Numerical solutions of electromagnetic wave model of fractional derivative using class of finite difference scheme

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Abstract

In this article, a numerical scheme is introduced for solving the fractional partial differential equation (FPDE) arising from electromagnetic waves in dielectric media (EMWDM) by using an efficient class of finite difference methods. The numerical scheme is based on the Hermite formula. The Caputo's fractional derivatives in time are discretized by a finite difference scheme of order $\mathcal{O}(k^{(4-\alpha)})$ & $\mathcal{O}(k^{(4-\beta)})$, $1 < \beta < \alpha \leq 2$. The stability and the convergence analysis of the proposed methods are given by a procedure similar to the standard von Neumann stability analysis under mild conditions. Also for FPDE, accuracy of order $\mathcal{O}\left(k^{(4-\alpha)} + k^{(4-\beta)} + h^2\right)$ is investigated. Finally, several numerical experiments with different fractional-order derivatives are provided and compared with the exact solutions to illustrate the accuracy and efficiency of the scheme. A comparative numerical study is also done to demonstrate the efficiency of the proposed scheme.

Keywords: Finite difference scheme, fractional wave model, Caputo fractional derivative, Hermite formula, stability and convergence analysis.

1. Introduction

We analyze and present a precise numerical scheme for tackling an FPDE emerging from EMWDM (see [1]). It is magnificent that the dielectric unwinding in solids described by the complex frequency-dependent dielectric sensitivity builds up the complete power-law dependence. Here, a conclusion seized FPDE for EMWDM with the support of Maxwell's condition. It is a power law reliance in the recurrence space that brings about the association between the electric field and the polarization thickness detailed as a feebly solitary integral. And thus the field conditions take a type of FPDE with Caputo derivative [1]:

$$({}_cD_t^{\alpha}B)(x,t) - \lambda_1({}_cD_t^{\beta}B)(x,t) - \lambda_2\nabla^2 B(x,t) = f(x,t), x \in \Omega, t \in (0,T], \tag{1.1}$$

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subject to initial condition

$$\begin{cases} B(x,0) = g(x), \\ \left[\frac{\partial B(x,t)}{\partial t}\right]_{t=0} = h(x), \end{cases} \tag{1.2}$$

and Dirichlet boundary conditions are

$$B(x,t) = 0, x \in \partial\Omega, t \in (0,T]. \tag{1.3}$$

where $B(t,x)$ is magnetic field induction, $\Omega = [0,L]$, the constant coefficient λ_1 and λ_2 depend on the frequency independent properties of a medium, $f(t,x)$ is the current density of free charges, and ∇^2 is Laplace operator with space variable x and B is unknown. And $({}_cD_t^{\alpha}\xi)(t,x)$ and $({}_cD_t^{\beta}\xi)(t,x)$denotes the Caputo fractional derivative of order α and β respectively with respect to time t and defined by ([**?**]):

$$({}_cD_t^{\alpha}B)(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{B_{ss}(x,s)}{(t-s)^{\alpha-1}}ds, \quad \alpha \in (1,2), \tag{1.4}$$

and

$$({}_cD_t^{\beta}B)(x,t) = \frac{1}{\Gamma(2-\beta)}\int_0^t \frac{B_{ss}(x,s)}{(t-s)^{\beta-1}}ds, \quad \beta \in (1,2). \tag{1.5}$$

Since the singular kernel is a major drawback of the fractional derivative operator due to which exact solutions of most of the fractional differential equation cannot be obtained. And all fractional differential equations cannot be solved analytically so it is necessary to establish some numerical schemes with symbolic accuracy to solve the type of problems numerically. So, this article provides a finite difference scheme to obtain the numerical solution of FPDE arising from EMWDM. We solve the FPDE using a finite difference scheme that is widely used for its simplicity and intuition based on Caputo's derivative in the temporal direction. Caputo's derivative follows the algebraic properties and possesses the Lipschitz condition. Also, a derivative of a constant function is zero by the Caputo derivative. For numerical solution, we have taken space domain at $L = 1$ and temporal domain at $T = 1$ throughout the article.

In recent years, a few scientists discovered that the fractional order models are more appropriate than the integer-order. FPDEs provide a powerful and flexible device for modeling and describing the behavior of real problems. Many problems in electromagnetic waves, electrochemical process, viscoelastic fluid, control theory, finance, biological system, etc., theory can be solved by the fractional calculus approach, which gives attractive applications as a new modeling tool in a variety of engineering and scientific fields. In this article, the scalar $Eq.(1)$ can be analyzed as a speculation of the alleged Szabo condition [2]. It has a more endorsed frame in the examination with $Eq.(1)$, since the case $1 < \beta < \alpha \leq 2$ has been viewed as and widely considered by [3]. The FPDE has been proposed for EMWDM by the operational matrix [4]-[5], Grunwald-Lenikov discretization scheme [6] and finite difference scheme [7] only. Therefore, this article introduced a numerical scheme for $Eq.(1)$ based on finite difference.

The rest of the article is organized into 8 sections. In Section 2, we introduce a finite difference scheme for a fractional partial differential equation. We give the analysis of stability and error estimates for the presented method in Section 3. Then in Section 4, we illustrate some examples to validate the theory presented in section 3. The conclusion of the paper is described in Section 5.

2. Finite difference scheme based on Hermite formula

Let $h = \triangle x = \frac{L}{M}, k = \triangle k = \frac{T}{N}$ be the space and time increment and let $x_i = ih, t_n = nk$ be a mesh point in $[0, L] \times [0, T]$, where M and N are the total number of intervals in $[0, L]$ and $[0, T]$, respectively. The set of mesh points (x_j, t_n) in $(0, L] \times (0, T]$ and $[0, L] \times [0, T]$ are denoted by Ω and $\bar{\Omega}$ respectively. Define $B_j^n = B(x_j, t_n)$, and when no confusion arises we write (j, n) instead of (x_j, t_n).

