(AB)^*ABB^{(1)} = (AB)^*ABB^{(1,2)} = (AB)^*ABB^{(1,4)} = (AB)^*ABB^{(1,2,4)} = (AB)^*A \\ &\text{are solvable for all } B^{(1)},\ B^{(1,2)},\ B^{(1,4)},\ \text{and } B^{(1,2,4)} \Leftrightarrow B = 0 \text{ or } r(B) = m, \end{aligned} \tag{4.17}$$

combining this facts with (4.14) leads to Result (f);

$$(AB)^*ABB^{\dagger} = (AB)^*A \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B). \tag{4.18}$$

combining it with (4.15) leads to Result (g).

Results (h) and (i) are also derived from Lemma 2.7. □

Armed with the preceding results and facts, we can derive the main results in the paper. For the sake of convenience of reference, we will present a complete list of results for all the situations in (1.10)–(1.13).

Theorem 4.2. *Let* $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, *and* $C \in \mathbb{C}^{n\times n}$ *be given and assume that* A *and* C *are nonsingular. Also denote* $M = ABC$. *Then the following 64 groups of results hold.*

(1) $\{M^{(1)}\} = \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(2a) $\{M^{(1)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\}$ *holds.*

(2b) $\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow r(B) = \min\{m, n\}$.

(3a) $\{M^{(1)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\}$ *holds.*

(3b) $\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m$.

(4a) $\{M^{(1)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\}$ *holds.*

(4b) $\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(5a) $\{M^{(1)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\}$ *holds.*

(5b) $\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow r(B) = m$.

(6a) $\{M^{(1)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\}$ *holds.*

(6b) $\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow r(B) = n.$

(7a) $\{M^{(1)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\}$ *holds.*

(7b) $\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n.$

(8) $\{M^{(1)}\} \ni C^{-1}B^{\dagger}A^{-1}$ *holds.*

(9a) $\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(9b) $\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} = \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow r(B) = m$ *or* $r(B) = n.$

(10) $\{M^{(1,2)}\} = \{C^{-1}B^{(1,2)}A^{-1}\}$ *holds.*

(11a) $\{M^{(1,2)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset$ *holds.*

(11b) $\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow r(B) = m$ *or* $r(B) = n.$

(11c) $\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m.$

(11d) $\{M^{(1,2)}\} = \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow r(B) = m.$

(12a) $\{M^{(1,2)}\} \cap \{C^{-1}B^{(1,4)}A^{-1}\} \neq \emptyset.$ *holds.*

(12b) $\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow r(B) = m$ *or* $r(B) = n.$

(12c) $\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n.$

(12d) $\{M^{(1,2)}\} = \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow r(B) = n.$

(13a) $\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\}$ *holds.*

(13b) $\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} = \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m.$

(14a) $\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\}$ *holds.*

(14c) $\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} = \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n.$

(15a) $\{M^{(1,2)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset$ *holds.*

(15b) $\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow r(B) = m$ *or* $r(B) = n.$

(15c) $\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n.$

(15d) $\{M^{(1,2)}\} = \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow r(B) = m = n.$

(16) $\{M^{(1,2)}\} \ni C^{-1}B^{\dagger}A^{-1}$ *holds.*

(17a) $\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(17b) $\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m.$

(18a) $\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,2)}A^{-1}\} \neq \emptyset$ *holds.*

(18b) $\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m.$

(18c) $\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow r(B) = m$ *or* $r(B) = n.$

(18d) $\{M^{(1,3)}\} = \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow r(B) = m.$

(19a) $\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset$ *holds.*

(19b) $\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B).$

(20a) $\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,4)}A^{-1}\} \neq \emptyset$ *holds.*

(20b) $\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m.$

(20c) $\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n.$

(20d) $\{M^{(1,3)}\} = \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n.$

(21a) $\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(21b) $\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $r(B) = \min\{m, n\}$.

(22a) $\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,2,4)}A^{-1}\} \neq \emptyset$ *holds.*

(22b) $\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m$.

(22c) $\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow r(B) = n$.

(22d) $\{M^{(1,3)}\} = \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow r(B) = m = n$.

(23a) $\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(23b) $\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3)}\} = \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{r(B) = n$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)\}$.

(24) $\{M^{(1,3)}\} \ni C^{-1}B^{\dagger}A^{-1} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(25a) $\{M^{(1,4)}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(25b) $\{M^{(1,4)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(26a) $\{M^{(1,4)}\} \cap \{C^{-1}B^{(1,2)}A^{-1}\} \neq \emptyset$ *holds.*

(26b) $\{M^{(1,4)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(26c) $\{M^{(1,4)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow r(B) = m$ *or* $r(B) = n$.

(26d) $\{M^{(1,4)}\} = \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow r(B) = n$.

(27a) $\{M^{(1,4)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset$ *holds.*

(27b) $\{M^{(1,4)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(27c) $\{M^{(1,4)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m$.

(27d) $\{M^{(1,4)}\} = \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n$.

(28a) $\{M^{(1,4)}\} \cap \{C^{-1}B^{(1,4)}A^{-1}\} \neq \emptyset$ *holds.*

(28b) $\{M^{(1,4)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} = \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(29a) $\{M^{(1,4)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset$ *holds.*

(29b) $\{M^{(1,4)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(29c) $\{M^{(1,4)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow r(B) = m$.

(29d) $\{M^{(1,4)}\} = \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow r(B) = m = n$.

(30a) $\{M^{(1,4)}\} \cap \{C^{-1}B^{(1,2,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,4)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(30b) $\{M^{(1,4)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} = \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$ *and* $r(B) = \min\{m, n\}$.

(31a) $\{M^{(1,4)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,4)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(31b) $\{M^{(1,4)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} = \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{r(B) = m$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)\}$.

(32) $\{M^{(1,4)}\} \ni C^{-1}B^{\dagger}A^{-1} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(33a) $\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(33b) $\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} = \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow r(B) = m$.

(34a) $\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\}$ *holds.*

(34b) $\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} = \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m$.

(35a) $\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(35b) $\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} = \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $r(B) = \min\{m, n\}$.

(36a) $\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,4)}A^{-1}\} \neq \emptyset$ *holds.*

(36b) $\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow r(B) = m$.

(36c) $\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(36d) $\{M^{(1,2,3)}\} = \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow r(B) = m = n$.

(37) $\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,3)}\} = \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(38a) $\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,2,4)}A^{-1}\} \neq \emptyset$ *holds.*

(38b) $\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m$.

(38c) $\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(38d) $\{M^{(1,2,3)}\} = \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n$.

(39a) $\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(39b) $\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $r(B) = \min\{m, n\}$.

(39c) $\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{\mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $r(B) = n\}$.

(39d) $\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $r(B) = n$.

(40) $\{M^{(1,2,3)}\} \ni C^{-1}B^{\dagger}A^{-1} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(41a) $\{M^{(1,2,4)}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(41b) $\{M^{(1,2,4)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} = \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow r(B) = n$.

(42a) $\{M^{(1,2,4)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\}$ *holds.*

(42b) $\{M^{(1,2,4)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} = \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow \Leftrightarrow B = 0$ *or* $r(B) = n$.

(43a) $\{M^{(1,2,4)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset$ *holds.*

(43b) $\{M^{(1,2,4)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow r(B) = n$.

(43c) $\{M^{(1,2,4)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m$.

(43d) $\{M^{(1,2,4)}\} = \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow r(B) = m = n$.

(44a) $\{M^{(1,2,4)}\} \cap \{C^{-1}B^{(1,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,4)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(44b) $\{M^{(1,2,4)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} = \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$ *and* $r(B) = \min\{m, n\}$.

(45a) $\{M^{(1,2,4)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset$ *holds.*

(45b) $\{M^{(1,2,4)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = n$.

(45c) $\{M^{(1,2,4)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m$.

(45d) $\{M^{(1,2,4)}\} = \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n$.

(46) $\{M^{(1,2,4)}\} \cap \{C^{-1}B^{(1,2,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,4)}\} = \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(47a) $\{M^{(1,2,4)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(47b) $\{M^{(1,2,4)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$ *and* $r(B) = \min\{m, n\}$.

(47c) $\{M^{(1,2,4)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$ *and* $r(B) = m\}$.

(47d) $\{M^{(1,2,4)}\} = \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$ *and* $r(B) = m$.

(48) $\{M^{(1,2,4)}\} \ni C^{-1}B^{\dagger}A^{-1} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(49a) $\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(49b) $\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} = \{C^{-1}B^{(1)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n$.

(50a) $\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,2)}A^{-1}\} \neq \emptyset$.

(50b) $\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow B = 0$ *or* $r(B) = m = n$.

(50c) $\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow r(B) = m$ *or* $r(B) = n$.

(50d) $\{M^{(1,3,4)}\} = \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow r(B) = m = n$.

(51a) $\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(51b) $\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} = \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{r(B) = n$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)\}$.

(52a) $\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(52b) $\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} = \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{r(B) = m$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)\}$.

(53a) $\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(53b) $\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{r(B) = n$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)\}$.

(53c) $\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $r(B) = \min\{m, n\}$.

