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Global existence and decay estimates for the viscoelastic kirchhoff equation with a delay term

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Summary

In this paper, we consider a viscoelastic kirchhoff equation with a delay term in the internal feedback. By using the Faedo-Galarkin approximation method we prove the well-posedness of the global solutions. Introducing suitable energy, we prove the general uniform decay results.

KEYWORDS:
Viscoelastic kirchhoff equation, global existence, delay term, general decay

1 | INTRODUCTION

In this paper we investigate the global existence and uniform decay rate of the energy for solutions to the nonlinear viscoelastic kirchhof problem with delay term in the internal feedback.

$$\begin{aligned}&|u'(x,t)|^{\rho}u''(x,t)+\Delta^2 u(x,t)-\Delta u''(x,t)-M(\|\nabla u\|^2)\Delta u(x,t)\\&-\int_0^t h(t-s)\Delta^2 u(x,s)ds+\mu_1 g(u'(x,t))+\mu_2 g(u'(x,t-\tau))=0,\ \ x\in\Omega,\ \ t>0,\end{aligned}\tag{1}$$

$$u(x,t)=\frac{\partial u(x,t)}{\partial n}=0,\quad x\in\partial\Omega,\ \ t>0,\tag{2}$$

$$u(x,0)=u_0(x),\quad u'(x,0)=u_1(x),\quad x\in\Omega,\tag{3}$$

$$u'(x,t-\tau)=f_0(x,t-\tau),\quad x\in\Omega,\quad t\in(0,\tau),\tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $\frac{\partial}{\partial n}$ represents the outward normal derivative on $\partial\Omega$. ρ, μ_1, μ_2 are three positive real numbers, $M\in C^1(\mathbb{R}^+)$, h is a positive non increasing function defined on $\mathbb{R}^+$ which represents the kernel of the memory term and g is an odd non-decreasing function of the class $C^1(\mathbb{R})$ which represents internal feedback.

In the absence of the delay term, many authors have investigated problem (1) and proved the stability, instability and the exponential decaying energy of the system under suitable assumptions, see for example[6,7,11,17,18,22]. In paper[6], the authors considered a related problem with strong damping

$$|u'(x,t)|^{\rho}u''(x,t)-\Delta u(x,t)-\Delta u''(x,t)-\int_0^t h(t-s)\Delta u(x,s)ds-\gamma\Delta u_t=0.$$

They obtained the global existence result for $\gamma>0$ and the uniform exponential decay of the energy for $\gamma>0$. Lately, the decay result has been extended by[18] to the case $\gamma=0$.

In a recent work[17], Messaoudi and Tatar studied the following problem:

$$|u'(x,t)|^{\rho}u''(x,t)-\Delta u(x,t)-\Delta u''(x,t)-\int\limits_0^t h(t-s)\Delta u(x,s)ds=b|u|^{p-2}u.$$

By introducing a new functional and using a potential well method, they obtained the global existence of solutions and the uniform decay of the energy where the initial data are stable in a suitable set. Han and Wang proved in[11] the global existence and the uniform decay for the following nonlinear viscoelastic equation with damping:

$$|u'(x,t)|^{\rho}u''(x,t)-\Delta u(x,t)-\Delta u''(x,t)-\int\limits_0^t h(t-s)\Delta u(x,s)ds+u'=0.$$

It is well known that delay effects often arise in many pratical problems because these phenomen depend not only on the present state but also on the past history of the system. In recent years, the behavior of solutions for PDEs with time delay effects has become an active area of research, see[8,9,19,20,21,24] and the references therein. Datko proved in[8] that a small delay in a boundary control is a source of instability. To stabilize a system involving delay terms, additional control terms will be necessary. In[19] Nicaise and Pignotti considered the following wave equation with a linear damping and delay term inside the domain

$$u_{tt}-\Delta u+\mu_1 u_t+\mu_2 u_t(t-\tau)=0.$$

The stability was proved in the case $0<\mu_1<\mu_2$. Kirane and Said Houari in[12] investigated the following linear viscoelastic wave equation with a linear damping and delay term

$$u_{tt}-\Delta u+\int\limits_0^t g(t-s)\Delta u(s)ds+\mu_1 u_t+\mu_2 u_t(t-\tau)=0.$$

They showed that its energy was exponentially decaying when $0<\mu_2<\mu_1$. For the plate equation with time delay term, Park consider in[21] the problem

$$u_{tt}+\Delta^2 u-M(\|\nabla u\|^2)\Delta u+\sigma(t)\int\limits_0^t g(t-s)\Delta u(s)ds+a_0 u_t+a_1 u_t(t-\tau)=0,$$

which can be considered as an extensive weak viscoelastic plate equation with a linear time delay term. The author obtained a general decay result of energy by using suitable energy and Lyapunov functionals. Yang in[24] studied initial boundary value problem of Euler-Bernoulli viscoelastic equation with a delay term in the internal feedbacks,

$$u_{tt}+\Delta^2 u-\int\limits_0^t g(t-s)\Delta^2 u(s)ds+\mu_1 u_t+\mu_2 u_t(t-\tau)=0.$$

A global existence and uniform decay rates for the energy was proved. Recently[9] showed the energy decay of solutions for the following nonlinear viscoelastic equation with a time delay term in the internal feedback

$$u_{tt}+\Delta^2 u-div(F(\nabla u))-\sigma(t)\int\limits_0^t g(t-s)\Delta^2 u(s)ds+\mu_1|u_t|^{m-1}u_t+\mu_2|u_t(t-\tau)|^{m-1}u_t(t-\tau)=0.$$

In the present paper, we devote our study to problem (1)-(4). We will prove the global existence of weak solutions and the uniform exponential decay of the energy for this problem by using Faedo-Galerkin method and the perturbed energy method, respectively. Our paper is organized as follows. In Section 2, we present the assumptions and main results. Section 3 we prove our main results.

2 | ASSUMPTIONS AND MAIN RESULT

Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product $(,)$ and the corresponding norm $\|\cdot\|$. We also consider the sobolev space $H_0^2(\Omega)$ endowed with the scalar product

$$(u,v)_{H_0^2(\Omega)} = (\Delta u, \Delta v).$$

We define for all $1 \le p < \infty$ and $u \in L^p(\Omega)$,

$$\|u\|_p^p = \int_\Omega |u(x)|^p dx, \quad \text{and} \quad \|u\| = \|u\|_2.$$

We introduce as in [19] a new variable

$$z(x,\varrho,t) = u_t(x, t - \tau\varrho), \quad x \in \Omega, \ \varrho \in (0,1), \ t > 0.$$

Then we have

$$\tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0.$$

Therefore problem (1)-(4) is equivalent to

$$\begin{aligned} &|u'(x,t)|^\rho u''(x,t) + \Delta^2 u(x,t) - \Delta u''(x,t) - M(\|\nabla u\|^2)\Delta u(x,t) \\ &- \int_0^t h(t-s)\Delta^2 u(x,s)ds + \mu_1 g(u'(x,t)) + \mu_2 g(z(x,1,t)) = 0, \end{aligned} \tag{5}$$

$$\tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0, \quad x \in \Omega, \ \varrho \in (0,1), \ t > 0, \tag{6}$$

$$z(x,0,t) = u'(x,t), \quad x \in \Omega, \ t > 0, \tag{7}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \ t > 0, \tag{8}$$