In the first step, the second order partial derivative operators are discretized as follows

$$\frac{\partial^2 B}{\partial x^2} = \frac{B_{j-1,n} - 2B_{j,n} + B_{j+1,n}}{h^2}. \tag{2.1}$$

Now, we can derive an approximation technique for the time fractional partial derivative operator $({}_cD_t^\alpha B)(x, t)$ and $({}_cD_t^\beta B)(x, t)$ with $1 < \beta < \alpha \leq 2$ at the points (x_i, t_n) as follows:

$$\begin{aligned}
({}_cD_t^\alpha B)(x_j, t_n) &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} B_{ss}(x_j, s)(t_n - s)^{-\alpha+1} ds, \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{r=1}^{n} \int_{(r-1)k}^{rk} \left[\frac{B_j^{r+1} - 2B_j^r + B_j^{r-1}}{k^2} + O(k^2) \right] (nk - s)^{-\alpha+1} ds, \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{1}{(2-\alpha)} \sum_{r=1}^{n} \left[\frac{B_j^{r+1} - 2B_j^r + B_j^{r-1}}{k^2} + O(k^2) \right] \\
&\quad \times \left[(n - r + 1)^{2-\alpha} - (n - r)^{2-\alpha} \right] (k)^{2-\alpha}, \qquad\qquad (2.2)\\
&= \frac{1}{\Gamma(3-\alpha)k^\alpha} \sum_{r=1}^{n} \left(B_j^{r+1} - 2B_j^r + B_j^{r-1} \right) \left[(n - r + 1)^{2-\alpha} - (n - r)^{2-\alpha} \right] \\
&\quad \times \frac{1}{\Gamma(3-\alpha)} \sum_{r=1}^{n} \left[(n - r + 1)^{2-\alpha} - (n - r)^{2-\alpha} \right] + O(k^{4-\alpha}), \\
&= P_{\alpha,k} \sum_{r=1}^{n} \delta_r^{(\alpha)} \left(B_j^{n-r+2} - 2B_j^{n-r+1} + B_j^{n-r} \right) + O(k^{4-\alpha}),
\end{aligned}$$

where $P_{\alpha,k} = \frac{1}{\Gamma(3-\alpha)k^\alpha}$, $\delta_r^{(\alpha)} = r^{2-\alpha} - (r-1)^{2-\alpha}, \delta_r^{(\alpha)} > 0$ and $\delta_r^{(\alpha)} > \delta_r^{(\alpha+1)}$ $\forall$ $r = 1, 2, ...$

Similarly we can obtain discretization of operator $({}_cD_t^{\beta}B)(x,t)$ as follows:

$$({}_cD_t^{\beta}B)(x,t) = P_{\beta,k}\sum_{r=1}^{n}\delta_r^{(\beta)}\left(B_j^{n-r+2} - 2B_j^{n-r+1} + B_j^{n-r}\right) + O(k^{4-\beta}), \tag{2.3}$$

where $P_{\beta,k} = \frac{1}{\Gamma(3-\beta)k^{\beta}}$, $\delta_r^{(\beta)} = r^{2-\beta} - (r-1)^{2-\beta}$, $\delta_r^{(\beta)} > 0$ and $\delta_r^{(\beta)} > \delta_r^{(\beta+1)}$ $\forall\ r = 1, 2, ...$ Now, we are going to establish the finite difference scheme of the $Eq.$(1.1) using Hermite formula. We calculate this equation to achieve this goal at the points of the grid (x_i, t_n):

$$({}_cD_t^{\alpha}B)(x_j,t_n) - \lambda_1({}_cD_t^{\beta}B)(x_j,t_n) - \lambda_2 B_{xx}(x_j,t_n) = f(x_j,t_n). \tag{2.4}$$

Using $Eqs.$(2.2) $-$ (2.4), we have

$$\begin{aligned} B_{xx}(x_j,t_n) = &\frac{1}{\lambda_2}P_{\alpha,k}\sum_{r=1}^{n}\delta_r^{(\alpha)}\left(B_j^{n-r+2} - 2B_j^{n-r+1} + B_j^{n-r}\right)\\ &+\frac{\lambda_1}{\lambda_2}P_{\beta,k}\sum_{r=1}^{n}\delta_r^{(\beta)}\left(B_j^{n-r+2} - 2B_j^{n-r+1} + B_j^{n-r}\right)\\ &-\frac{1}{\lambda_2}f(x_j,t_n) + O(k^{4-\alpha} + k^{4-\beta}). \end{aligned} \tag{2.5}$$

Now, in order to use Hermite formula to get two additional equation, replace i by $i-1$ and $i+1$, respectively, in the $Eq.$(2.5), we get

$$\begin{aligned} B_{xx}(x_{j-1},t_n) = &\frac{1}{\lambda_2}P_{\alpha,k}\sum_{r=1}^{n}\delta_r^{(\alpha)}\left(B_{j-1}^{n-r+2} - 2B_{j-1}^{n-r+1} + B_{j-1}^{n-r}\right)\\ &+\frac{\lambda_1}{\lambda_2}P_{\beta,k}\sum_{r=1}^{n}\delta_r^{(\beta)}\left(B_{j-1}^{n-r+2} - 2B_{j-1}^{n-r+1} + B_{j-1}^{n-r}\right)\\ &-\frac{1}{\lambda_2}f(x_{j-1},t_n) + O(k^{4-\alpha} + k^{4-\beta}). \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} B_{xx}(x_{j+1},t_n) = &\frac{1}{\lambda_2}P_{\alpha,k}\sum_{r=1}^{n}\delta_r^{(\alpha)}\left(B_{j+1}^{n-r+2} - 2B_{j+1}^{n-r+1} + B_{j+1}^{n-r}\right)\\ &+\frac{\lambda_1}{\lambda_2}P_{\beta,k}\sum_{r=1}^{n}\delta_r^{(\beta)}\left(B_{j}^{n-r+2} - 2B_{j+1}^{n-r+1} + B_{j+1}^{n-r}\right)\\ &-\frac{1}{\lambda_2}f(x_{j+1},t_n) + O(k^{4-\alpha} + k^{4-\beta}). \end{aligned} \tag{2.7}$$

The Hermite formula for fractional partial derivatives at the grid point (x_i, t_n) is

$$\left[\frac{\partial^2 B}{\partial x^2} + 10\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial x^2} - \frac{12}{h^2}(B_{j-1}^n - 2B_j^n + B_{j+1}^n)\right](x_j,t_n) + O(h^4) = 0. \tag{2.8}$$