(53d) $\{M^{(1,3,4)}\} = \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow r(B) = n$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(54a) $\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,2,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(54b) $\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow B = 0$ *or* $\{r(B) = m$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)\}$.

(54c) $\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$ *and* $r(B) = \min\{m, n\}$.

(54d) $\{M^{(1,3,4)}\} = \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow r(B) = m$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(55) $\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3,4)}\} = \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(56) $\{M^{(1,3,4)}\} \ni C^{-1}B^{\dagger}A^{-1} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(57) $M^{\dagger} \in \{C^{-1}B^{(1)}A^{-1}\}$ *holds.*

(58) $M^{\dagger} \in \{C^{-1}B^{(1,2)}A^{-1}\}$ *holds.*

(59) $M^{\dagger} \in \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(60) $M^{\dagger} \in \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(61) $M^{\dagger} \in \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

(62) $M^{\dagger} \in \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(63) $M^{\dagger} \in \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*)$.

(64) [12, Case 2] $M^{\dagger} = C^{-1}B^{\dagger}A^{-1} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $\mathscr{R}(CC^*B^*) = \mathscr{R}(B^*) \Leftrightarrow \mathscr{R}(AA^*M) = \mathscr{R}(M)$ *and* $\mathscr{R}(C^*CM^*) = \mathscr{R}(M^*) \Leftrightarrow (A^*AB)(A^*AB)^{\dagger} = BB^{\dagger}$ *and* $(BCC^*)^{\dagger}(BCC^*) = B^{\dagger}B \Leftrightarrow A^*ABB^*$ *and* B^*BCC^* *are EP* $\Leftrightarrow AA^*MM^*$ *and* M^*MC^*C *are EP.*

Proof. By (4.12) and (4.13),

$$\{M^{(1)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \text{ and } \{CM^{(1)}A\} \subseteq \{B^{(1)}\} \tag{4.19}$$

hold. Combining (4.19) with (4.10) yields Result (1).

Result (2a) follows from (4.12). By (4.10),

$$\begin{aligned}
&\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow \{CM^{(1)}A\} \subseteq \{B^{(1,2)}\} \\
&\Leftrightarrow BCM^{(1)}AB = B \text{ and } r(CM^{(1)}A) = r(B) \text{ for all } M^{(1)} \text{ (by (2.39))} \\
&\Leftrightarrow r(M^{(1)}) = r(B) \text{ for all } M^{(1)} \\
&\Leftrightarrow \min\{m, n\} = r(B) \quad \text{(by (2.28) and (2.29))} \\
&\Leftrightarrow r(B) = m \text{ or } r(B) = n,
\end{aligned} \tag{4.20}$$

establishing the equivalence the first and third terms in Result (2b). Combining this fact with Result (2a) leads to the second equivalence in Result (2b).

Result (3a) follows from (4.12). By (2.40) and (4.10),

$$\begin{aligned}&\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{CM^{(1)}A\} \subseteq \{B^{(1,3)}\}\\ &\Leftrightarrow B^*BCM^{(1)}A = B^* \text{ for all } M^{(1)}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(h))},\end{aligned} \tag{4.21}$$

establishing the equivalence the first and third terms in Result (3b). Combining this fact with Result (3a) leads to the second equivalence in Result (3b).

Result (5a) follows from (4.12). By (4.10),

$$\begin{aligned}&\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{CM^{(1)}A\} \subseteq \{B^{(1,2,3)}\}\\ &\Leftrightarrow B^*BCM^{(1)}A = B^* \text{ and } r(CM^{(1)}A) = r(B) \text{ for all } M^{(1)} \text{ (by (2.42))}\\ &\Leftrightarrow \{B = 0 \text{ or } r(B) = m\} \text{ and } r(B) = \min\{m,\, n\} \text{ (by (2.28), (2.29), (4.1), and Lemma 4.1(h))}\\ &\Leftrightarrow r(B) = m,\end{aligned} \tag{4.22}$$

establishing the equivalence the first and third terms in Result (5b). Combining this fact with Result (5a) leads to the second equivalence in Result (5b).

Result (7a) follows from (4.12). By (4.10),

$$\begin{aligned}&\{M^{(1)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \{CM^{(1)}A\} \subseteq \{B^{(1,3,4)}\}\\ &\Leftrightarrow B^*BCM^{(1)}A = B^* \textit{ and } CM^{(1)}ABB^* = B^* \text{for all } M^{(1,3,4)} \text{ (by (2.44))}\\ &\Leftrightarrow \{B = 0 \text{ or } r(B) = m\} \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(h))}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m = n,\end{aligned} \tag{4.23}$$

establishing the equivalence the first and third terms in Result (7b). Combining this fact with Result (7a) leads to the second equivalence in Result (7b).

Result (8) follows from (4.12).

Result (9a) follows from (2.19) and Result (1). By (4.10),

$$\begin{aligned}&\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\}\\ &\Leftrightarrow \{M^{(1)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} \text{ and } r(C^{-1}B^{(1)}A^{-1}) = r(M) \text{ for all } B^{(1)} \text{ (by (2.39))}\\ &\Leftrightarrow r(B^{(1)}) = r(B) \text{ for all } B^{(1)} \text{ (by (4.1) and (4.12))}\\ &\Leftrightarrow r(B) = \min\{m,\, n\} \text{ (by (2.28) and (2.29))}\\ &\Leftrightarrow r(B) = m \text{ or } r(B) = n,\end{aligned} \tag{4.24}$$

establishing the equivalence the first and third terms in Result (9b). Combining this fact with Result (9a) leads to the second equivalence in Result (9b).

By (4.1),

$$r(C^{-1}B^{(1,2)}A^{-1}) = r(B^{(1,2)}) = r(B) \tag{4.25}$$

holds for all $B^{(1,2)}$. Combining this fact with Result (2a) leads to

$$\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\}. \tag{4.26}$$

On the other hand, both $BCM^{(1,2)}AB = B$ and $r(CM^{(1,2)}A) = r(M^{(1,2)}) = r(M) = r(B)$ for all $M^{(1,2)}$ hold by (4.12), which implies $\{CM^{(1,2)}A\} \subseteq \{B^{(1,2)}\}$ by (2.39), so that $\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\}$ by (4.10). Combining this fact with (4.25) leads to the set equality in Result (10).

By (2.29), there exists a $B^{(1,3)}$ such that $r(C^{-1}B^{(1,3)}A^{-1}) = r(B^{(1,3)}) = r(B) = r(M)$. Combining this fact with (4.12), we see that the product satisfies $C^{-1}B^{(1,3)}A^{-1} \in \{M^{(1,2)}\}$, thus establishing Result (11a). By (2.39),

$$\begin{aligned}&\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\}\\ &\Leftrightarrow MC^{-1}B^{(1,3)}A^{-1}M = M \textit{ and } r(C^{-1}B^{(1,3)}A^{-1}) = r(M) \text{ for all } B^{(1,3)}\\ &\Leftrightarrow r(B^{(1,3)}) = r(B) \text{ for all } B^{(1,3)} \text{ (by (4.1) and (4.12))}\\ &\Leftrightarrow \min\{m,\, n\} = r(B) \text{ (by (2.28) and (2.29))}\\ &\Leftrightarrow r(B) = m \text{ or } r(B) = n,\end{aligned} \tag{4.27}$$

thus establishing Result (11b). By (4.10),

$$\begin{aligned}&\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{CM^{(1,2)}A\} \subseteq \{B^{(1,3)}\}\\ &\Leftrightarrow B^*BCM^{(1,2)}A = B^* \text{ for all } M^{(1,2)} \text{ (by (2.40))}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(h))},\end{aligned} \tag{4.28}$$

thus establishing Result (11c). Combining Results (11b) and (11c) leads to Result (11d).

Results (13a) follows from (4.7) and Result (10). By (4.10),

$$\begin{aligned}&\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{CM^{(1,2)}A\} \subseteq \{B^{(1,2,3)}\}\\ &\Leftrightarrow B^*BCM^{(1,2)}A = B^* \textit{ and } r(CM^{(1,2)}A) = r(B) \text{ for all } M^{(1,2)} \text{ (by (2.42))}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(h))},\end{aligned} \tag{4.29}$$

establishing the equivalence the first and third terms in Result (13b). Combining this fact with Result (13a) leads to the second equivalence in Result (13b).

By (2.29), there exists a $B^{(1,3,4)}$ such that $r(C^{-1}B^{(1,3,4)}A^{-1}) = r(B^{(1,3,4)}) = r(B) = r(M)$. Combining this fact with (4.12), we see that the product satisfies $C^{-1}B^{(1,3,4)}A^{-1} \in \{M^{(1,2)}\}$, thus establishing Result (15a). By (2.39),

$$\begin{aligned}&\{M^{(1,2)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\}\\ &\Leftrightarrow MC^{-1}B^{(1,3,4)}A^{-1}M = M \text{ and } r(C^{-1}B^{(1,3,4)}A^{-1}) = r(M) \text{ for all } B^{(1,3,4)}\\ &\Leftrightarrow r(B^{(1,3,4)}) = r(B) \text{ for all } B^{(1,3,4)} \text{ (by (4.1) and (4.12))}\\ &\Leftrightarrow r(B) = \min\{m,\, n\} \text{ (by (2.28) and (2.29))}\\ &\Leftrightarrow r(B) = m \text{ or } r(B) = n,\end{aligned} \tag{4.30}$$

thus establishing Result (15b). By (4.10),

$$\begin{aligned}&\{M^{(1,2)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,2)}A\} \subseteq \{B^{(1,3,4)}\}\\ &\Leftrightarrow B^*BCM^{(1,2)}A = B^* \text{ and } CM^{(1,2)}ABB^* = B^* \text{ for all } M^{(1,2)} \text{ (by (2.44))}\\ &\Leftrightarrow \{B = 0 \text{ or } r(B) = m\} \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(h))}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m = n,\end{aligned} \tag{4.31}$$

thus establishing Result (15c). Combining Results (11b) and (15c) leads to Result (15d).