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega, \tag{9}$$

$$z(x,\varrho,0) = f_0(x, -\varrho\tau), \quad x \in \Omega, \ \varrho \in (0,1). \tag{10}$$

To state and prove our result, we assume the following hypothesis

(A1) Assume that ρ satisfies

$$0 < \rho \le \frac{2}{n-2} \quad \text{if } n \ge 3, \quad 0 < \rho < \infty \quad \text{if } n = 1,2;$$

(A2) Assume that $M \in C^1(\mathbb{R}_+)$ satisfies

$$\exists m_0 > 0, \quad M(\lambda) \ge m_0, \quad \forall\, \lambda \ge 0.$$
$$\exists \gamma, \delta, \quad M(\lambda) \le \delta\lambda^\gamma, \quad \forall\, \lambda \ge 0.$$
$$\exists \alpha, \beta, \quad |M'(\lambda)| \le \beta\lambda^\alpha, \quad \forall\ \lambda \ge 0.$$

(A3) The kernel function $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ is a bounded C^1 function such that

$$1 - \int_0^\infty h(s)\,ds = \beta_1 > 0,$$

and we assume that there exist a positive constant ζ satisfying

$$h'(t) \le -\zeta h(t), \quad t \ge 0.$$

(A4) $g : \mathbb{R} \to \mathbb{R}$ is an odd non decreasing function of class C^1 such that there exist c_1,α_1, α_2 positives satisfying

$$|g'(s)| \le c_1, \ \ \forall\, s \in \mathbb{R},$$
$$\alpha_1 s g(s) \le G(s) \le \alpha_2 s g(s), \ \ \forall\, s \in \mathbb{R},$$

where $G(s) = \int_0^s g(t)dt$, $\lim_{s\to+\infty} g(s) = +\infty$ and $\alpha_2\mu_2 \le \alpha_1\mu_1$.

First We state some lemmas which will be used in the next sections.

Lemma 2.1. For $h \in C^1([0,+\infty[,\mathbb{R})$ and $\varphi \in C^1(0,T; L^2(\Omega))$, we have

$$\int_\Omega \int_0^t h(t-s)\varphi(x,s)\varphi'(x,t)dsdx = -\frac{1}{2}h(t)\|\varphi(t)\|^2 + \frac{1}{2}(h'\Box\varphi)(t) + \frac{1d}{2dt}\left[\left(\int_0^t h(s)ds\right)\|\varphi(t)\|^2 - (h\Box\varphi)(t)\right],$$

where $(h\Box\varphi)(t) = \int_0^t h(t-s)\|\varphi(s) - \varphi(t)\|^2 ds$.

Lemma 2.2. Let Φ is a convex function of class $C^1(\mathbb{R})$. The Legendre transformation of Φ is defined as follows

$$\Phi^*(s) = \sup_{t\in\mathbb{R}}(st - \Phi(t)).$$

If Φ' is an odd and $\lim_{s\to+\infty}\Phi'(s) = +\infty$, then

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi\left((\Phi')^{-1}(s)\right), \ \ \forall\, s \in \mathbb{R},$$

and satisfies the inequality

$$st \le \Phi^*(s) + \Phi(t), \ \ \forall\, s,t \in \mathbb{R}.$$

The energy associated with problem (5)-(10) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\nabla u'(t)\|^2 \\ & + \frac{1}{2}\widehat{M}(\|\nabla u(t)\|^2) - \frac{1}{2}\left(\int_0^t h(s)\,ds\right)\|\Delta u(t)\|^2 \\ & + \frac{1}{2}(h\Box\Delta u)(t) + \xi\int_\Omega\int_0^1 G(z(x,\varrho,t))\,d\varrho\,dx, \end{aligned} \tag{11}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(t)dt$ and ξ is a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}.$$

Theorem 2.3. Let $u_0 \in H_0^2(\Omega)\cap H^3(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $f_0 \in H_0^1(\Omega, H^1(0,1))$ satisfy the compatibility condition $f(\cdot,0) = u_1$. Assume that (A1)-(A4) hold. Then (5)-(10) admits a weak solution

$$u \in L^\infty([0,\infty); H_0^2(\Omega)\cap H^3(\Omega)), \ \ u' \in L^\infty([0,\infty); H_0^2(\Omega)), \ \ u'' \in L^2([0,\infty);\ H_0^1(\Omega)).$$
$$z \in L^\infty([0,\infty); H_0^1(\Omega\times(0,1))), \ \ z' \in L^\infty([0,\infty); L^2(\Omega\times(0,1))),$$
$$G(z(x,\varrho,t)) \in L^\infty([0,\infty); L^1(\Omega\times(0,1))).$$

Moreover, if $E(0)$ is positive and bounded, then for every $t_0 > 0$, there exist positive constants k and K such that the energy defined by (2.11) possesses the following decay:

$$E(t) \le Ke^{-kt}, \ \ \forall t \ge t_0. \tag{12}$$

3 | PROOF OF THE MAIN RESULT

We will divide the proof into two steps: in the first step, we will use the Faedo-Galerkin method to prove the existence of global solutions, where the second step is devoted to proving the uniform decay of the energy by the perturbed energy method.

Step 1. Existence of weak solutions.

Let $T > 0$ be fixed and let $(w_i)_{i\in\mathbb{N}^*}$ be an orthogonal basis of $H^3(\Omega)\cap H_0^2(\Omega)$ with w_i being the eigenfunctions of the bi-Laplacien operator subject to the boundary condition

$$\Delta^2 w_i = \lambda_i w_i, \quad \text{in } \Omega, \quad w_i = \frac{\partial w_i}{\partial n} = 0 \quad \text{in } \partial\Omega.$$

By the linear elliptic operator theory described in[25], we have $w_j \in H^m(\Omega)\cap H_0^2(\Omega)$, $m\in\mathbb{N}$. Now we denote by $W_k = \text{span}\{w_1, w_2, ..., w_k\}$ the subspace generated by the first k vectors of the basis $(w_i)_{i\in\mathbb{N}^*}$. By normalization, we get $\|w_i\| = 1$. Now we definie for $1\le i\le k$ the sequence as follows $\varphi_i(x,0) = w_i(x)$. Then we may extend $\varphi_i(x,0)$ by $\varphi_i(x,\varrho)$ over $L^2(\Omega\times]0,1[)$ and denote Z_k the space generated by $\varphi_1, ..., \varphi_k$. For any given integer k, we consider the approximate solution (u_k, z_k)

$$u_k(x,t) = \sum_{i=1}^{k} g_{ik}(t) w_i(x), \quad z_k(x,\varrho,t) = \sum_{i=1}^{k} h_{ik}(t)\varphi_i(x,\varrho),$$

which satisfies

$$\begin{aligned}&(|u_k'(t)|^\rho u_k''(t), w_i) + (\Delta u_k(t), \Delta w_i) + (\nabla u_k''(t), \nabla w_i) + M(\|\nabla u_k(t)\|^2)(\nabla u_k(t), \nabla w_i)\\ &- \int_0^t h(t-s)(\Delta u_k(s), \Delta w_i)ds + \mu_1(g(u_k'(t)), w_i) + \mu_2(g(z_k(.,1,t)), w_i) = 0,\end{aligned} \tag{13}$$