Now using $Eqs.(2.5) - (2.7)$ into $Eq.(2)$, denote $B(x_j, t_n)$ by B_i^n and after some simplifications, we can get the following form:

$$\frac{h^2}{6\lambda_2} X_{\alpha,\beta} \sum_{r=2}^{n} W_{r,\alpha,\beta} \left[B_{j-1}^{n-r+2} - 2B_{j-1}^{n-r+1} + B_{j-1}^{n-r} + 10 \left(B_j^{n-r+2} - 2B_j^{n-r+1} + B_j^{n-r} \right) \right.$$
$$\left. + B_{j+1}^{n-r+2} - 2B_{j+1}^{n-r+1} + B_{j+1}^{n-r} \right] + \frac{h^2}{6} \left(f_{j-1}^n + 10 f_j^n + f_{j+1}^n \right) - 2 \left[1 + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \right] B_{j-1}^n$$
$$-2 \left[-2 + \frac{5h^2}{3\lambda_2} X_{\alpha,\beta} \right] B_j^n - 2 \left[1 + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \right] B_{j+1}^n + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \left[B_{j-1}^{n+1} + B_{j-1}^{n-1} + B_{j+1}^{n+1} + B_{j+1}^{n-1} \right]$$
$$+ \frac{5h^2}{3\lambda_2} X_{\alpha,\beta} \left[B_j^{n+1} + B_j^{n-1} \right] + O(h^4 + k^{4-\alpha} + k^{4-\beta}) = 0. \tag{2.9}$$

where $W_{r,\alpha,\beta} = P_{\alpha,k}\delta_r^\alpha + \lambda_1 P_{\beta,k}\delta_r^\beta$ and $X_{\alpha,\beta} = P_{\alpha,k} + \lambda_1 P_{\beta,k}$.

Omitting the high order terms from $Eq.(2.9)$, we get the following difference scheme for the problem $(1.1) - (1.3)$:

$$\frac{h^2}{6\lambda_2} X_{\alpha,\beta} \sum_{r=2}^{n} W_{r,\alpha,\beta} \left[B_{j-1}^{n-r+2} - 2B_{j-1}^{n-r+1} + B_{j-1}^{n-r} + 10 \left(B_j^{n-r+2} - 2B_j^{n-r+1} + B_j^{n-r} \right) \right.$$
$$\left. + B_{j+1}^{n-r+2} - 2B_{j+1}^{n-r+1} + B_{j+1}^{n-r} \right] + \frac{h^2}{6} \left(f_{j-1}^n + 10 f_j^n + f_{j+1}^n \right) - 2 \left[1 + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \right] B_{j-1}^n$$
$$-2 \left[-2 + \frac{5h^2}{3\lambda_2} X_{\alpha,\beta} \right] B_j^n - 2 \left[1 + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \right] B_{j+1}^n + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \left[B_{j-1}^{n+1} + B_{j-1}^{n-1} + B_{j+1}^{n+1} + B_{j+1}^{n-1} \right]$$
$$+ \frac{5h^2}{3\lambda_2} X_{\alpha,\beta} \left[B_j^{n+1} + B_j^{n-1} \right] = 0. \tag{2.10}$$

Let b_i^n be the approximate solution, and let $T_j^n = B_j^n - b_j^n, j = 1, 2, ..., M, n = 1, 2, ..., N$ be the error, then we have the error formula

$$\frac{h^2}{6\lambda_2} X_{\alpha,\beta} \sum_{r=2}^{n} W_{r,\alpha,\beta} \left[T_{j-1}^{n-r+2} - 2T_{j-1}^{n-r+1} + T_{j-1}^{n-r} + 10 \left(T_j^{n-r+2} - 2T_j^{n-r+1} + T_j^{n-r} \right) \right.$$
$$\left. + T_{j+1}^{n-r+2} - 2T_{j+1}^{n-r+1} + T_{j+1}^{n-r} \right] + \frac{h^2}{6} \left(f_{j-1}^n + 10 f_j^n + f_{j+1}^n \right) - 2 \left[1 + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \right] T_{j-1}^n$$
$$-2 \left[-2 + \frac{5h^2}{3\lambda_2} X_{\alpha,\beta} \right] T_j^n - 2 \left[1 + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \right] T_{j+1}^n + \frac{h^2}{6\lambda_2} X_{\alpha,\beta} \left[T_{j-1}^{n+1} + T_{j-1}^{n-1} + T_{j+1}^{n+1} + T_{j+1}^{n-1} \right]$$
$$+ \frac{5h^2}{3\lambda_2} X_{\alpha,\beta} \left[T_j^{n+1} + T_j^{n-1} \right] = 0. \tag{2.11}$$

with $T_0^n = T_M^n = 0, \qquad n = 1, 2, ..., N.$

3. Stability and convergence analysis

We use the Von Neumann method to study th stability analysis of the finite difference scheme (2.10) with the force free case (i.e., $f(x,t)=0$). For stability, we need lemma as follows

Lemma 3.1. *Let the solution of (2.11) has the form $T_j^n = A_n e^{i\theta jh}$, then*

$$G_n = \frac{\frac{h^2}{6\lambda_2} X_{\alpha,\beta} \left(cos(\theta h)+5\right) \left(G_{n+1}+G_{n-1}+\sum_{r=2}^{n} W_{r,\alpha,\beta} \left(G_{n-r+2}-2Gn-r+1+G_{n-r}\right)\right)}{Y_{\alpha,\beta}}, \tag{3.1}$$

where $Y_{\alpha,\beta} = 2\left(1+\frac{h^2}{6\lambda_2}X_{\alpha,\beta}\right) cos(\theta h) + \frac{5h^2}{3\lambda_2}X_{\alpha,\beta} - 2$, $\theta = 2\pi n$ and $X_{\alpha,\beta} = P_{\alpha,k} + \lambda_1 P_{\beta,k}$.