Result (16) follows from (4.12), $r(C^{-1}B^{\dagger}A^{-1}) = r(B)$, and (2.39).

Result (17a) follows from (4.10) and (4.13). By (2.40),

$$\begin{aligned}\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\} &\Leftrightarrow M^*MC^{-1}B^{(1)}A^{-1} = M^* \text{ for all } B^{(1)}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(f))},\end{aligned} \tag{4.32}$$

thus establishing the equivalence the first and third terms in Result (17b). Combining this fact with Result (17a) leads to the second equivalence in Result (17b).

Result (18a) follows from Lemma 4.1(e). By (2.40),

$$\begin{aligned}\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\} &\Leftrightarrow M^*MC^{-1}B^{(1,2)}A^{-1} = M^* \text{ for all } B^{(1,2)}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(f))},\end{aligned} \tag{4.33}$$

thus establishing Result (18b). By (4.10),

$$\begin{aligned}&\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow \{CM^{(1,3)}A\} \subseteq \{B^{(1,2)}\}\\ &\Leftrightarrow BCM^{(1,3)}AB = B \text{ and } r(CM^{(1,3)}A) = r(B)\text{for all } M^{(1,3)}\\ &\Leftrightarrow r(M^{(1,3)}) = r(B) \text{ for all } M^{(1,3)} \text{ (by (4.1) and (4.13))}\\ &\Leftrightarrow r(B) = m \text{ or } r(B) = n \text{ (by (2.28) and (2.29))},\end{aligned} \tag{4.34}$$

thus establishing Result (18c). Combining Results (18b) and (18c) leads to Result (18d).

Result (19a) follows from Lemma 4.1(e). By (2.40),

$$\begin{aligned}\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} &\Leftrightarrow M^*MC^{-1}B^{(1,3)}A^{-1} = M^* \text{ for all } B^{(1,3)}\\ &\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(g))}.\end{aligned} \tag{4.35}$$

Also by (4.10),

$$\begin{aligned}
&\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{CM^{(1,3)}A\} \subseteq \{B^{(1,3)}\} \\
&\Leftrightarrow B^*BCM^{(1,3)}A = B^* \text{ for all } M^{(1,3)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(i))}, \qquad (4.36)
\end{aligned}$$

Combining (4.35) and (4.36) leads to Result (19b).

Result (20a) follows from Lemma 4.1(e). By (2.40),

$$\begin{aligned}
\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\} &\Leftrightarrow M^*MC^{-1}B^{(1,4)}A^{-1} = M^* \text{ for all } B^{(1,4)} \\
&\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(f))}, \qquad (4.37)
\end{aligned}$$

thus establishing Result (20b). By (4.10),

$$\begin{aligned}
&\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,3)}A\} \subseteq \{B^{(1,4)}\} \\
&\Leftrightarrow CM^{(1,3)}ABB^* = B^* \text{ for all } M^{(1,3)} \\
&\Leftrightarrow B = 0 \text{ or } r(B) = n \text{ (by Lemma 4.1(f))}, \qquad (4.38)
\end{aligned}$$

thus establishing Result (20c).

Combining (4.41) and (4.42) leads to Result (20c). Combining Results (20b) and (20c) leads to Result (20d).

By (2.40),

$$\begin{aligned}
\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset &\Leftrightarrow M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^* \text{ for a } B^{(1,2,3)} \\
&\Leftrightarrow \{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \\
&\Leftrightarrow M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^* \text{ for all } B^{(1,2,3)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(g))}. \qquad (4.39)
\end{aligned}$$

establishing Result (21a). By (4.10) and By (2.42),

$$\begin{aligned}
&\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{CM^{(1,3)}A\} \subseteq \{B^{(1,2,3)}\} \\
&\Leftrightarrow B^*BCM^{(1,3)}A = B^* \text{ and } r(CM^{(1,3)}A) = r(B) \text{ for all } M^{(1,3)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } r(B) = \min\{m,\, n\} \text{ (by Lemma 4.1(i), (2.28) and (2.29))}, \qquad (4.40)
\end{aligned}$$

establishing the equivalence the first and third terms in Result (21b). Combining this fact with Result (21a) leads to the second equivalence in Result (21b).

Result (22a) follows from Lemma 4.1(e). By (2.40),

$$\begin{aligned}
\{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} &\Leftrightarrow M^*MC^{-1}B^{(1,2,4)}A^{-1} = M^* \text{ for all } B^{(1,2,4)} \\
&\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(f))}, \qquad (4.41)
\end{aligned}$$

thus establishing Result (22b). By (4.10),

$$\begin{aligned}
&\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,3)}A\} \subseteq \{B^{(1,2,4)}\} \\
&\Leftrightarrow CM^{(1,3)}ABB^* = B^* \text{ and } r(CM^{(1,3)}A) = r(B) \text{ for all } M^{(1,3)} \\
&\Leftrightarrow \{B = 0 \text{ or } r(B) = n\} \text{ and } r(B) = \min\{m,\, n\} \text{ (by Lemma 4.1(f), (2.28) and (2.29))} \\
&\Leftrightarrow r(B) = n, \qquad (4.42)
\end{aligned}$$

thus establishing Result (22c). Combining Results (22b) and (22c) leads to Result (22d).

By (2.40),

$$\begin{aligned}
\{M^{(1,3)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset &\Leftrightarrow M^*MC^{-1}B^{(1,3,4)}A^{-1} = M^* \text{ for a } B^{(1,3,4)} \\
&\quad \{M^{(1,3)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \\
&\Leftrightarrow M^*MC^{-1}B^{(1,3,4)}A^{-1} = M^* \text{ for all } B^{(1,3,4)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(g))}, \qquad (4.43)
\end{aligned}$$

thus establishing Result (23a).

Also by (4.10),

$$\begin{aligned}
&\{M^{(1,3)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,3)}A\} \subseteq \{B^{(1,3,4)}\} \\
&\Leftrightarrow B^*BCM^{(1,3)}A = B^* \text{ and } CM^{(1,3)}ABB^* = B^* \text{for all } M^{(1,3)} \text{ (by (2.44))} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(h) and (i))} \\
&\Leftrightarrow B = 0 \text{ or } \{r(B) = n \text{ and } \mathscr{R}(A^*AB) = \mathscr{R}(B)\}. \qquad (4.44)
\end{aligned}$$

establishing the equivalence the first and third terms in Result (23b). Combining this fact with Result (23a) leads to the second equivalence in Result (23b).

Result (24) follows from (2.40) and Lemma 4.1(g).

Results (25a)–(32) are obtained from Lemma 2.1, Results (17a)–(24), and matrix replacements.

Result (33a) follows from (4.10) and (4.13). By (2.42),

$$\begin{aligned}
&\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\}\\
&\Leftrightarrow M^*MC^{-1}B^{(1)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1)}A^{-1}) = r(M) \text{ for all } B^{(1)}\\
&\Leftrightarrow \{B = 0 \text{ or } r(B) = m\} \text{ and } r(B) = \min\{m,\, n\} \text{ (by Lemma 4.1(f), (2.28) and (2.29))}\\
&\Leftrightarrow r(B) = m, \qquad (4.45)
\end{aligned}$$

thus establishing the equivalence the first and third terms in Result (33b). Combining this fact with Result (33a) leads to the second equivalence in Result (33b).

Result (34a) follows from (4.10), (4.13) and $r(M^{(1,2,3)}) = r(M) = r(B)$. By (2.42),

$$\begin{aligned}
&\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\}\\
&\Leftrightarrow M^*MC^{-1}B^{(1,2)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1,2)}A^{-1}) = r(M) \text{ for all } B^{(1,2)}\\
&\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(f))}, \qquad (4.46)
\end{aligned}$$

thus establishing the equivalence the first and third terms in Result (34b). Combining this fact with Result (34a) leads to the second equivalence in Result (34b).