$$z_k(x,0,t) = u_k'(x,t),$$

$$(\tau z_{kt}(t) + z_{k\varrho}(t), \varphi_i)_{L^2(\Omega\times]0,1[)} = 0 \tag{14}$$

$$u_k(0) = u_{0k}, \;\; u_k'(0) = u_{1k}, \;\; z_k(0) = z_{0k} \tag{15}$$

where $i = \overline{1,k}$

$$u_{0k} = \sum_{i=1}^{k}(u_0, w_i)w_i, \;\; u_{1k} = \sum_{i=1}^{k}(u_1, w_i)w_i, \;\; z_{0k} = \sum_{i=1}^{k}(f_0, \varphi_i)_{L^2(\Omega\times]0,1[)}\varphi_i$$

and for $k\to+\infty$

$$\begin{array}{lll} u_{0k}\to u_0 & \text{in} & H_0^2(\Omega)\cap H^3(\Omega),\\ u_{1k}\to u_1 & \text{in} & H_0^2(\Omega),\\ z_{0k}\to f_0 & \text{in} & H_0^1(\Omega, H^1(0,1)).\end{array} \tag{16}$$

Taking account of assumption (A1), $H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$, then $u_k''\in L^{2(\rho+1)}(\Omega)$, $|u_k'|^\rho \in L^{\frac{2(\rho+1)}{\rho}}(\Omega)$ and $w_i\in L^2(\Omega)$, from the generalized Hölder inequality, the nonlinear term in (3.1)

$$(|u_k'(t)|^\rho u_k''(t), w_i) = \int_\Omega |u_k'|^\rho u_k'' w_i dx \le \|u_k'\|_{2(\rho+1)}^\rho \|u_k''\|_{2(\rho+1)}\|w_i\|$$

make sens. According to the standard of ordinary differential equations theory, the finite dimensional problem (13)-(15) has a solution (g_{ik}, h_{ik}) defined on $[0,t_k[$. Then we can obtain an approximate solution u_k and z_k of (13)-(15) in W_k and Z_k respectively over $[0,t_k[$. Moreover, the solution can be extended to $[0,T]$ for any given T by the first estimates below.

Now we derive the first estimate. Multiplying (13) by $g_{ik}'(t)$ and summing with respect to i, we conclude that

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{\rho+2}\|u_k'(t)\|_{\rho+2}^{\rho+2} + \frac12\|\Delta u_k(t)\|^2 + \frac12\|\nabla u_k'(t)\|^2 + \frac12\widehat{M}(\|\nabla u_k(t)\|^2)\right] - \int_0^t h(t-s)(\Delta u_k(s), \Delta u_k'(t))ds\\ &+\mu_1\int_\Omega u_k'(x,t)g(u_k'(x,t))dx + \mu_2\int_\Omega u_k'(x,t)g(z_k(x,1,t))dx = 0.\end{aligned}$$

Applying lemma 2.1 with $\varphi = \Delta u_k$, (17) become

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{\rho+2}\|u_k'(t)\|_{\rho+2}^{\rho+2} + \frac12\|\Delta u_k(t)\|^2 + \frac12\|\nabla u_k'(t)\|^2 + \frac12\widehat{M}(\|\nabla u_k(t)\|^2) - \frac12\left(\int_0^t h(s)\,ds\right)\|\Delta u_k(t)\|^2 + \frac12(h\square\Delta u_k)(t)\right]\\ &= -\mu_1\int_\Omega u_k'(x,t)g(u_k'(x,t))dx - \mu_2\int_\Omega u_k'(x,t)g(z_k(x,1,t))dx - \frac12 h(t)\|\Delta u_k(t)\|^2 + \frac12(h'\square\Delta u_k)(t).\end{aligned} \tag{17}$$

We multiply equation (14) by $\xi g(z(x,\varrho,t))$ and integrating over $\Omega\times(0,1)$, we obtain

$$\xi\int_{\Omega}\int_0^1 z_{kt}(x,\varrho,t)g(z_k(x,\varrho,t))d\varrho dx = -\frac{\xi}{\tau}\int_{\Omega}\int_0^1 z_{k\varrho}(x,\varrho,t)g(z_k(x,\varrho,t))d\rho dx$$
$$= -\frac{\xi}{\tau}\int_{\Omega}\int_0^1 \frac{\partial}{\partial\varrho}\Big(G(z_k(x,\varrho,t))\Big)d\varrho dx.$$

Hence

$$\xi\frac{d}{dt}\int_{\Omega}\int_0^1 G(z_k(x,\varrho,t))d\varrho dx = -\frac{\xi}{\tau}\int_{\Omega}G(z_k(x,1,t))dx + \frac{\xi}{\tau}\int_{\Omega}G(u_k'(x,t))dx. \tag{18}$$

Combining (17) and (18), we obtain

$$\begin{aligned} E_k'(t) = &-\mu_1\int_{\Omega}u_k'(x,t)g(u_k'(x,t))dx - \mu_2\int_{\Omega}u_k'(x,t)g(z_k(x,1,t))dx - \frac{1}{2}h(t)\|\Delta u_k(t)\|^2 + \frac{1}{2}(h'\Box\Delta u_k)(t) \\ &- \frac{\xi}{\tau}\int_{\Omega}G(z_k(x,1,t))dx + \frac{\xi}{\tau}\int_{\Omega}G(u_k'(x,t))dx. \end{aligned} \tag{19}$$

From assumption (A4),we knew that G is a convex function of classe C^2, $G'=g$ is an odd and $\lim_{s\to+\infty}G'(s)=+\infty$, then by lemma 2.2, we deduce

$$G^*(s) = sg^{-1}(s) - G(g^{-1}(s)), \quad \forall s\in\mathbb{R}.$$

Applying these equality with $s=g(z_k(x,1,t))$, we obtain

$$G^*(g(z_k(x,1,t))) = z_k(x,1,t)g(z_k(x,1,t)) - G(z_k(x,1,t)),$$

By using inequality in lemma 2.2 together with (A4) for $s=g(z_k(x,1,t))$, $t=-u_k'(x,t)$ and G is even function, we get

$$\begin{aligned} -u_k'(x,t)g(z_k(x,1,t)) &\le G^*(g(z_k(x,1,t))) + G(-u_k'(x,t)), \\ &= z_k(x,1,t)g(z_k(x,1,t)) - G(z_k(x,1,t)) + G(u_k'(x,t)), \\ &\le (1-\alpha_1)z_k(x,1,t)g(z_k(x,1,t)) + \alpha_2 u_k'g(u_k'). \end{aligned} \tag{20}$$

From (19), (20) and assumption (A4), we have

$$\begin{aligned} E_k'(t) \le &-\frac{1}{2}h(t)\|\Delta u_k(t)\|^2 + \frac{1}{2}(h'\Box\Delta u_k)(t) \\ &-\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_{\Omega}u_k'(x,t)g(u_k'(x,t))dx \\ &-\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_{\Omega}z_k(x,1,t)g(z_k(x,1,t))dx. \end{aligned} \tag{21}$$