Proof. Substituting $T_j^n = G_n e^{i\theta} jh$ in (2.11) and after simplification, we get

$$\begin{aligned}
&\frac{h^2}{3\lambda_2} X_{\alpha,\beta} \left(\frac{e^{i\theta jh}+e^{-i\theta jh}}{2}+5\right) \sum_{r=2}^{n} W_{r,\alpha,\beta} \left[G_{n-r+2}-2Gn-r+1+G_{n-r}\right] \\
&-2\left[2\left(1+\frac{h^2}{6\lambda_2}X_{\alpha,\beta}\right)\left(\frac{e^{i\theta jh}+e^{-i\theta jh}}{2}\right)+\frac{5h^2}{3\lambda_2}X_{\alpha,\beta}-2\right] G_n \\
&+\frac{h^2}{3\lambda_2} X_{\alpha,\beta} \left(\frac{e^{i\theta jh}+e^{-i\theta jh}}{2}+5\right)\left[G_{n+1}+G_{n-1}\right]=0,
\end{aligned} \tag{3.2}$$

Now using some trigonometric formulas in (**??**) we can obtain as follows:

$$\begin{aligned}
&\frac{h^2}{6\lambda_2} X_{\alpha,\beta} \left(cos(\theta h)+5\right) \sum_{r=2}^{n} W_{r,\alpha,\beta} \left[G_{n-r+2}-2Gn-r+1+G_{n-r}\right] \\
&-\left[2\left(1+\frac{h^2}{6\lambda_2}X_{\alpha,\beta}\right) cos(\theta h)+\frac{5h^2}{3\lambda_2}X_{\alpha,\beta}-2\right] G_n \\
&+\frac{h^2}{6\lambda_2} X_{\alpha,\beta} \left(cos(\theta h)+5\right)\left[G_{n+1}+G_{n-1}\right]=0,
\end{aligned}$$

This implies

$$G_n = \frac{\frac{h^2}{6\lambda_2} X_{\alpha,\beta} \left(cos(\theta h)+5\right) \left(G_{n+1}+G_{n-1}+\sum_{r=2}^{n} W_{r,\alpha,\beta} \left(G_{n-r+2}-2Gn-r+1+G_{n-r}\right)\right)}{Y_{\alpha,\beta}} \tag{3.3}$$

where $Y_{\alpha,\beta} = 2\left(1+\frac{h^2}{6\lambda_2}X_{\alpha,\beta}\right) cos(\theta h) + \frac{5h^2}{3\lambda_2}X_{\alpha,\beta} - 2$. ☐

Lemma 3.2. *Let $H^\alpha < \frac{h^2}{6\Gamma(3-\alpha)}$ and $H^\beta < \frac{h^2}{6\Gamma(3-\beta)}$, then*

$$\frac{\frac{h^2}{6\lambda_2} X_{\alpha,\beta}\left(cos(\theta h)+5\right)}{4\left(1+\frac{h^2}{6\lambda_2}X_{\alpha,\beta}\right) cos(\theta h)+\frac{5h^2}{3\lambda_2}X_{\alpha,\beta}-2} < 1,$$

where $X_{\alpha,\beta} = P_{\alpha,k} + \lambda_1 P_{\beta,k}$.

Proof. Since $P_{\alpha,k} = \frac{1}{k^{\alpha}\Gamma(3-\alpha)}$ and $P_{\beta,k} = \frac{1}{k^{\beta}\Gamma(3-\beta)}$ so this implies $k^{\alpha} = \frac{1}{P_{\alpha,k}\Gamma(3-\alpha)}$ and $k^{\beta} = \frac{1}{P_{\beta,k}\Gamma(3-\beta)}$. Hence by given assumptions we can obtain as follow:

$$\frac{1}{P_{\alpha,k}} < \frac{h^2}{6} \qquad \text{and} \qquad \frac{1}{P_{\beta,k}} < \frac{h^2}{6}. \tag{3.4}$$

So,

$$2\left(1+\frac{1}{6}h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)\right) < \frac{5}{6}h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-1, \tag{3.5}$$

and hence

$$2\left(1+\frac{1}{6}h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)\right)(1-cos(\theta h)) < 2\left(\frac{5}{6}h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-1\right), \tag{3.6}$$

which implies that

$$2\left(1+\frac{1}{6}h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)\right)(1-cos(\theta h)) + \left(\frac{5}{3}h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-2\right) > 0. \tag{3.7}$$

Instantly, since
$-3h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right) < 15h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)$ and $h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right) - 4h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right) - 24 < 20h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)$ then we have

$$\begin{aligned}&\left(h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-4h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-24\right)cos(\theta h)\\ &\qquad < 20h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-5h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-24.\end{aligned} \tag{3.8}$$

this implies

$$\begin{aligned}&h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)(cos(\theta h))\\ &\qquad < 2\left[\left(12+2h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)\right)cos(\theta h)+10h^2\left(P_{\alpha,k}+\lambda_1 P_{\beta,k}\right)-12\right]\end{aligned} \tag{3.9}$$

i.e.

$$h^2 X_{\alpha,\beta}\,(cos(\theta h)) < 12\left[\left(2+\frac{1}{6}h^2 X_{\alpha,\beta}\right)cos(\theta h)+10h^2 X_{\alpha,\beta}-2\right], \tag{3.10}$$

where $X_{\alpha,\beta} = P_{\alpha,k}+\lambda_1 P_{\beta,k}$.
Using (3.7) and (3.10) completes the proof of lemma. □

Lemma 3.3. *Let G_N be the solution of (3.1), with the conditions $P_{\alpha,k} = \frac{1}{k^{\alpha}\Gamma(3-\alpha)}$ and $P_{\beta,k} = \frac{1}{k^{\beta}\Gamma(3-\beta)}$ then G_n is bounded.*

Proof. Since from 3.1 we have

$$G_n = \frac{\frac{h^2}{6\lambda_2}X_{\alpha,\beta}\left(cos(\theta h)+5\right)\left(G_{n+1}+G_{n-1}+\sum_{r=2}^{n}W_{r,\alpha,\beta}\left(G_{n-r+2}-2Gn-r+1+G_{n-r}\right)\right)}{Y_{\alpha,\beta}}. \tag{3.11}$$