By (2.42),

$$\begin{aligned}
\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset &\Leftrightarrow M^*MC^{-1}B^{(1,3)}A^{-1} = M^* \text{ for a } B^{(1,3)}\\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(g))}, \qquad (4.47)
\end{aligned}$$

and by (4.10),

$$\begin{aligned}
&\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} \Leftrightarrow \{CM^{(1,2,3)}A\} \subseteq \{B^{(1,3)}\}\\
&\Leftrightarrow B^*BCM^{(1,2,3)}A = B^* \text{ for all } M^{(1,2,3)}\\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(i))}, \qquad (4.48)
\end{aligned}$$

Combining (4.47) and (4.48) leads to Result (35a). By (2.42),

$$\begin{aligned}
&\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\}\\
&M^*MC^{-1}B^{(1,3)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1,3)}A^{-1}) = r(M) \text{ for all } B^{(1,3)}\\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } r(B) = \min\{m,\, n\} \text{ (by Lemma 4.1(g), (2.28) and (2.29))}. \qquad (4.49)
\end{aligned}$$

thus establishing the equivalence the first and third terms in Result (35b). Combining this fact with Result (35a) leads to the second equivalence in Result (35b).

Result (36a) follows from (4.10) and Lemma 4.1(e). By (2.42),

$$\begin{aligned}
&\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,4)}A^{-1}\}\\
&\Leftrightarrow M^*MC^{-1}B^{(1,4)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1,4)}A^{-1}) = r(M) \text{for all } B^{(1,4)}\\
&\Leftrightarrow M^*MC^{-1}B^{(1,4)}A^{-1} = M^* \text{ and } r(B^{(1,4)}) = r(B) \text{for all } B^{(1,4)} \text{ (by Lemma 4.1(g))}\\
&\Leftrightarrow \{B = 0 \text{ or } r(B) = m\} \text{ and } r(B) = \min\{m,\, n\} \text{ (by Lemma 4.1(f), (2.28) and (2.29))}\\
&\Leftrightarrow r(B) = m, \qquad (4.50)
\end{aligned}$$

thus establishing Result (36b). Also by (4.10),

$$\begin{aligned}
&\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,2,3)}A\} \subseteq \{B^{(1,4)}\}\\
&\Leftrightarrow CM^{(1,2,3)}ABB^* = B^* \text{ for all } M^{(1,2,3)}\\
&\Leftrightarrow B = 0 \text{ or } r(B) = n \text{ (by Lemma 4.1(h))}, \qquad (4.51)
\end{aligned}$$

thus establishing Result (36c). Combining Results (36b) and (36c) leads to Result (36d).

By (4.10),

$$\begin{aligned}
&\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset\\
&\Leftrightarrow M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1,2,3)}A^{-1}) = r(M) \text{ for a } B^{(1,2,3)}\\
&\Leftrightarrow \{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\}\\
&\Leftrightarrow M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1,2,3)}A^{-1}) = r(M) \text{ for all } B^{(1,2,3)}\\
&\Leftrightarrow M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^*\\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(g))}, \qquad (4.52)
\end{aligned}$$

and by (4.10),

$$\begin{aligned}&\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{CM^{(1,2,3)}A\} \subseteq \{B^{(1,2,3)}\}\\ &\Leftrightarrow B^*BCM^{(1,2,3)}A = B^* \text{ for all } M^{(1,2,3)}\\ &\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(i))},\end{aligned} \tag{4.53}$$

Combining (4.52) and (4.53) leads to Result (37).

Result (38a) follows from (4.10) and Lemma 4.1(e). By (2.42),

$$\begin{aligned}\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} &\Leftrightarrow M^*MC^{-1}B^{(1,2,4)}A^{-1} = M^* \text{ for all } B^{(1,2,4)}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m \text{ (by Lemma 4.1(h))},\end{aligned} \tag{4.54}$$

thus establishing Result (38b). By (4.10),

$$\begin{aligned}&\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,2,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,2,3)}A\} \subseteq \{B^{(1,2,4)}\}\\ &\Leftrightarrow CM^{(1,2,3)}ABB^* = B^* \text{ for all } M^{(1,2,3)} \text{ (by (2.43))}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = n \text{ (by Lemma 4.1(h))},\end{aligned} \tag{4.55}$$

thus establishing Result (38c). Combining Results (38b) and (38c) leads to Result (38d).

By (2.42),

$$\begin{aligned}&\{M^{(1,2,3)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset\\ &\Leftrightarrow M^*MC^{-1}B^{(1,3,4)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1,3,4)}A^{-1}) = r(M) \text{for a } B^{(1,3,4)}\\ &\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(g))},\end{aligned} \tag{4.56}$$

thus establishing Result (39a). Also by (2.42),

$$\begin{aligned}&\{M^{(1,2,3)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\}\\ &\Leftrightarrow M^*MC^{-1}B^{(1,3,4)}A^{-1} = M^* \text{ and } r(C^{-1}B^{(1,3,4)}A^{-1}) = r(M) \text{ for all } B^{(1,3,4)}\\ &\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } r(B) = \min\{m, n\} \text{ (by Lemma 4.1(g), (2.28) and (2.29))},\end{aligned} \tag{4.57}$$

thus establishing Result (39b). and by (4.10),

$$\begin{aligned}&\{M^{(1,2,3)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,2,3)}A\} \subseteq \{B^{(1,3,4)}\}\\ &\Leftrightarrow B^*BCM^{(1,2,3)}A = B^* \text{ and } CM^{(1,2,3)}ABB^* = B^* \text{ for all } M^{(1,2,3)} \text{ (by (2.44))}\\ &\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(h) and (i))},\end{aligned} \tag{4.58}$$

thus establishing Result (39c).

Result (40) follows from (2.42) and Lemma 4.1(g).

Results (41a)–(48) are obtained from Lemma 2.1, Results (33a)–(40), and matrix replacements.

Result (49a) follows from (2.38), (4.10), and (4.13). By (2.44),

$$\begin{aligned}&\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1)}A^{-1}\}\\ &\Leftrightarrow M^*MC^{-1}B^{(1)}A^{-1} = M^* \text{ and } C^{-1}B^{(1)}A^{-1}MM^* = M^* \text{ for all } B^{(1)}\\ &\Leftrightarrow \{B = 0 \text{ or } r(B) = m\} \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(f))}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m = n,\end{aligned} \tag{4.59}$$

thus establishing the equivalence the first and third terms in Result (49b). Combining this fact with Result (49a) leads to the second equivalence in Result (49b).

Result (50a) follows from (4.8) and (4.13). By (2.44),

$$\begin{aligned}&\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,2)}A^{-1}\}\\ &\Leftrightarrow M^*MC^{-1}B^{(1,2)}A^{-1} = M^* \text{ and } C^{-1}B^{(1,2)}A^{-1}MM^* = M^* \text{ for all } B^{(1,2)}\\ &\Leftrightarrow \{B = 0 \text{ or } r(B) = m\} \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(f))}\\ &\Leftrightarrow B = 0 \text{ or } r(B) = m = n,\end{aligned} \tag{4.60}$$

thus establishing Result (50b). By (4.10),

$$\begin{aligned}&\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,2)}A^{-1}\} \Leftrightarrow \{CM^{(1,3,4)}A\} \supseteq \{B^{(1,2)}\}\\ &\Leftrightarrow BCM^{(1,3,4)}AB = B \text{ and } r(CM^{(1,3,4)}AB) = r(B) \text{ for all } M^{(1,3,4)}\\ &\Leftrightarrow r(B) = \min\{m, n\} \text{ (by (4.13), (2.28) and (2.29))}\\ &\Leftrightarrow r(B) = m \text{ or } r(B) = n,\end{aligned} \tag{4.61}$$

thus establishing Result (50c). Combining Results (50b) and (50c) leads to Result (50d).

By (2.44),

$$\begin{aligned}
&\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,3)}A^{-1}\} \neq \emptyset \\
&\Leftrightarrow M^*MC^{-1}B^{(1,3)}A^{-1} = M^* \text{ and } C^{-1}B^{(1,3)}A^{-1}MM^* = M^* \text{for a } B^{(1,3)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(e) and (g))},
\end{aligned} \tag{4.62}$$

and by (4.10),

$$\begin{aligned}
\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,3)}A^{-1}\} &\Leftrightarrow \{CM^{(1,3,4)}A\} \subseteq \{B^{(1,3)}\} \\
&\Leftrightarrow B^*BCM^{(1,3,4)}A = B^* \text{ for all } M^{(1,3,4)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(i))}.
\end{aligned} \tag{4.63}$$

Combining (4.62) and (4.63) leads to Result (51a). by (4.10),

$$\begin{aligned}
&\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,3)}A^{-1}\} \\
&\Leftrightarrow M^*MC^{-1}B^{(1,3)}A^{-1} = M^* \text{ and } C^{-1}B^{(1,3)}A^{-1}MM^* = M^* \text{ for all } B^{(1,3)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(f) and (g))} \\
&\Leftrightarrow B = 0 \text{ or } \{r(B) = n \text{ and } \mathscr{R}(A^*AB) = \mathscr{R}(B)\},
\end{aligned} \tag{4.64}$$

thus establishing Result (51b).

Results (52a) and (52b) are obtained from Lemma 2.1(e), Results (51a) and (51), and matrix replacements.