Integrating (21) over $(0,t)$ and using assumption $(A3)$, we conclude that

$$E_k(t) + \theta_1\int_0^t\int_{\Omega}u_k'(x,t)g(u_k'(x,t))dx + \theta_2\int_0^t\int_{\Omega}z_k(x,1,t)g(z_k(x,1,t))dx \le C_1. \tag{22}$$

where

$$\theta_1 = \Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big), \quad \theta_2 = \Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big),$$

and C_1 is a positive constant depending only on $\|u_0\|_{H_0^2(\Omega)}$, $\|u_1\|_{H_0^1(\Omega)}$ and $\|f_0\|_{L^2(\Omega\times(0,1))}$. Noting (A2), (A3) and (22), we obtain the first estimate

$$\|u_k'(t)\|_{\rho+2}^{\rho+2} + \|\Delta u_k(t)\|^2 + \|\nabla u_k'(t)\|^2 + \|\nabla u_k(t)\|^2 + (h\Box\Delta u_k)(t) + \xi \int_\Omega \int_0^1 G(z_k(x,\varrho,t))\, d\varrho\, dx \tag{23}$$
$$+ \int_0^t \int_\Omega u_k'(x,s) g(u_k'(x,s)) dx ds + \int_0^t \int_\Omega z_k(x,1,s) g(z_k(x,1,s)) dx ds \le C_2,$$

where C_2 is a positive constant depending only on $\|u_0\|_{H_0^2(\Omega)}$, $\|u_1\|_{H_0^1(\Omega)}$, $\|f_0\|_{L^2(\Omega\times(0,1))}$, m_0, β_1, ξ, τ, θ_1 and θ_2.
It follows from (23) that

u_k is uniformly bounded in $L^\infty(0,T; H_0^2(\Omega))$,
u_k' is uniformly bounded in $L^\infty(0,T; H_0^1(\Omega))$,
$G(z_k(x,\rho,t))$ is uniformly bounded in $L^\infty(0,T; L^1(\Omega\times(0,1)))$,
$u_k' g(u_k')$ is uniformly bounded in $L^1(\Omega\times(0,T))$,
$z_k(.,1,.)g(z_k(.,1,.))$ is uniformly bounded in $L^1(\Omega\times(0,T)))$.

Then, we derive the second estimate. Substituting w_i by $-\Delta w_i$ in (13), multiplying by g_{ik}' and then summing with respect to i, it holds that

$$\frac{1}{2}\frac{d}{dt}\left[\|\nabla\Delta u_k(t)\|^2 + \|\Delta u_k'(t)\|^2\right] - \int_\Omega |u_k'(x,t)|^\rho u_k''(x,t)\Delta u_k'(x,t) dx$$
$$+ \mu_1 \int_\Omega |\nabla u_k'(x,t)|^2 g'(u_k(x,t)) dx + \mu_2 \int_\Omega g'(z_k(x,1,t))\nabla u_k'(x,t)\nabla z_k(x,1,t) dx$$
$$- \int_0^t h(t-s) \int_\Omega \nabla\Delta u_k(x,s)\nabla\Delta u_k'(x,t) dx ds + \int_\Omega M(\|\nabla u_k(t)\|^2)\Delta u_k(x,t)\Delta u_k'(x,t) dx = 0$$

Using lemma 2.1 with $\varphi = \nabla\Delta u_k$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left(1 - \int_0^t h(s) ds\right)\|\nabla\Delta u_k\|^2 + \|\Delta u_k'\|^2 + M(\|\nabla u_k\|^2)\|\Delta u_k\|^2 + h\Box\nabla\Delta u_k(t)\right] \\
&- \int_\Omega |u_k'|^\rho u_k'' \Delta u_k' dx - M'(\|\nabla u_k\|^2)\|\Delta u_k\|^2 (\nabla u_k, \nabla u_k') \\
&+ \mu_1 \int_\Omega |\nabla u_k'|^2 g'(u_k) dx + \mu_2 \int_\Omega g'(z_k(x,1,t))\nabla u_k' \nabla z_k(x,1,t) dx \\
&= -\frac{1}{2} h(t)\|\nabla\Delta u_k\|^2 + \frac{1}{2}(h'\Box\nabla\Delta u_k)(t).
\end{aligned} \tag{24}$$

Substituting φ_i by $-\Delta\varphi_i$ in (14), multiplying by h_{ik}, summing with respect to i and integratin over $\varrho \in (0,1)$, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))} + \frac{1}{2}\|\nabla z_k(.,1,t)\|^2 - \frac{1}{2}\|\nabla u_k'(t)\|^2 = 0. \tag{25}$$

Using the Green's formula, we get

$$- \int_\Omega |u_k'|^\rho u_k'' \Delta u_k' dx = \frac{d}{dt}\int_\Omega |u_k'(x,t)|^\rho |\nabla u_k'(x,t)|^2 dx - \int_\Omega |u_k'(x,t)|^\rho \nabla u_k''(x,t)\nabla u_k'(x,t) dx. \tag{26}$$

Combining (24) to (26), we obtain

$$\frac{1}{2}\frac{d}{dt}\Bigg[\left(1-\int_0^t h(s)ds\right)\|\nabla\Delta u_k\|^2+\|\Delta u_k'\|^2+M(\|\nabla u_k\|^2)\|\Delta u_k\|^2+h\Box\nabla\Delta u_k(t)+2\int_\Omega |u_k'|^\rho|\nabla u_k'|^2dx$$
$$+\tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}\Bigg]+\frac{1}{2}\|\nabla z_k(.,1,t)\|^2+\mu_1\int_\Omega |\nabla u_k'|^2 g'(u_k)dx \tag{27}$$
$$=\int_\Omega |u_k'|^\rho\nabla u_k''\nabla u_k' dx-\mu_2\int_\Omega g'(z_k(x,1,t))\nabla u_k'\nabla z_k(x,1,t)dx+\frac{1}{2}\|\nabla u_k'(t)\|^2$$
$$+M'(\|\nabla u_k\|^2)\|\Delta u_k\|^2(\nabla u_k,\nabla u_k')-\frac{1}{2}h(t)\|\nabla\Delta u_k\|^2+\frac{1}{2}(h'\Box\nabla\Delta u_k)(t)$$

From Young inequality we have, for all $\eta>0$, that

$$ab\le \eta a^2+\frac{b^2}{4\eta},\quad \text{where}\ \ a,b\in\mathbb{R}_+^*.$$

Assumption $(A2)$, (23) and using Young's inequality with $\eta=1/2$, we obtain

$$\begin{aligned}
&M'(\|\nabla u_k\|^2)\|\Delta u_k\|^2(\nabla u_k,\nabla u_k')\\
&\quad\le M'(\|\nabla u_k\|^2)\|\Delta u_k\|^2\|\nabla u_k\|\|\nabla u_k'\|\\
&\quad\le \frac{1}{2}\left(M'(\|\nabla u_k\|^2)\right)^2\|\Delta u_k\|^4\|\nabla u_k\|^2+\frac{1}{2}\|\nabla u_k'\|^2\\
&\quad\le \frac{1}{2}\beta^2\|\nabla u_k\|^{4\alpha+2}\|\Delta u_k\|^4+\frac{C_2}{2}\\
&\quad\le \frac{\beta^2C_2^{2\alpha+3}+C_2}{2}.
\end{aligned}$$