Taking absolute value of (3.11) as follow:

$$|G_n| \leq \frac{\left|\frac{h^2}{\lambda_2} X_{\alpha,\beta}\left(G_{n+1}+G_{n-1}\right)\right|}{|Y_{\alpha,\beta}|} + \frac{\sum_{r=2}^{n}|W_{r,\alpha,\beta}|\left(|G_{n-r+2}|+2\,|Gn-r+1|+|G_{n-r}|\right)}{|Y_{\alpha,\beta}|}. \tag{3.12}$$

Also,

$$\begin{aligned}
&|X_{\alpha,\beta}| \leq |P_{\alpha,k}| + |\lambda_1|\,|P_{\beta,k}| \leq K_1(\text{say}),\\
&|Y_{\alpha,\beta}| \leq 2\left|1+\frac{h^2 X_{\alpha,\beta}}{6\lambda_2}\right| + \left|2+\frac{5h^2 X_{\alpha,\beta}}{3\lambda_2}\right| \leq K_2(\text{say}),\\
&|W_{r,\alpha,\beta}| \leq |P_{\alpha,k}\delta_r^{\alpha}| + |\lambda_1|\left|P_{\beta,k}\delta_r^{\beta}\right| \leq K_3(\text{say}).
\end{aligned} \tag{3.13}$$

Since K_1, K_2 and K_3 are positive real constant. So by using (3.12) and (3.13), G_n is bounded. □

Theorem 3.1. *If $P_{\alpha,k} = \frac{1}{k^{\alpha}\Gamma(3-\alpha)}$ and $P_{\beta,k} = \frac{1}{k^{\beta}\Gamma(3-\beta)}$ then the finite difference scheme (2.10) is stable.*

Proof. We know that

$$\|T^n\|_2^2 = \sum_{N=-\infty}^{\infty} |G_n(N)|^2. \tag{3.14}$$

So, from *Lemma* 3.3 and (3.14), $T^n, n = 1, 2, ..., N$ is bounded, which means that the difference scheme is stable. And by using the Lax equivalence theorem [8] we can obtain that the numerical solution converges to the exact solution as $h, k \to 0$. □

4. Numerical examples

Now, we have discussed some examples to show the accuracy and efficiency of the proposed schemes to validate our scheme which verifies the stability and convergence of the finite difference scheme. Let B be the exact solution and $\bar{B}$ be the numerical solution then

$$\text{Absolute } \; error = \left|B-\bar{B}\right|,$$

$$\left\|B-\bar{B}\right\|_2 = \sqrt{\sum_{j=1}^{M-1}\left|B(x_j,T)-\bar{B}(x_j,T)\right|^2},$$

and

$$\left\|B-\bar{B}\right\|_\infty = \max_{1\leq j\leq N}\left|B(x_j,T)-\bar{B}(x_j,T)\right|.$$

We have used the following formula for calculating the computational orders (COs) of the proposed finite difference scheme:

$$CO(N) = \frac{log\frac{E_N}{E_{N+1}}}{log\frac{h_N}{h_{N+1}}},$$

where E_N and E_{N+1} are errors corresponding to the grids with mesh size h_N and h_{N+1} respectively.

Example 4.1.

$$({}_cD_t^{\alpha}B)(x,t) - ({}_cD_t^{\beta}B)(x,t) - \nabla^2 B(x,t) = f(x,t), \tag{4.1}$$

subject to initial condition

$$\begin{cases} B(x,0) = g(x), \\ \left[\frac{\partial B(x,t)}{\partial t}\right]_{t=0} = h(x), \end{cases} \tag{4.2}$$

and Dirichlet boundary conditions are

$$B(x,t) = 0, x \in \partial\Omega, t \in (0,T], \tag{4.3}$$

and the exact solution of the above test problem is $B(x,t) = t^{\alpha+\beta}x^{1+\alpha+\beta}(1-x)$. *The value of source term* $f(x,t)$ *is varies for different choices of* α *and* β.

We have solve Example 4.1 using proposed finite difference scheme is given in $Eq.$(2.10). For Example 4.1, Figure 1 shows the behavior of exact solution, Figure 2 shows the behavior of numerical solution, Figure 3 shows the comparison of exact and numerical solution at $T = 1$ and Figure 4 shows absolute errors between exact and numerical solutions with $h = 1/100, k = 1/1000$ and different values of α and β, where E_1, E_2 and E_3 represent the absolute errors corresponding to $\alpha = 1.5,\ \beta = 1.1;\ \alpha = 1.7,\ \beta = 1.5$ and $\alpha = 1.9,\ \beta = 1.3$, respectively. L_2 & L_∞ errors and temporal order of convergence for Example 4.1 with time $T = 1$ at fixed temporal step size $k = 1/1000$ and varies spatial step size h are given in Table 1 and Table 2 respectively.

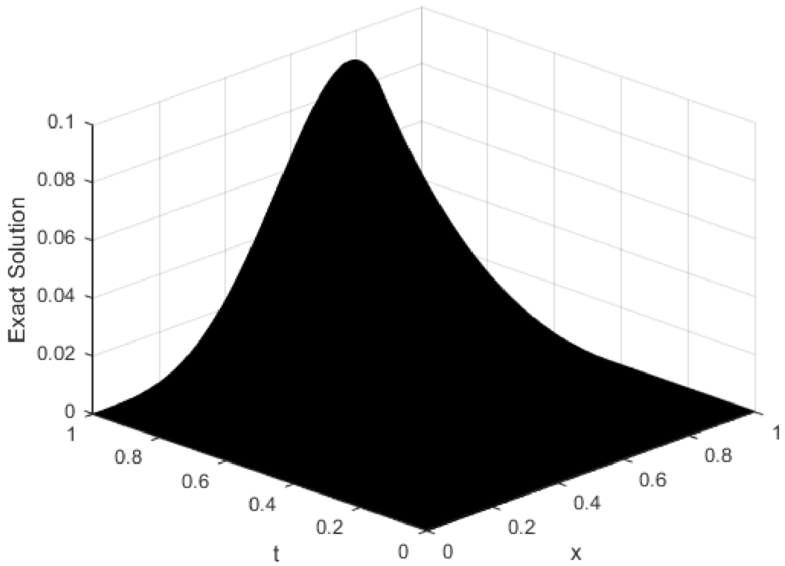

Figure 1: The behavior of the exact solution of Example 4.1 of the fractional wave model (1.1) by means of proposed scheme for $\alpha = 1.5, \beta = 1.1$, and $h = 100, k = 1000$.