By (2.44),

$$\begin{aligned}
&\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,2,3)}A^{-1}\} \neq \emptyset \\
&\Leftrightarrow M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^* \text{ and } C^{-1}B^{(1,2,3)}A^{-1}MM^* = M^* \text{for a } B^{(1,2,3)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ (by Lemma 4.1(e) and (g))},
\end{aligned} \tag{4.65}$$

thus establishing Result (53a). By (4.10),

$$\begin{aligned}
&\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \\
&\Leftrightarrow M^*MC^{-1}B^{(1,2,3)}A^{-1} = M^* \text{ and } C^{-1}B^{(1,2,3)}A^{-1}MM^* = M^* \text{ for all } B^{(1,2,3)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \{B = 0 \text{ or } r(B) = n\} \text{ (by Lemma 4.1(f) and (g))} \\
&\Leftrightarrow B = 0 \text{ or } \{r(B) = n \text{ and } \mathscr{R}(A^*AB) = \mathscr{R}(B)\},
\end{aligned} \tag{4.66}$$

thus establishing Result (53b). By (4.10),

$$\begin{aligned}
&\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,2,3)}A^{-1}\} \Leftrightarrow \{CM^{(1,3,4)}A\} \subseteq \{B^{(1,2,3)}\} \\
&\Leftrightarrow B^*BCM^{(1,3,4)}A = B^* \text{ and } r(CM^{(1,3,4)}A) = r(B) \text{ for all } M^{(1,3,4)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } r(B) = \min\{m, n\} \text{ (by Lemma 4.1(g), (2.28) and (2.29))},
\end{aligned} \tag{4.67}$$

thus establishing Result (53c). Combining Results (53b) and (53c) leads to Result (53d).

Results (54a)–(54d) are obtained from Lemma 2.1(e), Results (53a)–(53d), and matrix replacements.

By (2.44),

$$\begin{aligned}
&\{M^{(1,3,4)}\} \cap \{C^{-1}B^{(1,3,4)}A^{-1}\} \neq \emptyset \\
&\Leftrightarrow M^*MC^{-1}B^{(1,3,4)}A^{-1} = M^* \text{ and } C^{-1}B^{(1,3,4)}A^{-1}MM^* = M^* \text{ for a } B^{(1,3,4)} \\
&\{M^{(1,3,4)}\} \supseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \\
&\Leftrightarrow M^*MC^{-1}B^{(1,3)}A^{-1} = M^* \text{ and } C^{-1}B^{(1,3)}A^{-1}MM^* = M^* \text{ for all } B^{(1,3,4)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*) \text{ (by Lemma 4.1(g))},
\end{aligned} \tag{4.68}$$

and by (4.10),

$$\begin{aligned}
&\{M^{(1,3,4)}\} \subseteq \{C^{-1}B^{(1,3,4)}A^{-1}\} \Leftrightarrow \{CM^{(1,3,4)}A\} \subseteq \{B^{(1,3,4)}\} \\
&\Leftrightarrow B^*BCM^{(1,3,4)}A = B^* \text{ and } CM^{(1,3,4)}ABB^* = B^* \text{ for all } M^{(1,3,4)} \\
&\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*) \text{ (by Lemma 4.1(i))}.
\end{aligned} \tag{4.69}$$

Combining (4.68) and (4.69) leads to Result (55).

By (4.10),

$$\begin{aligned}\{M^{(1,3,4)}\} \ni C^{-1}B^{\dagger}A^{-1} &\Leftrightarrow M^*MC^{-1}B^{\dagger}A^{-1} = M^* \text{ and } C^{-1}B^{\dagger}A^{-1}MM^* = M^* \\ &\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \mathscr{R}(CC^*B^*) = \mathscr{R}(B^*) \text{ (by Lemma 4.1(g))}, \end{aligned} \tag{4.70}$$

establishing Result (56).

Finally, we leave the verifications of Results (57)–(64) to the reader. □

We have presented a classification analysis to (1.5) using the elementary matrix range and rank method and established hundreds of necessary and sufficient conditions for (1.5) to hold for the eight commonly-used types of generalized inverses of matrices. With doubt, we can use the previous results to solve many calculation problems on generalized inverses of matrix products, for example, when both A and C are unitary matrices, that is, If $AA^* = A^*A = I_m$ and $CC^* = C^*C = I_n$, then Theorem 4.2 reduces to a family of trivial results. If A, B, and C happen to be square matrices of the same size, and $C = A^{-1}$, then (1.5) can be written as

$$(ABA^{-1})^{(i,\dots,j)} = AB^{(k,\dots,l)}A^{-1}, \tag{4.71}$$

which are covariance equalities for generalized inverses of matrices. The special case $(ABA^{-1})^{\dagger} = AB^{\dagger}A^{-1}$ was approached by several authors; see, e.g., [1, 16, 20–22], and the relevant literature quoted there.

Corollary 4.3. *Let A, $B \in \mathbb{C}^{m\times m}$ and assume that A is nonsingular. Also denote $M = ABA^{-1}$. Then*

(a) *The following results hold*

$$\begin{array}{ll}
\{M^{(1)}\} = \{AB^{(1)}A^{-1}\}, & \{M^{(1)}\} \supseteq \{AB^{(1,2)}A^{-1}\}, \\
\{M^{(1)}\} \supseteq \{AB^{(1,3)}A^{-1}\} & \{M^{(1)}\} \supseteq \{AB^{(1,4)}A^{-1}\}, \\
\{M^{(1)}\} \supseteq \{AB^{(1,2,3)}A^{-1}\}, & \{M^{(1)}\} \supseteq \{AB^{(1,2,4)}A^{-1}\}, \\
\{M^{(1)}\} \supseteq \{AB^{(1,3,4)}A^{-1}\}, & \{M^{(1)}\} \ni AB^{\dagger}A^{-1}, \\
\{M^{(1,2)}\} \subseteq \{AB^{(1)}A^{-1}\}, & \{M^{(1,2)}\} = \{AB^{(1,2)}A^{-1}\}, \\
\{M^{(1,2)}\} \cap \{AB^{(1,3)}A^{-1}\} \neq \emptyset, & \{M^{(1,2)}\} \cap \{AB^{(1,4)}A^{-1}\} \neq \emptyset, \\
\{M^{(1,2)}\} \supseteq \{AB^{(1,2,3)}A^{-1}\}, & \{M^{(1,2)}\} \supseteq \{AB^{(1,2,4)}A^{-1}\}, \\
\{M^{(1,2)}\} \cap \{AB^{(1,3,4)}A^{-1}\} \neq \emptyset, & \{M^{(1,2)}\} \ni AB^{\dagger}A^{-1}, \\
\{M^{(1,3)}\} \subseteq \{AB^{(1)}A^{-1}\}, & \{M^{(1,3)}\} \cap \{AB^{(1,2)}A^{-1}\} \neq \emptyset, \\
\{M^{(1,3)}\} \cap \{AB^{(1,3)}A^{-1}\} \neq \emptyset, & \{M^{(1,3)}\} \cap \{AB^{(1,4)}A^{-1}\} \neq \emptyset, \\
\{M^{(1,3)}\} \cap \{AB^{(1,2,4)}A^{-1}\} \neq \emptyset, & \{M^{(1,4)}\} \subseteq \{AB^{(1)}A^{-1}\}, \\
\{M^{(1,4)}\} \cap \{AB^{(1,2)}A^{-1}\} \neq \emptyset, & \{M^{(1,4)}\} \cap \{AB^{(1,3)}A^{-1}\} \neq \emptyset, \\
\{M^{(1,4)}\} \cap \{AB^{(1,4)}A^{-1}\} \neq \emptyset, & \{M^{(1,4)}\} \cap \{AB^{(1,2,3)}A^{-1}\} \neq \emptyset, \\
\{M^{(1,2,3)}\} \subseteq \{AB^{(1)}A^{-1}\}, & \{M^{(1,2,3)}\} \subseteq \{AB^{(1,2)}A^{-1}\}, \\
\{M^{(1,2,3)}\} \cap \{AB^{(1,4)}A^{-1}\} \neq \emptyset, & \{M^{(1,2,3)}\} \cap \{AB^{(1,2,4)}A^{-1}\} \neq \emptyset, \\
\{M^{(1,2,4)}\} \subseteq \{AB^{(1)}A^{-1}\}, & \{M^{(1,2,4)}\} \subseteq \{AB^{(1,2)}A^{-1}\}, \\
\{M^{(1,2,4)}\} \cap \{AB^{(1,3)}A^{-1}\} \neq \emptyset, & \{M^{(1,2,4)}\} \cap \{AB^{(1,2,3)}A^{-1}\} \neq \emptyset, \\
\{M^{(1,3,4)}\} \subseteq \{AB^{(1)}A^{-1}\}, & \{M^{(1,3,4)}\} \cap \{AB^{(1,2)}A^{-1}\} \neq \emptyset, \\
M^{\dagger} \in \{AB^{(1)}A^{-1}\}, & M^{\dagger} \in \{AB^{(1,2)}A^{-1}\}.
\end{array}$$