From the generalized Hölder inequality and Sobolev embedding theorem $H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$, we get

$$\begin{aligned}
\int_\Omega |u_k'|^\rho\nabla u_k''\nabla u_k' dx &\le \|u_k'\|^\rho_{2(\rho+1)}\|\nabla u_k'\|_{2(\rho+1)}\|\nabla u_k''\|\\
&\le C_s^{\rho+1}\|\nabla u_k'\|^\rho\|\Delta u_k'\|\|\nabla u_k''\|.
\end{aligned}$$

From Young inequality and (23), we deduce

$$\int_\Omega |u_k'|^\rho\nabla u_k''\nabla u_k' dx\le \eta\|\nabla u_k''\|^2+\frac{C_s^{2(\rho+1)}C_2^\rho}{4\eta}\|\Delta u_k'\|^2.$$

Similary, Young inequality and assumption $(A4)$ leads to

$$-\mu_2\int_\Omega \nabla u_k'\nabla z_k(x,1,t)g'(z_k(x,1,t))\le \eta\|\nabla z_k(.,1,t)\|^2+\frac{(\mu_2c_1)^2C_2}{4\eta}. \tag{28}$$

Taking account to (28)-(28) into (27) yields

$$\frac{1}{2}\frac{d}{dt}\Bigg[\left(1-\int_0^t h(s)ds\right)\|\nabla\Delta u_k\|^2+\|\Delta u_k'\|^2+M(\|\nabla u_k\|^2)\|\Delta u_k\|^2+h(\Box\nabla\Delta u_k)(t)+2\int_\Omega |u_k'|^\rho|\nabla u_k'|^2dx$$
$$+\tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}\Bigg]+(\frac{1}{2}-\eta)\|\nabla z_k(.,1,t)\|^2+\mu_1\int_\Omega |\nabla u_k'|^2g'(u_k)dx \tag{29}$$
$$\le \eta\|\nabla u_k''\|^2+\frac{C_s^{2(\rho+1)}C_2^\rho}{4\eta}\|\Delta u_k'\|^2-\frac{1}{2}h(t)\|\nabla\Delta u_k\|^2+\frac{1}{2}(h'\Box\nabla\Delta u_k)(t)+C_2(\eta).$$

Multiplying (13) by g''_{ik} and summing with respect to i, it holds that

$$\begin{aligned}
&\int_\Omega |u'_k|^\rho |u''_k|^2 dx + \|\nabla u''_k\|^2 \\
&= -\int_\Omega u''_k \Delta^2 u_k dx + \int_0^t h(t-s) \int_\Omega \Delta u_k(x,s)\Delta u''_k(x,t) dx ds \\
&- \int_\Omega M(\|\nabla u_k\|^2)\nabla u_k \nabla u''_k dx - \mu_1 \int_\Omega u''_k g(u'_k) dx - \mu_2 \int_\Omega u''_k g(z^k(x,1,t)) dx.
\end{aligned}$$

Differentiating (14) with respect to t, multiplying by h'_{jk} and summing with respect i, it follows

$$\frac{\tau}{2}\frac{d}{dt}\|z'_k\|^2 + \frac{1}{2}\frac{d}{d\varrho}\|z'_k\|^2 = 0, \tag{30}$$

Integrating (30) with respect to $\varrho \in (0,1)$ and summing with (3.28), we obtain that

$$\begin{aligned}
&\int_\Omega |u'_k|^\rho |u''_k|^2 dx + \|\nabla u''_k\|^2 + \frac{\tau}{2}\frac{d}{dt}\|z'_k\|^2_{L^2(\Omega\times(0,1))} + \frac{1}{2}\|z'_k(1,t)\|^2 \\
&= -\int_\Omega u''_k \Delta^2 u_k dx + \int_0^t h(t-s) \int_\Omega \Delta u_k(x,s)\Delta u''_k(x,t) dx ds + \frac{1}{2}\|u''_k\|^2 \\
&- \int_\Omega M(\|\nabla u_k\|^2)\nabla u_k \nabla u''_k dx - \mu_1 \int_\Omega u''_k g(u'_k) dx - \mu_2 \int_\Omega u''_k g(z^k(x,1,t)) dx.
\end{aligned} \tag{31}$$

In what follows, we will estimate the right hand side in (31). Using Green formula and Young's inequality, we get

$$\begin{aligned}
\int_\Omega u''_k(x,t)\Delta^2 u_k(x,t) dx &= -\int_\Omega \nabla u''_k(x,t)\nabla\Delta u_k(x,t) dx \\
&\le \eta\|\nabla u''_k(t)\|^2 + \frac{1}{4\eta}\|\nabla\Delta u_k(t)\|^2.
\end{aligned} \tag{32}$$

Applying Cauchy-Schwarz inequality and Young's inequality, we obtain from assumption $(A2)$ and (23) that

$$\begin{aligned}
-\int_\Omega M(\|\nabla u_k\|^2)\nabla u_k \nabla u''_k dx &\le \delta\|\nabla u_k\|^{2\gamma+1}\|\nabla u''_k\| \\
&\le \eta\|\nabla u''_k\|^2 + \frac{\delta^2 C_2^{2\gamma+1}}{4\eta}.
\end{aligned} \tag{33}$$

Similarly, we have

$$\begin{aligned}
&\int_0^t h(t-s) \int_\Omega \Delta u_k(x,s)\Delta u''_k(x,t) dx ds \\
&\le \eta\|\nabla u''_k\|^2 + \frac{1}{4\eta}\int_\Omega \Big(\int_0^t h(t-s)|\nabla\Delta u_k(s)| ds\Big)^2 dx \\
&\le \eta\|\nabla u''_k\|^2 + \frac{1}{4\eta}\int_\Omega \Big(\int_0^t h(t-s)\big(|\nabla\Delta u_k(s) - \nabla\Delta u_k(t)| + |\nabla\Delta u_k(t)|\big) ds\Big)^2 dx \\
&= \eta\|\nabla u''_k\|^2 + \frac{1}{4\eta} I.
\end{aligned}$$

Applying Hölder's inequality and Young's, we get

$$
\begin{aligned}
|I| &\leq \left(\int_0^t h(s)ds\right)^2 \|\nabla\Delta u_k(t)\|^2 + \int_\Omega \left(\int_0^t h(t-s)|\nabla\Delta u_k(s) - \nabla\Delta u_k(t)|ds\right)^2 dx \\
&\quad + 2\int_\Omega \left(\int_0^t h(t-s)|\nabla\Delta u_k(s) - \nabla\Delta u_k(t)|ds\right)\left(\int_0^t h(t-s)|\nabla\Delta_k u(t)|ds\right)dx \\
&\leq 2\left(\int_0^t h(s)ds\right)^2 \|\nabla\Delta_k u(t)\|^2 + 2\int_\Omega \left(\int_0^t h(t-s)|\nabla\Delta u_k(s) - \nabla\Delta u_k(t)|ds\right)^2 dx. \\
&\leq 2(1-\beta_1)^2\|\nabla\Delta_k u(t)\|^2 + 2(1-\beta_1)(h\Box\nabla\Delta u_k)(t),
\end{aligned}
$$

then, we obtain estimation

$$
\begin{aligned}
&\int_0^t h(t-s)\int_\Omega \Delta u_k(x,s)\Delta u_k''(x,t)dxds \\
&\leq \eta\|\nabla u_k''\|^2 + \frac{(1-\beta_1)^2}{2\eta}\|\nabla\Delta_k u(t)\|^2 + \frac{(1-\beta_1)}{2\eta}(h\Box\nabla\Delta u_k)(t).
\end{aligned} \tag{34}
$$