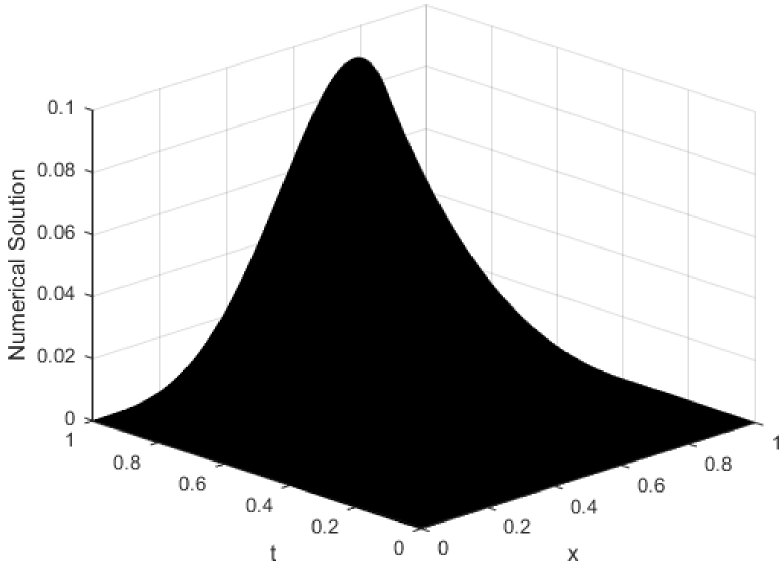

Figure 2: The behavior of the numerical solution of Example 4.1 of the fractional wave model (1.1) by means of proposed scheme for $\alpha = 1.5, \beta = 1.1$, and $h = 100, k = 1000$.

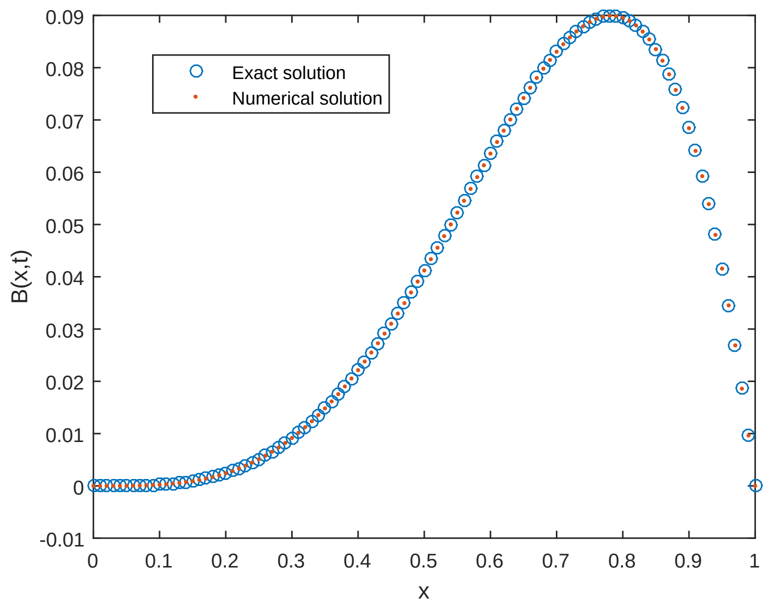

Figure 3: The behavior of the exact solution and the numerical solution of Example 4.1 of the fractional wave model (1.1) by means of proposed scheme for $\alpha = 1.5, \beta = 1.1, h = 100, k = 1000$ and $T = 1$.

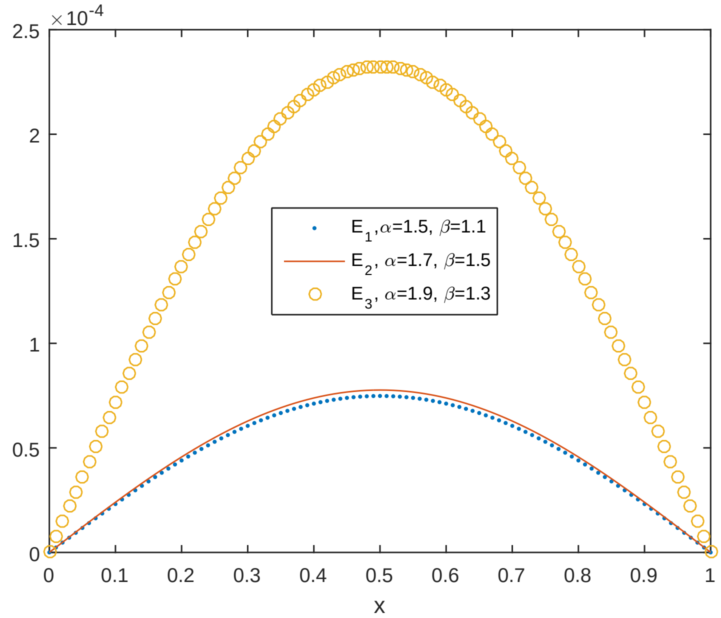

Figure 4: Absolute errors of $Example$4.1 at time $T = 1$ with spatial step size $h = 100$, temporal step size $k = 1000$ and different values of α & β.

Table 1: L_2 errors and order of convergence of Example 4.1 for different values of α and β at time $T = 1$ with $k = 1/1000$

h	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\left\|B - \bar{B}\right\|_2$	COs	$\left\|B - \bar{B}\right\|_2$	COs	$\left\|B - \bar{B}\right\|_2$	COs
$1/5$	2.1500E-02	2.02027	2.1700E-02	2.00666	1.3900E-02	1.83354
$1/10$	5.3000E-03	2.02748	5.4000E-03	2.05445	3.9000E-03	1.82595
$1/20$	1.3000E-03	1.97500	1.3000E-03	1.92337	1.1000E-03	1.75900
$1/40$	3.3068E-04	2.00584	3.4273E-04	1.89541	3.0632E-04	1.84439
$1/80$	8.2336E-05	2.02275	9.2125E-05	1.64386	9.5034E-05	1.04771
$1/160$	2.0262E-05	-	2.9480E-05	-	4.1134E-05	-