(b) *The following equivalent statements hold*

$$\begin{aligned}
&\{M^{(1)}\} \subseteq \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} \subseteq \{AB^{(1,2,3)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1)}\} = \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} \subseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{AB^{(1,2,4)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,2)}\} \supseteq \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} = \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} \supseteq \{AB^{(1,3)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,2)}\} = \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} \supseteq \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} = \{AB^{(1,4)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,2)}\} \supseteq \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} = \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1,2)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,3)}\} \subseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \subseteq \{AB^{(1,2)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,4)}\} = \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \subseteq \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} = \{AB^{(1,2,3)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,2,3)}\} \supseteq \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} = \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} \supseteq \{AB^{(1,4)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,2,3)}\} = \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} \supseteq \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} \subseteq \{AB^{(1,3,4)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,2,4)}\} \supseteq \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} = \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} \supseteq \{AB^{(1,3)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,2,4)}\} = \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} \supseteq \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} = \{AB^{(1,3,4)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,3,4)}\} \subseteq \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} = \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \subseteq \{AB^{(1,2,3)}A^{-1}\} \\
&\Leftrightarrow \{M^{(1,3,4)}\} = \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \subseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} = \{AB^{(1,2,4)}A^{-1}\} \\
&\Leftrightarrow r(B) = m.
\end{aligned}$$

(c) *The following equivalent statements hold*

$$\{M^{(1)}\} \subseteq \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} \subseteq \{AB^{(1,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1)}\} = \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} \subseteq \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1)}\} = \{AB^{(1,3,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2)}\} \subseteq \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} \subseteq \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} \subseteq \{AB^{(1,2,3)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2)}\} = \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} \subseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2)}\} = \{AB^{(1,2,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2)}\} \subseteq \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \supseteq \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,3)}\} \supseteq \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \supseteq \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \subseteq \{AB^{(1,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \supseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \supseteq \{AB^{(1)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \supseteq \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \supseteq \{AB^{(1,3)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,4)}\} \subseteq \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} = \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \supseteq \{AB^{(1,2,3)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2,3)}\} \supseteq \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} = \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} \subseteq \{AB^{(1,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2,3)}\} \supseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} \subseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} = \{AB^{(1,2,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2,3)}\} \subseteq \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} \supseteq \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} = \{AB^{(1,2)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2,4)}\} \subseteq \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} \supseteq \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} \subseteq \{AB^{(1,2,3)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2,4)}\} = \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} \subseteq \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \supseteq \{AB^{(1)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,3,4)}\} = \{AB^{(1)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \supseteq \{AB^{(1,2)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \supseteq \{AB^{(1,3)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,3,4)}\} = \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \supseteq \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} = \{AB^{(1,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,3,4)}\} \supseteq \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \supseteq \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow B = 0 \text{ or } r(B) = m.$$

(d) *The following equivalent statements hold*

$$\{M^{(1,3)}\} \supseteq \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \subseteq \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1,3)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,3)}\} \cap \{AB^{(1,2,3)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \cap \{AB^{(1,3,4)}A^{-1}\} \neq \emptyset$$
$$\Leftrightarrow \{M^{(1,3)}\} = \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3)}\} \ni AB^{\dagger}A^{-1} \Leftrightarrow \{M^{(1,2,3)}\} \cap \{AB^{(1,3)}A^{-1}\} \neq \emptyset$$
$$\Leftrightarrow \{M^{(1,2,3)}\} = \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,2,3)}\} \cap \{AB^{(1,2,3)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,3)}\} = \{AB^{(1,2,3)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2,3)}\} \cap \{AB^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,3)}\} \ni AB^{\dagger}A^{-1} \Leftrightarrow \{M^{(1,3,4)}\} \cap \{AB^{(1,3)}A^{-1}\} \neq \emptyset$$
$$\Leftrightarrow \{M^{(1,3,4)}\} \subseteq \{AB^{(1,3)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \cap \{AB^{(1,2,3)}A^{-1}\} \neq \emptyset \Leftrightarrow M^{\dagger} \in \{AB^{(1,3)}A^{-1}\}$$
$$\Leftrightarrow M^{\dagger} \in \{AB^{(1,2,3)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B).$$

(e) *The following equivalent statements hold*

$$\{M^{(1,4)}\} \supseteq \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \subseteq \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} = \{AB^{(1,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,4)}\} \cap \{AB^{(1,2,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,4)}\} = \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \cap \{AB^{(1,3,4)}A^{-1}\} \neq \emptyset$$
$$\Leftrightarrow \{M^{(1,4)}\} = \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,4)}\} \ni AB^{\dagger}A^{-1} \Leftrightarrow \{M^{(1,2,4)}\} \cap \{AB^{(1,4)}A^{-1}\} \neq \emptyset$$
$$\Leftrightarrow \{M^{(1,2,4)}\} = \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,2,4)}\} \cap \{AB^{(1,2,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,4)}\} = \{AB^{(1,2,4)}A^{-1}\}$$
$$\Leftrightarrow \{M^{(1,2,4)}\} \cap \{AB^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,2,4)}\} \ni AB^{\dagger}A^{-1} \Leftrightarrow \{M^{(1,3,4)}\} \cap \{AB^{(1,4)}A^{-1}\} \neq \emptyset$$
$$\Leftrightarrow \{M^{(1,3,4)}\} \subseteq \{AB^{(1,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \cap \{AB^{(1,2,4)}A^{-1}\} \neq \emptyset \Leftrightarrow M^{\dagger} \in \{AB^{(1,4)}A^{-1}\}$$
$$\Leftrightarrow M^{\dagger} \in \{AB^{(1,2,4)}A^{-1}\} \Leftrightarrow \mathscr{R}(A^*AB^*) = \mathscr{R}(B^*).$$

(f) *The following equivalent statements hold*

$$\{M^{(1,3,4)}\} \cap \{AB^{(1,3,4)}A^{-1}\} \neq \emptyset \Leftrightarrow \{M^{(1,3,4)}\} = \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow \{M^{(1,3,4)}\} \ni AB^{\dagger}A^{-1}$$
$$\Leftrightarrow M^{\dagger} \in \{AB^{(1,3,4)}A^{-1}\} \Leftrightarrow M^{\dagger} = AB^{\dagger}A^{-1} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B) \text{ and } \mathscr{R}(A^*AB^*) = \mathscr{R}(B^*).$$

(g) *Under $A^*A = I_m$, the following equalities hold*

$$\begin{aligned}
&\{(ABA^*)^{(1)}\} = \{AB^{(1)}A^*\}, && \{(ABA^*)^{(1,2)}\} = \{AB^{(1,2)}A^*\},\\
&\{(ABA^*)^{(1,3)}\} = \{AB^{(1,3)}A^*\}, && \{(ABA^*)^{(1,4)}\} = \{AB^{(1,4)}A^*\},\\
&\{(ABA^*)^{(1,2,3)}\} = \{AB^{(1,2,3)}A^*\}, && \{M^{(1,2,4)}\} = \{AB^{(1,2,4)}A^*\},\\
&\{(ABA^*)^{(1,3,4)}\} = \{AB^{(1,3,4)}A^*\}, && (ABA^*)^{\dagger} = AB^{\dagger}A^*.
\end{aligned}$$

Finally, we give two consequences. Let $A,\, B \in \mathbb{C}^{m\times n}$. Then it is easy to verify

$$A+B = \frac{1}{2}[I_m,\, I_m]\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} I_n \\ I_n \end{bmatrix} = \frac{1}{2}PNQ, \quad P^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m \\ I_m \end{bmatrix}, \quad Q^{\dagger} = \frac{1}{\sqrt{2}}[I_n,\, I_n]. \tag{4.72}$$

Corollary 4.4. *Let $A,\, B \in \mathbb{C}^{m\times n}$ and N be as given in* (4.72). *Then the following seven set equalities and a matrix equality hold*

$$\begin{aligned}
&\{(A+B)^{(1)}\} = \left\{\frac{1}{2}[I_n,\, I_n]N^{(1)}\begin{bmatrix} I_m \\ I_m \end{bmatrix}\right\}, && \{(A+B)^{(1,2)}\} = \left\{\frac{1}{2}[I_n,\, I_n]N^{(1,2)}\begin{bmatrix} I_m \\ I_m \end{bmatrix}\right\}, \\
&\{(A+B)^{(1,3)}\} = \left\{\frac{1}{2}[I_n,\, I_n]N^{(1,3)}\begin{bmatrix} I_m \\ I_m \end{bmatrix}\right\}, && \{(A+B)^{(1,4)}\} = \left\{\frac{1}{2}[I_n,\, I_n]N^{(1,4)}\begin{bmatrix} I_m \\ I_m \end{bmatrix}\right\}, \\
&\{(A+B)^{(1,2,3)}\} = \left\{\frac{1}{2}[I_n,\, I_n]N^{(1,2,3)}\begin{bmatrix} I_m \\ I_m \end{bmatrix}\right\}, && \{(A+B)^{(1,2,4)}\} = \left\{\frac{1}{2}[I_n,\, I_n]N^{(1,2,4)}\begin{bmatrix} I_m \\ I_m \end{bmatrix}\right\}, \\
&\{(A+B)^{(1,3,4)}\} = \left\{\frac{1}{2}[I_n,\, I_n]N^{(1,3,4)}\begin{bmatrix} I_m \\ I_m \end{bmatrix}\right\}, && (A+B)^{\dagger} = \frac{1}{2}[I_n,\, I_n]N^{\dagger}\begin{bmatrix} I_m \\ I_m \end{bmatrix}.
\end{aligned}$$

Substituting concrete expressions of generalized inverse of N into the above equalities will yield various formulas for calculating generalized inverses of $A+B$. Some previous results on this topic can be found in [31].