And also by Young's inequality and Sobolev embedding theorem, we obtain

$$
-\mu_1\int_\Omega u_k'' g(u_k')dx \leq \eta C_s^2\|\nabla u_k''\|^2 + \frac{\mu_1^2}{4\eta}\int_\Omega |g(u_k')^2dx. \tag{35}
$$

$$
-\mu_2\int_\Omega u_k'' g(z_k(x,1,t))dx \leq \eta C_s^2\|\nabla u_k''\|^2 + \frac{\mu_2^2}{4\eta}\int_\Omega |g(z^k(x,1,t))|^2dx. \tag{36}
$$

Taking into account (32)-(36) into (31) yields

$$
\begin{aligned}
&\int_\Omega |u_k'|^\rho|u_k''|^2dx + \left(1 - 3\eta - 2\eta C_s^2 - \frac{C_s^2}{2}\right)\|\nabla u_k''(t)\|^2 + \frac{\tau}{2}\frac{d}{dt}\|z_k'(t)\|^2_{L^2(\Omega\times(0,1))} + \frac{1}{2}\|z_k'(1,t)\|^2 \\
&\leq \frac{2(1-\beta_1)^2+1}{4\eta}\|\nabla\Delta u_k(t)\|^2 + \frac{\mu_1^2}{4\eta}\int_\Omega |g(u_k')|^2dx + \frac{\mu_2^2}{4\eta}\int_\Omega |g(z_k(x,1,t))|^2dx + \frac{\delta^2C_2^{2\gamma+1}}{4\eta} + \frac{1-\beta_1}{2\eta}(h\Box\nabla\Delta u_k)(t).
\end{aligned} \tag{37}
$$

Thus from (29), (37), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg[\left(1 - \int_0^t h(s)ds\right)\|\nabla\Delta u_k\|^2 + \|\Delta u_k'\|^2 + M(\|\nabla u_k\|^2)\|\Delta u_k\|^2 + (h\Box\nabla\Delta u_k)(t) + \tau\|z_k'\|^2_{L^2(\Omega\times(0,1))} \\
&\quad + 2\int_\Omega |u_k'|^\rho|\nabla u_k'|^2dx + \tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}\Bigg] + (\frac{1}{2}-\eta)\|\nabla z_k(.,1,t)\|^2 + \mu_1\int_\Omega |\nabla u_k'|^2 g'(u_k)dx + \int_\Omega |u_k'|^\rho|u_k''|^2dx \\
&\quad + \left(1 - 4\eta - 2\eta C_s^2 - \frac{C_s^2}{2}\right)\|\nabla u_k''\|^2 + \frac{1}{2}\|z_k'(1,t)\|^2 \\
&\leq \frac{2(1-\beta_1)^2+1}{4\eta}\|\nabla\Delta u_k(t)\|^2 + \frac{\mu_1^2}{4\eta}\int_\Omega |g(u_k')|^2dx + \frac{\mu_2^2}{4\eta}\int_\Omega |g(z_k(x,1,t))|^2dx + \frac{C_s^{2(\rho+1)}C_2^\rho}{4\eta}\|\Delta u_k'\|^2 \\
&\quad + \frac{1-\beta_1}{2\eta}(h\Box\nabla\Delta u_k)(t) + \frac{1}{2}(h'\Box\nabla\Delta u_k)(t) - \frac{1}{2}h(t)\|\nabla\Delta u_k\|^2 + \frac{\delta^2C_2^{2\gamma+1}}{4\eta} + C_2(\eta).
\end{aligned} \tag{38}
$$

Using (A4), (23) and the mean value theorem, we obtain

$$
\begin{aligned}
\int_{Q_T} |g(u_k')|^2dxds &= \int_{Q_T} |g(u_k') - g(0)||g(u_k')|dxds \\
&\leq c_1\int_{Q_T} g(u_k')u_k'dxds \leq c_1C_2,
\end{aligned} \tag{39}
$$

$$\text{and} \int_{Q_T} |g(z_k(x,1,s))|^2 dxds \le c_1 C_2, \quad \text{where } Q_T = \Omega \times (0,T). \tag{40}$$

Integrating (38) over $(0,T)$, using (39), (40), and $(A3)$, it yields

$$\begin{aligned}
&\left(1 - \int_0^t h(s)ds\right) \|\nabla\Delta u_k\|^2 + \|\Delta u_k'\|^2 + M(\|\nabla u_k\|^2)\|\Delta u_k\|^2 + (h\Box\nabla\Delta u_k)(t) + \tau\|z_k'\|^2_{L^2(\Omega\times(0,1))} + 2\int_\Omega |u_k'|^\rho |\nabla u_k'|^2 dx \\
&+ \tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))} + (1-2\eta)\int_0^t \|\nabla z_k(.,1,s)\|^2 ds + \int_0^t \|z_k'(.,1,s)\|^2 ds + 2\int_0^t\int_\Omega |u_k'|^\rho |u_k''|^2 dxds \\
&+ \left(2 - 8\eta - 4\eta C_s^2 - C_s^2\right)\int_0^t \|\nabla u_k''\|^2 ds + 2\mu_1 \int_0^t\int_\Omega |\nabla u_k'|^2 g'(u_k) dxds \\
&\le \frac{2(1-\beta_1)^2+1}{2\eta}\int_0^t \|\nabla\Delta u_k\|^2 ds + \frac{\mu_1^2 c_1 C_2}{2\eta} + \frac{\mu_2^2 c_1 C_2}{2\eta} + \frac{C_s^{2(\rho+1)} C_2^\rho}{2\eta}\int_0^t \|\Delta u_k'\|^2 ds + \frac{(\delta^2 C_2^{2\gamma+1} + 4C_2(\eta))T}{2\eta} \\
&\quad + \frac{1-\beta_1}{\eta}\int_0^t (h\Box\nabla\Delta u_k)(s) ds + C_3,
\end{aligned} \tag{41}$$

where C_3 is positive constant depending on $\|u_0\|_{H^3\cap H_0^2(\Omega)}$, $\|u_1\|_{H_0^2}$ and $\|f_0\|_{H_0^1(\Omega\times(0,1))}$. Taking η suitably small in (41) and using Gronwall lemma, we obtained the second estimate.

$$\begin{aligned}
&\|\nabla\Delta u_k(t)\|^2 + \|\Delta u_k'(t)\|^2 + \|z_k'(t)\|^2_{L^2(\Omega\times(0,1))} + \|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))} + \int_0^t \|\nabla z_k(.,1,s)\|^2 ds \\
&+ \int_0^t \|\nabla u_k''(s)\|^2 ds + \int_0^t \|z_k'(.,1,s)\|^2 ds \le C_4,
\end{aligned} \tag{42}$$

where C_4 is positive constant depending on $\|u_0\|_{H^3\cap H_0^2(\Omega)}$, $\|u_1\|_{H_0^1}$, $\|f_0\|_{H_0^1(\Omega\times(0,1))}$, ρ, $g(0)$, m_0, β_1, τ, and T. Estimate (42) implies

$$\begin{aligned}
u_k &\text{ is uniformly bounded in } L^\infty(0,T;H^3(\Omega)\cap H_0^2(\Omega)), \\
u_k' &\text{ is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)), \\
u_k'' &\text{ is uniformly bounded in } L^2(0,T;H_0^1(\Omega)), \\
z_k &\text{ is uniformly bounded in } L^\infty(0,T;L^2((0,1);H_0^1(\Omega))), \\
z_k' &\text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega\times(0,1))), \\
z_k(.,1,.) &\text{ is uniformly bounded in } L^2(0,T;H_0^1(\Omega)), \\
z_k'(.,1,.) &\text{ is uniformly bounded in } L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{43}$$