Table 2: L_∞ errors and order of convergence of Example 4.1 for different values of α and β at time $T=1$ with $k=1/1000$

h	$\alpha=1.5,\beta=1.1$		$\alpha=1.7,\beta=1.5$		$\alpha=1.9,\beta=1.3$	
	$\left\|B-\bar{B}\right\|_2$	COs	$\left\|B-\bar{B}\right\|_2$	COs	$\left\|B-\bar{B}\right\|_2$	COs
1/5	2.8900E-02	1.94611	2.9200E-02	1.94190	1.8700E-02	1.76553
1/10	7.5000E-03	1.98089	7.6000E-03	2.00000	5.5000E-03	1.45943
1/20	1.9000E-03	2.02247	1.9000E-03	1.97084	2.0000E-03	0.86249
1/40	4.6766E-04	2.00587	4.8470E-04	1.89548	1.1000E-03	1.39004
1/80	1.1644E-04	2.02273	1.3028E-04	1.64353	4.1971E-04	1.35078
1/160	2.8655E-05	-	4.1699E-05	-	1.6456E-04	-

Example 4.2.

$$({}_cD_t^\alpha B)(x,t) - ({}_cD_t^\beta B)(x,t) - \nabla^2 B(x,t) = f(x,t), \tag{4.4}$$

subject to initial condition

$$\begin{cases} B(x,0) = g(x), \\ \left[\frac{\partial B(x,t)}{\partial t}\right]_{t=0} = h(x), \end{cases} \tag{4.5}$$

and Dirichlet boundary conditions are

$$B(x,t) = 0, x \in \partial\Omega, t \in (0,T], \tag{4.6}$$

and the exact solution of the above test problem is $B(x,t) = t^{\alpha+\beta} sin(\pi x)$. *The value of source term* $f(x,t)$ *is varies for different choices of* α *and* β.

We have solve Example 4.2 using proposed finite difference scheme is given in $Eq.$(2.10). For Example 4.2, Figure 5 shows the behavior of exact solution, Figure 6 shows the behavior of numerical solution, Figure 7 shows the comparison of exact and numerical solution at $T = 1$ and Figure 8 shows absolute errors between exact and numerical solutions with $h = 1/100, k = 1/1000$ and different values of α and β, where E_1, E_2 and E_3 represent the absolute errors corresponding to $\alpha = 1.5,\ \beta = 1.1;\ \alpha = 1.7,\ \beta = 1.5$ and $\alpha = 1.9,\ \beta = 1.3$, respectively. L_2 & L_∞ errors and temporal order of convergence for Example 4.2 with time $T = 1$ at fixed temporal step size $k = 1/1000$ and varies spatial step size h are given in Table 3 and Table 4 respectively.

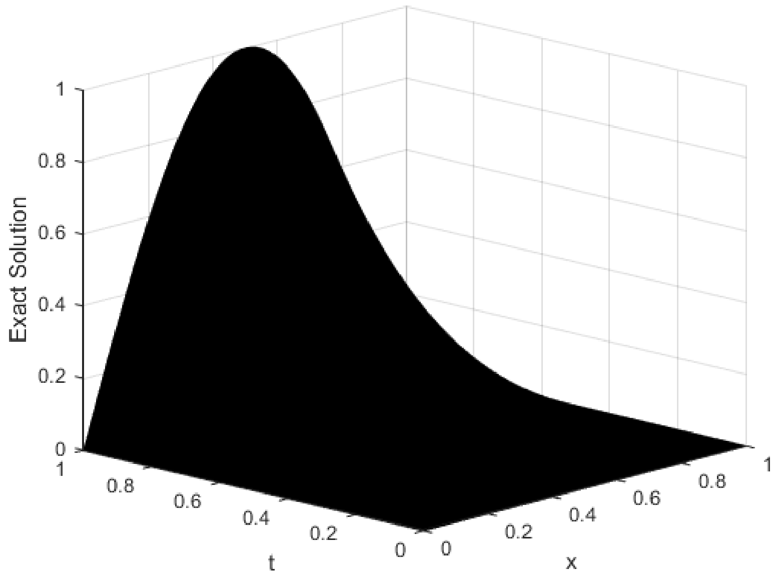

Figure 5: The behavior of the exact solution of Example 4.2 of the fractional wave model (1.1) by means of proposed scheme for $\alpha = 1.5, \beta = 1.1$ and $h = 100, k = 1000$.

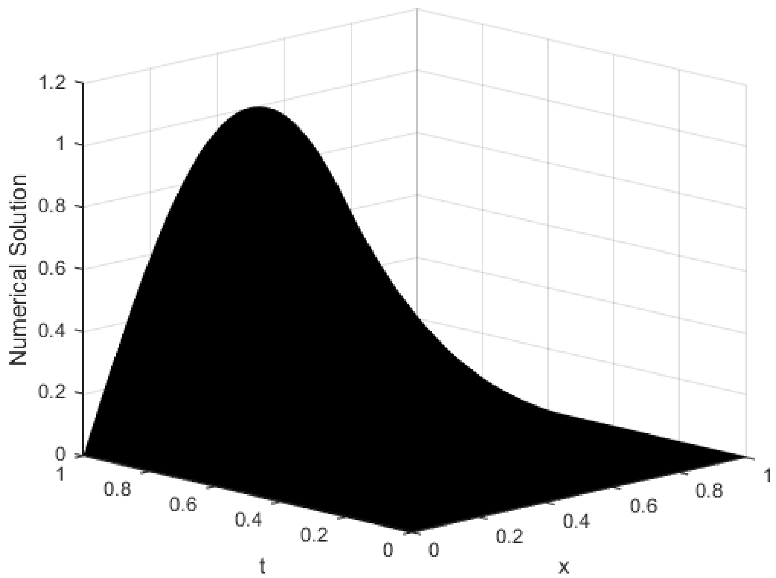

Figure 6: The behavior of the numerical solution of Example 4.2 of the fractional wave model (1.1) by means of proposed scheme for $\alpha = 1.5, \beta = 1.1$ and $h = 100, k = 1000$.