Another pair of examples are given below. It is easy to verify that following two identities

$$(I_m + \alpha A + \beta B) = (I_m + \alpha A)[I_m - (\alpha\beta)(1+\alpha)^{-1}(1+\beta)^{-1}AB](I_m + \beta B), \tag{4.73}$$

$$(I_m + \alpha A + \beta B) = (I_m + \beta B)[I_m - (\alpha\beta)(1+\alpha)^{-1}(1+\beta)^{-1}BA](I_m + \alpha A) \tag{4.74}$$

hold for two idempotents A and B, where $\alpha \neq -1, 0$ and $\beta \neq -1, 0$; see [39]. In this case, $I_m + \alpha A$ and $I_m + \beta B$ are nonsingular, and thus two families of reverse order laws for generalized inverses associated with (4.73) and (4.74) are given by

$$\begin{aligned}
(I_m - \lambda AB)^{(i,\ldots,j)} &= (I_m + \beta B)(I_m + \alpha A + \beta B)^{(k,\ldots,l)}(I_m + \alpha A), \\
(I_m - \lambda BA)^{(i,\ldots,j)} &= (I_m + \alpha A)(I_m + \alpha A + \beta B)^{(k,\ldots,l)}(I_m + \beta B),
\end{aligned}$$

where $\lambda = (\alpha\beta)(1+\alpha)^{-1}(1+\beta)^{-1}$. Applying Theorem 4.2 to (4.73), we obtain the following consequence.

Corollary 4.5. *Let $A,\, B \in \mathbb{C}^{m\times m}$ be two idempotent matrices, assume $\alpha \neq -1, 0$ and $\beta \neq -1, 0$, and denote $\lambda = \alpha\beta(1-\alpha)^{-1}(1-\beta)^{-1}$. Then the following matrix set equality*

$$\{(I_m - \lambda AB)^{(1)}\} = \{(I_m + \beta B)(I_m + \alpha A + \beta B)^{(1)}(I_m + \alpha A)\}$$

holds. In particular, the reverse order law

$$(I_m - \lambda AB)^{\dagger} = (I_m + \beta B)(I_m + \alpha A + \beta B)^{\dagger}(I_m + \alpha A)$$

holds if and only if $\mathscr{R}[(I_m + \beta B)(I_m + \beta B)^*(I_m + \alpha A + \beta B)] = \mathscr{R}(I_m + \alpha A + \beta B)$ and $\mathscr{R}[(I_m + \alpha A)^*(I_m + \alpha A)(I_m + \alpha A + \beta B)^*] = \mathscr{R}[(I_m + \alpha A + \beta B)^*]$.

5 Cancellation Laws for Generalized Inverses of a Triple Matrix Product

Let $P \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{n\times p}$, and $Q \in \mathbb{C}^{p\times q}$. Then the product PAQ is defined. In particular, if P and Q satisfy the orthogonality conditions $P^*P = I_n$ and $QQ^* = I_p$, which may occur in decompositions of matrices involving orthogonal matrices, then it is easy to verify by definition that the following two reverse order laws and cancellation laws

$$(PAQ)^{\dagger} = Q^*A^{\dagger}P^*, \quad Q(PAQ)^{\dagger}P = A^{\dagger} \tag{5.1}$$

hold. In addition, the following matrix identities

$$PAQ(PAQ)^{\dagger} = PAA^{\dagger}P^*, \quad (PAQ)^{\dagger}(PAQ) = Q^*A^{\dagger}AQ, \tag{5.2}$$

$$AQ(PAQ)^{\dagger}P = AA^{\dagger}, \quad Q(PAQ)^{\dagger}PA = A^{\dagger}A \tag{5.3}$$

hold as well. In this section, we consider the following several equalities

$$(PAQ)^{(1)} = Q^*A^{(1)}P^*, \quad Q(PAQ)^{(1)}P = A^{(1)}, \tag{5.4}$$

$$PAQ(PAQ)^{(1)} = PAA^{(1)}P^*, \quad (PAQ)^{(1)}PAQ = Q^*A^{(1)}AQ, \tag{5.5}$$

$$AQ(PAQ)^{(1)}P = AA^{(1)}, \quad Q(PAQ)^{(1)}PA = A^{(1)}A \tag{5.6}$$

under the assumptions $P^*P = I_n$ and $QQ^* = I_p$, which are the extensions of (5.1)–(5.3) to $\{1\}$-inverse situations. Since g-inverses of a singular matrix are not unique, we need to describe the relationships between the matrix sets composed by both sides of (5.4)–(5.6), and derive necessary and sufficient conditions for the set inclusions and set equalities to hold.

Lemma 5.1 ([37]). *Let $X \in \mathbb{C}^{m\times n}$, $N \in \mathbb{C}^{k\times l}$, $S \in \mathbb{C}^{n\times l}$ and $T \in \mathbb{C}^{k\times m}$ be given with $r(S) = n$ and $r(T) = m$. Then the set inclusion*

$$\{X^{(1)}\} \subseteq \{SN^{(1)}T\}$$

holds if and only if $\mathscr{R}(N) \cap \mathscr{R}(T) = \{0\}$, or $\mathscr{R}(N^*) \cap \mathscr{R}(S^*) = \{0\}$, or

$$r(N - TXS) = r\begin{bmatrix} N \\ S \end{bmatrix} + r[N,\, T] - r(N) + r(X) - m - n.$$

The main results are given below.

Theorem 5.2. *Let $P \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{n\times p}$, and $Q \in \mathbb{C}^{p\times q}$ be given with $P^*P = I_n$ and $QQ^* = I_p$. Also denote $M = PAQ$. Then the following results hold.*

(a) $\{M^{(1)}\} \supseteq \{Q^*A^{(1)}P^*\}$ *always holds.*

(b) $\{M^{(1)}\} \subseteq \{Q^*A^{(1)}P^*\} \Leftrightarrow \{M^{(1)}\} = \{Q^*A^{(1)}P^*\} \Leftrightarrow m = n$ *and* $p = q$.

(c) $\{MM^{(1)}\} \supseteq \{PAA^{(1)}P^*\}$ *always holds.*

(d) $\{MM^{(1)}\} \subseteq \{PAA^{(1)}P^*\} \Leftrightarrow \{MM^{(1)}\} = \{PA^{(1)}AP^*\} \Leftrightarrow A = 0$ *or* $m = n$.

(e) $\{M^{(1)}M\} \supseteq \{Q^*A^{(1)}AQ\}$ *always holds.*

(f) $\{M^{(1)}M\} \subseteq \{Q^*A^{(1)}AQ\} \Leftrightarrow \{M^{(1)}M\} = \{Q^*A^{(1)}AQ\} \Leftrightarrow A = 0$ *or* $p = q$.

Proof. By definition, $MQ^*A^{(1)}P^*M = PAQQ^*A^{(1)}P^*PAQ = PAA^{(1)}AQ = PAQ = M$ holds for all $A^{(1)}$, thus establishing (a).

Applying Lemma 5.1 to $\{M^{(1)}\} \subseteq \{Q^*A^{(1)}P^*\}$ and simplifying leads to the equivalence of the first term and last term in (b). Combining this fact with (a) leads to the second equivalence in (b).

Pre- and post-multiplying the set inclusion in (a) with M yields (c) and (e), respectively.

By Lemma 2.1(b),

$$\{PAA^{(1)}P^*\} = \{PAA^{\dagger}P^* + PAU_1E_AP^*\}, \tag{5.7}$$

$$\{Q^*A^{(1)}AQ\} = \{Q^*A^{\dagger}AQ + Q^*F_AU_2AQ\}, \tag{5.8}$$

$$\{MM^{(1)}\} = \{MM^{\dagger} + MV_1E_M\}, \tag{5.9}$$

$$\{M^{(1)}M\} = \{M^{\dagger}M + F_MV_2M\}, \tag{5.10}$$

where $U_1,\, U_2 \in \mathbb{C}^{p\times n}$ and $V_1,\, V_2 \in \mathbb{C}^{q\times m}$ are arbitrary. In these cases, applying Lemma 4.2 to (5.7)–(5.10) and simplifying by Lemma 2.1 leads to (d) and (f), respectively. □

Theorem 5.3. *Let $P \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{n\times p}$, and $Q \in \mathbb{C}^{p\times q}$ be given with $P^*P = I_n$ and $QQ^* = I_p$. Also denote $M = PAQ$. Then the following three set equalities*

$$\{QM^{(1)}P\} = \{A^{(1)}\}, \quad \{AQM^{(1)}P\} = \{AA^{(1)}\}, \quad \{QM^{(1)}PA\} = \{A^{(1)}A\} \tag{5.11}$$

always hold.