By (43) and $z_{k\varrho} = -\tau z_k'$, then

$$z_k \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega\times(0,1))).$$

Applying Dunford-Pettis theorem, we infer that there exists a subsequence (u_j) of (u_k) and u such that

$$u_k \rightharpoonup u \quad \text{weakly star in } L^\infty(0,T;H^3(\Omega)\cap H_0^2(\Omega)), \tag{44}$$

$$u_k' \rightharpoonup u' \quad \text{weakly star in } L^\infty(0,T;H_0^2(\Omega)), \tag{45}$$

$$u_k'' \rightharpoonup u'' \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)). \tag{46}$$

By Aubin's lemma, it follows from (44)-(46) that there exist a subsquence of u_j still denote by u_j such that

$$u_j \to u \quad \text{strongly in } L^2(0,T;H_0^2(\Omega)) \tag{47}$$

$$u_j' \to u' \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)) \tag{48}$$

which implies $\nabla u_j \to \nabla u$, $\Delta u_j \to \Delta u$ and $u'_j \rightharpoonup u'$ almost everywhere in $\Omega \times (0,T)$. Hence,

$$|u'_j|^\rho u'_j \to |u'|^\rho u' \quad \text{almost everywhere in } \Omega \times (0,T). \tag{49}$$

$$M(\|\nabla u_j\|^2)\Delta u_j \to M(\|\nabla u\|^2)\Delta u \quad \text{almost everywhere in } \Omega \times (0,T) \tag{50}$$

On the other hand, by the Sobolev embedding theorem and the first estimate, this yields

$$\begin{aligned} \| |u'_j|^\rho u'_j \|_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|u'_j\|_{2(\rho+1)}^{2(\rho+1)} dt \\ &\le c_s^{2(\rho+1)} \int_0^T \|\nabla u'_j\|_2^{2(\rho+1)} dt \\ &\le c_s^{2(\rho+1)} C_2^{2(\rho+1)} T. \end{aligned} \tag{51}$$

Thus using (A2), (23), (39), (49), (50) and the Lions lemma ([15],p 12.), we derive

$$|u'_j|^\rho u'_j \rightharpoonup |u'|^\rho u' \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{52}$$

$$M(\|\nabla u_j\|^2)\Delta u_j \rightharpoonup M(\|\nabla u\|^2)\Delta u \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \tag{53}$$

$$g(u_j) \rightharpoonup g(u) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{54}$$

Similarly, by applying the Dunford-Pettis theorem, we infer that there exists a subsequence (z_j) of (z_k) and z such that

$$z_k \rightharpoonup z \quad \text{weakly star in } L^\infty(0,T;H_0^1(\Omega \times (0,1))), \tag{55}$$

$$z'_k \rightharpoonup z' \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega \times (0,1))), \tag{56}$$

$$\begin{gathered} z_{k\varrho} \rightharpoonup z_\varrho \quad \text{weakly in } L^2(0,T;L^2(\Omega \times (0,1))), \\ z_k(.,1,.) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \\ z'_k(.,1,.) \text{ is bounded in } L^2(0,T;L^2(\Omega)). \end{gathered} \tag{57}$$

By Aubins lemma, it follows from (51), (52), (54) and (55) that there exists a subsequence of z_j still denoted by z_j and subsequence of $z_j(.,1,.)$ still denoted by $z_j(.1,.)$, such that

$$z_k \to z \ \text{ strongly in } L^2(0,T;L^2(\Omega \times (0,1))),$$

$$z_k(.,1,.) \to z(.,1,.) \ \text{ strongly in } L^2(0,T;L^2(\Omega))$$

Also by (40), (57) and Lions lemma, then

$$g(z_k(.,1,.)) \rightharpoonup g(z(.,1,.)) \ \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

Let $\mathcal{D}(0,T)$ be the space of C^∞ functions with compact support in $(0,T)$. Multiplying (13), (14) by $\theta \in \mathcal{D}(0,T)$ and integrating over $(0,T)$, it follows that

$$\begin{aligned} &-\frac{1}{\rho+1}\int_0^T (\|u'_k(t)\|^\rho u'_k(t), w_i)\theta'(t)dt + \int_0^T (\Delta u_k(t), \Delta w_i)\theta(t)dt + \int_0^T (\nabla u''_k(t), \nabla w_i)\theta(t)dt \\ &+ \int_0^T M(\|\nabla u_k(t)\|^2)(\nabla u_k(t), \nabla w_i)\theta(t)dt + \int_0^T\int_0^t h(t-s)(\nabla \Delta u_k(t), \nabla w_i)\theta(t)dsdt \\ &+ \mu_1 \int_0^T (g(u'_k(t)), w_i)\theta(t)dt + \mu_2 \int_0^T (g(z_k(.,1,t)), w_i)\theta(t)dt = 0, \end{aligned} \tag{58}$$

and

$$\tau \int_0^T (z'_k, \varphi_i)_{L^2(\Omega\times(0,1))}\theta(t)dt + \int_0^T (z_{k\varrho}, \varphi_i)_{L^2(\Omega\times(0,1))}\theta(t)dt = 0. \tag{59}$$

Noting that $\{w_i\}_{i=1}^{\infty}$ is basis of $H^3(\Omega)\cap H_0^2(\Omega)$ and $\{\varphi_i\}_{i=1}^{\infty}$ is basis of $L^2(\Omega\times(0,1))$, we can pass to the limit in (58), (59) and obtain

$$
\begin{aligned}
&|u'|^\rho u'' + \Delta^2 u - \Delta u'' - M(\|\nabla u\|^2)\Delta u - \int_0^t h(t-s)\Delta^2 u(s)ds \\
&+ \mu_1 g(u') + \mu_2 g(z(.,1,.)) = 0, \text{ in } L^2(0,T;H^{-1}(\Omega)), \\
&\tau z' + z_\varrho = 0, \text{ in } L^2(0,T;L^2(\Omega\times(0,1)))
\end{aligned}
$$

for arbitrary $T>0$. From (44)- (45), (55)- (56) and lemma 3.3.7 in[25], we conclude $u_j(0)\to u(0)$ weakly in $H_0^2(\Omega)$, $u_j'(0)\rightharpoonup u'(0)$ weakly in $H_0^1(\Omega)$ and $z_j(0)\rightharpoonup z(0)$ weakly in $L^2(\Omega\times(0,1))$. Hence by (16), we have $u(0)=u_0$, $u'(0)=u_1$ and $z(0)=f_0$. Consequently, the global existence of weak solutions is established.