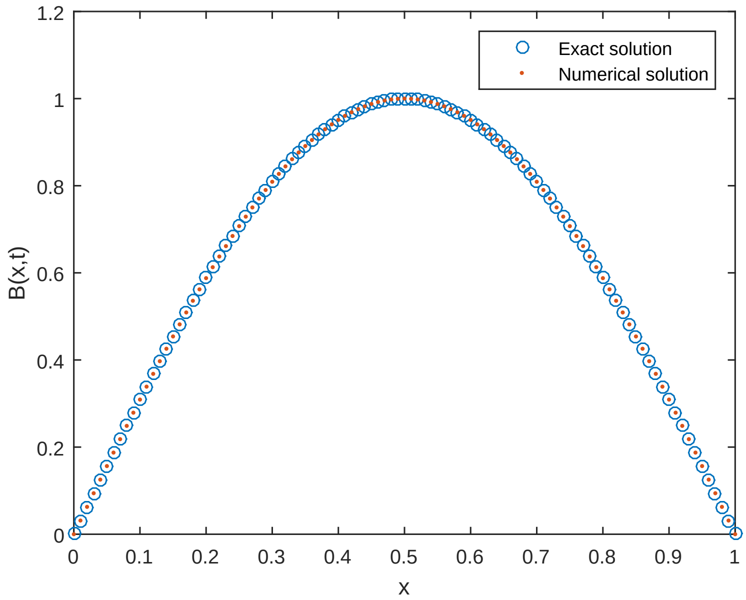

Figure 7: The behavior of the exact solution and the numerical solution of Example 4.2 of the fractional wave model (1.1) by means of proposed scheme for $\alpha = 1.5, \beta = 1.1, h = 100, k = 1000$ and $T = 1$.

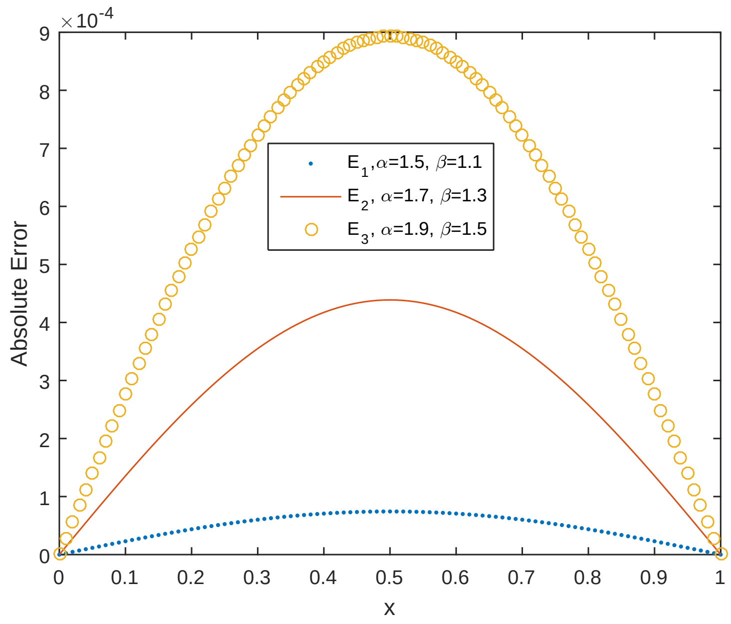

Figure 8: Absolute errors of Example 4.2 at time $T = 1$ with spatial step size $h = 100$, temporal step size $k = 1000$ and different values of α & β.

Table 3: L_2 errors and order of convergence of Example 4.2 for different values of α and β at time $T = 1$ with $k = 1/1000$

h	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\|B - \bar{B}\|_2$	COs	$\|B - \bar{B}\|_2$	COs	$\|B - \bar{B}\|_2$	COs
$1/5$	2.0100E-02	2.00720	2.2400E-02	2.02600	1.4700E-02	1.87774
$1/10$	5.0000E-03	2.05889	5.5000E-03	1.97400	4.0000E-03	1.62149
$1/20$	1.2000E-03	1.92253	1.4000E-03	2.05683	1.3000E-03	0.94124
$1/40$	3.1655E-04	1.91478	3.3648E-04	2.10794	6.7702E-04	1.14317
$1/80$	8.3953E-05	1.70148	7.8056E-05	2.53551	3.0653E-04	1.66347
$1/160$	2.5813E-05	-	1.3463E-05	-	9.6765E-05	-

Table 4: L_∞ errors and order of convergence of Example 4.2 for different values of α and β at time $T = 1$ with $k = 1/1000$

h	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\|B - \bar{B}\|_2$	COs	$\|B - \bar{B}\|_2$	COs	$\|B - \bar{B}\|_2$	COs
$1/5$	2.7000E-02	1.94753	3.0100E-02	1.94822	1.9800E-02	1.79647
$1/10$	7.0000E-03	1.95936	7.8000E-03	2.03747	5.7000E-03	1.58496
$1/20$	1.8000E-03	2.00746	1.9000E-03	1.99739	1.9000E-03	0.98871
$1/40$	4.4768E-04	1.91478	4.7586E-04	2.10793	9.5746E-04	1.16640
$1/80$	1.1873E-04	1.70152	1.1039E-04	2.53550	4.2658E-04	1.40253
$1/160$	3.6505E-05	-	1.9040E-05	-	1.6136E-04	-

5. Conclusions

This article presents a class of numerical schemes for solving the FPDE (1.1) arising from EMWDM. This class of schemes depends on the finite difference scheme based on the Hermite formula. To solve the proposed FPDE (1.1), first we applied a difference scheme of order $\mathcal{O}(t^{(4-\alpha)})$ & $\mathcal{O}(t^{(4-\beta)})$, $1 < \beta < \alpha \leq 2$ and the Caputo's derivative in time. Special consideration is given to the study of the stability and the convergence of the fractional finite difference scheme. To carry out this aim we have resorted to a kind of fractional von Neumann stability analysis. From the theoretical examination, we can conclude that the proposed scheme is suitable for FPDE (1.1) and leads to very good predictions for the stability bounds. Moreover, numerical estimation of the COs and absolute errors are also presented through tables 1-4 to validate the effectiveness of the proposed scheme. In this article, we could not provide the results for nonlinear source terms and unbounded domain, which is a topic for future study.

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