Proof. Pre- and post-multiplying the set inclusion in Theorem 5.2(a) with Q and P yields $\{QM^{(1)}P\} \supseteq \{A^{(1)}\}$. By definition, $AQM^{(1)}PA = P^*PAQM^{(1)}PAQQ^* = P^*PAQM^{(1)}PAQQ^* = P^*MQ^* = A$ holds for all $M^{(1)}$, thus $\{QM^{(1)}P\} \subseteq \{A^{(1)}\}$ holds. Combining the two facts leads to the first set equality in (5.11). Pre- and post-multiplying first set equality in (5.11) with A yields the second and third set equalities in (5.11). □

6 Final Remarks

We have formulated some general research problems on equalities and reverse order laws for generalized inverses of matrices, approached many specified reverse order laws for the products of two and triple matrices the definitions of generalized inverses, the BMRM, and the MRM; obtained various identifying conditions for the reverse order laws to hold under various assumptions; and featured several examples that involve generalized inverses of matrices. We believe all the preceding results and facts can be used in the computations of various matrix expressions that involve products of matrices and their generalized inverses.

As a direct extension of the Moore–Penrose inverse, the weighted Moore–Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ with respect to two Hermitian positive semi-definite matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ is defined to be the solution X satisfying the following four equations

$$\text{(i) } MAXA = MA, \quad \text{(ii) } NXAX = NX, \quad \text{(iii) } (MAX)^* = MAX, \quad \text{(iv) } (NXA)^* = NXA,$$

see page 118, Exercise 33 in [4]. A matrix X is called a weighted $\{i,\dots,j\}$-generalized inverse of A, denoted by $A^{(i,\dots,j)}_{M,N}$, if it satisfies the ith,$\dots$, jth equations in (1.1). The collection of all weighted $\{i,\dots,j\}$-generalized inverses of A is denoted by $\{A^{(i,\dots,j)}_{M,N}\}$. There are also 15 types of weighted $\{i,\dots,j\}$-generalized inverses of A by definitions. In this situation, it would be of interest to consider the extensions of the preceding results to various reverse order laws for weighted generalized inverses of a matrix product. In addition to (5.1)–(5.6), it would also be of interest to consider the following reasonable matrix equalities

$$\begin{aligned} &(PAQ)^{(i,\dots,j)} = Q^*A^{(i,\dots,j)}P^*, && Q(PAQ)^{(i,\dots,j)}P = A^{(i,\dots,j)},\\ &PAQ(PAQ)^{(i,\dots,j)} = PAA^{(i,\dots,j)}P^*, && (PAQ)^{(i,\dots,j)}\}PAQ = Q^*A^{(i,\dots,j)}AQ,\\ &AQ(PAQ)^{(i,\dots,j)}P = AA^{(i,\dots,j)}, && Q(PAQ)^{(i,\dots,j)}PA = A^{(i,\dots,j)}A \end{aligned}$$

for other types of generalized inverses of the matrices under the conditions $P^*P = I_n$ and $QQ^* = I_p$. Both sides of these equalities are all linear or multilinear matrix-valued functions that involve one or two variables. Thus more matrix analysis tools are needed to characterize these equalities that involve variable matrices.

Recall moreover that generalized inverses of elements can be defined in many other algebraic structures in the same manner as in matrix case. Thus it would be of interest to consider the equality problems for generalized inverses of elements in other algebraic structures.

References

[1] M.H. Alizadeh. Note on the covariance coset of the Moore–Penrose inverses in C^*-algebras. J. Math. Extens. 7(2013), 1–7.

[2] E. Arghiriade. Remarques sur l'inverse généralisée d'un produit de matrices. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Ser. VIII 42(1967), 621–625.

[3] A. Ben-Israel, T.N.E. Greville. Generalized Inverses: Theory and Applications. 2nd ed., Springer, New York, 2003.

[4] S.L. Campbell, C.D. Meyer. Generalized Inverses of Linear Transformations. SIAM, Philadephia, 2009.

[5] T. Damm, H.K. Wimmer. A cancellation property of the Moore–Penrose inverse of triple products. J. Aust. Math. Soc. 86(2009), 33–44.

[6] N.Č. Dinčić, D.S. Djordjević. Basic reverse order law and its equivalencies. Aequat. math. 85(2013), 505–517.

[7] N.Č. Dinčić, D.S. Djordjević, D. Mosić. Mixed-type reverse order law and its equivalents. Studia Math. 204(2011), 123–136.

[8] N.Č. Dinčić, D.S. Djordjević. Hartwig's triple reverse order law revisited. Linear Multilinear Algebra 62(2014), 918–924.

[9] I. Erdelyi. On the "reverse order law" related to the generalized inverse of matrix products. J. ACM 13(1966), 439–443.

[10] A.M. Galperin, Z. Waksman. On pseudo inverse of operator products. Linear Algebra Appl. 33(1980), 123–131.

[11] T.N.E. Greville. Note on the generalized inverse of a matrix product. SIAM Rev. 8(1966), 518–521.

[12] R.E. Hartwig. The reverse order law revisited. Linear Algebra Appl. 76(1986), 241–246.

[13] S. Izumino. The product of operators with closed range and an extension of the reverse order law. Tôhoku Math. J. 34(1982), 43–52.

[14] Y. Liu, Y. Tian. A mixed-type reverse order law for generalized inverse of a triple matrix product (in Chinese). Acta Math. Sinica 52(2009), 197–204.

[15] G. Marsaglia, G.P.H. Styan. Equalities and inequalities for ranks of matrices. Linear Multilinear Algebra 2(1974), 269–292.

[16] A.R. Meenakshi, V. Chinadurai. Some remarks on the covariance of the Moore–Penrose inverse. Houtson J. Math. 18(1992), 167–174.

[17] H. Neudecker, S. Liu. Moore–Penrose inverse of a matrix product. Econometric Theory 8(1992), 584.

[18] R. Penrose. A generalized inverse for matrices. Proc. Cambridge Phil. Soc. 51(1955), 406–413.

[19] C.R. Rao, S.K. Mitra. Generalized Inverse of Matrices and Its Applications. Wiley, New York, 1971.

[20] D.W. Robinson. On the covariance of the Moore–Penrose inverse. Linear Algebra Appl. 61(1984), 91–99.

[21] D.W. Robinson. Covariance of the Moore–Penrose inverses with respect to an invertible matrix. Linear Algebra Appl. 71(1985), 275–281.

[22] H. Schwerdtfeger. On the covariance of the Moore–Penrose inverse. Linear Algebra Appl. 52/53(1983), 629–643.

[23] N. Shinozaki, M. Sibuya. The reverse order law $(AB)^- = B^-A^-$. Linear Algebra Appl. 9(1974), 29–40.

[24] N. Shinozaki, M. Sibuya. Further results on the reverse order law. Linear Algebra Appl. 27(1979), 9–16.

[25] M. Sibuya. Subclasses of generalized inverses of matrices. Ann. Instit. Statist. Math. 22(1970), 543–556.

[26] Y. Tian. Reverse order laws for the generalized inverses of multiple matrix products. Linear Algebra Appl. 211(1994), 185–200.

[27] Y. Tian. Upper and lower bounds for ranks of matrix expressions using generalized inverses. Linear Algebra Appl. 355(2002), 187–214.

[28] Y. Tian. More on maximal and minimal ranks of Schur complements with applications. Appl. Math. Comput. 152(2004), 675–692.
[29] Y. Tian. On mixed-type reverse-order laws for the Moore–Penrose inverse of a matrix product. Internat. J. Math. Math. Sci. 58(2004), 3103–3116.
[30] Y. Tian. The reverse-order law $(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$ and its equivalent equalities. J. Math. Kyoto Univ. 45(2005), 841–850.
[31] Y. Tian. The Moore–Penrose inverse for sums of matrices under rank additivity conditions. Linear Multilinear Algebra 53(2005), 45–65.
[32] Y. Tian. The equivalence between $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and other mixed-type reverse-order laws. Internat. J. Math. Edu. Sci. Tech. 37(2007), 331–339.
[33] Y. Tian. Classification analysis to the equalities $A^{(i,\dots,j)} = B^{(k,\dots,l)}$ for generalized inverses of two matrices. Linear Multilinear Algebra, 2019, doi:10.1080/03081087.2019.1627279.
[34] Y. Tian, S. Cheng. Some identities for Moore–Penrose inverses of matrix products. Linear Multilinear Algebra 52(2004), 405–420.
[35] Y. Tian, B. Jiang. Closed-form formulas for calculating the max-min ranks of a triple matrix product composed by generalized inverses. Comp. Appl. Math. 37(2018), 5876–5919.
[36] Y. Tian, Y. Liu. On a group of mixed-type reverse-order laws for generalized inverses of a triple matrix product with applications. Electron. J. Linear Algebra 16(2007), 73–89.
[37] Y. Tian, G.P.H. Styan. Some rank equalities for idempotent and involutory matrices. Linear Algebra Appl. 335(2001), 101–117.
[38] G. Trenkler. Moore–Penrose inverse of a matrix product with normal matrix. Econometric Theory 11(1995), 653–654.
[39] A.M. Vetoshkin. Jordan form of the difference of projectors. Comput. Math. Math. Phys. 54(2014), 382–396.
[40] H.J. Werner. When is $B^{-}A^{-}$ a generalized inverse of AB? Linear Algebra Appl. 210(1994), 255–263.
[41] E.A. Wibker, R.B. Howe, J.D. Gilbert. Explicit solutions to the reverse order law $(AB)^{+} = B^{-}_{mr}A^{-}_{lr}$. Linear Algebra Appl. 25(1979), 107–114.