Step 2. Uniform decay of the energy.
To continue our proof, we need to introduce three new functionals

$$\Phi(t) = \frac{1}{\rho+1}\int_\Omega |u'|^\rho u' u dx + \int_\Omega \nabla u'\nabla u dx.$$

$$\Psi(t) = -\int_\Omega \nabla u' \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))dsdx - \frac{1}{\rho+1}\int_\Omega |u'|^\rho u' \int_0^t h(t-s)(u(t)-u(s))dsdx. \tag{60}$$

$$\Upsilon(t) = \int_\Omega\int_0^1 e^{-2\tau\varrho} G(z(x,\varrho,t))d\varrho dx$$

We set

$$F(t) = NE(t) + \epsilon_1\Phi(t) + \Psi(t) + \epsilon_2\Upsilon(t), \tag{61}$$

where N, ϵ_1 and ϵ_2 are suitable positive constants to be determined later.

Proposition 3.1. There exist positive numbers k_0 and k_1 such that

$$k_0 E(t) \le F(t) \le k_1 E(t). \tag{62}$$

Proof. Using (11), we get

$$|\Upsilon(t)| \le \frac{1}{\xi}E(t). \tag{63}$$

Thanks to Young inequality and the Sobolev embedding theorem, we deduce

$$
\begin{aligned}
|\Phi(t)| &\le \frac{1}{\rho+1}\left|\int_\Omega |u'|^\rho u' u dx\right| + \left|\int_\Omega \nabla u'\nabla u dx\right| \\
&\le \frac{1}{\rho+2}\|u'\|_{\rho+2}^{\rho+2} + \frac{(\rho+1)^{-1}}{(\rho+2)}\|u\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u'\|^2 + \frac{1}{2}\|\nabla u\|^2 \\
&\le \frac{1}{\rho+2}\|u'\|_{\rho+2}^{\rho+2} + \left(\frac{(\rho+1)^{-1}}{(\rho+2)}C_s^{\rho+2}(2E(0)/\beta_1)^{\rho/2}\right)\|\nabla u'\|^2 + \frac{1}{2}\|\nabla u\|^2.
\end{aligned}
\tag{64}
$$

By Youngs inequality and Sobolev embedding theorem, the second term in the right hand side (60) can be estimated as follows

$$
\begin{aligned}
&\left|\tfrac{1}{\rho+1}\int_\Omega |u'|^\rho u' \int_0^t h(t-s)(u(t)-u(s))dsdx\right| \\
&\le \tfrac{1}{\rho+2}\|u'\|_{\rho+2}^{\rho+2} + \tfrac{(\rho+1)^{-1}}{(\rho+2)}\int_\Omega\left(\int_0^t h(t-s)|u(t)-u(s)|ds\right)^{\rho+2}dx \\
&\le \tfrac{1}{\rho+2}\|u'\|_{\rho+2}^{\rho+2} + \tfrac{(\rho+1)^{-1}}{(\rho+2)}\left(\int_0^t h(s)ds\right)^{\rho+1}\int_0^t h(t-s)\int_\Omega |u(t)-u(s)|^{\rho+2}dxds \\
&\le \tfrac{1}{\rho+2}\|u'\|_{\rho+2}^{\rho+2} + \tfrac{(\rho+1)^{-1}}{(\rho+2)}(1-\beta_1)^{\rho+1}C_s^{\rho+2}(8E(0)/\beta_1)^{\rho/2}(h\Box\Delta u)(t).
\end{aligned}
\tag{65}
$$

Thus, from (65) we obtain

$$\Psi(t) \le \frac{1}{2}\|\nabla u'\|^2 + \frac{(1-\beta_1)C_s^2}{2}(h\Box\Delta u)(t) + \frac{1}{\rho+2}\|u'\|_{\rho+2}^{\rho+2} + \frac{(\rho+1)^{-1}}{(\rho+2)}(1-\beta_1)^{\rho+1}C_s^{\rho+2}(8E(0)/\beta_1)^{\rho/2}(h\Box\Delta u)(t). \tag{66}$$

From (63),(64), (66) and the choice of ϵ_1, ϵ_2 and N, (62) can be established.

In order to obtain the exponential decay result of $E(t)$ via (41), it is sufficient to prove that of $F(t)$. To this end, we need to estimate the derivative of $F(t)$ first. Using (1), we obtain

$$\begin{aligned}\Phi'(t) =& \frac{1}{\rho+1}\|u'\|_{\rho+2}^{\rho+2} + \|\nabla u'\|^2 - \|\Delta u\|^2 - M(\|\nabla u\|^2)\|\nabla u\|^2 + \int_0^t h(t-s)(\Delta u(s), \Delta u(t))ds \\ & - \mu_1\int_\Omega u(x,t)g(u'(x,t))dx - \mu_2\int_\Omega u(x,t)g(z(x,1,t))dx.\end{aligned} \tag{67}$$

By use of Youngs inequality and sobolev embedding theorem, we can estimate the right hand side of (67) as follows:

$$\begin{aligned}&\int_\Omega\int_0^t h(t-s)\Delta u(x,t)\Delta u(x,s)dsdx \le \int_\Omega\int_0^t h(t-s)|\Delta u(x,t)|(|\Delta u(x,s)-\Delta u(x,t)| + |\Delta u(x,t)|)dsdx \\ &\le \int_0^t h(s)\|\Delta u(t)\|^2ds + \int_\Omega\int_0^t h(t-s)|\Delta u(x,t)||\Delta u(x,s)-\Delta u(x,t)|dsdx \\ &\le (1+\eta)\int_0^t h(s)\|\Delta u(t)\|^2ds + \frac{1}{4\eta}(h\Box\Delta u)(t),\end{aligned} \tag{68}$$

$$-\mu_1\int_\Omega u(x,t)g(u'(x,t))dx \le \mu_1\eta C_s^2\|\Delta u(t)\|^2 + \frac{\mu_1}{4\eta}\|g(u'(t))\|^2, \tag{69}$$

$$-\mu_2\int_\Omega ug(z(x,1,t))dx \le \mu_2\eta C_s^2\|\Delta u(t)\|^2 + \frac{\mu_2}{4\eta}\|g(z(.,1,t))\|^2, \tag{70}$$

where $\eta > 0$. Here and in the following we use C_s to represent the Poincare constant. From (A3), (68)-(70), we obtain

$$\begin{aligned}\Phi'(t) \le& \frac{1}{\rho+1}\|\nabla u'\|_{\rho+2}^{\rho+2} + \|\nabla u'\|^2 - \left(1-(1-\beta_1+1)(1+\eta) - (\mu_1+\mu_2)\eta C_s^2\right)\|\Delta u\|^2 \\ \le& -M(\|\nabla u\|^2)\|\nabla u\|^2 + \frac{\mu_1}{4\eta}\|g(u'(t))\|^2 + \frac{\mu_2}{4\eta}\|g(z(.,1,t))\|^2 + \frac{1}{4\eta}(h\Box\Delta u)(t).\end{aligned} \tag{71}$$

Taking the derivative of $\Psi(t)$, it follows from that (6)

$$\begin{aligned}\Psi'(t) =& \int_0^t h(t-s)(\Delta u(t), \Delta u(t)-\Delta u(s))dsdx \\ & - \int_0^t h'(t-s)(\nabla u'(t), \nabla u(t)-\nabla u(s))ds \\ & + \int_\Omega\int_0^t h(t-s)M(\|\nabla u\|^2)(\nabla u(t), \nabla u(t)-\nabla u(s))dsdx \\ & - \int_\Omega\left(\int_0^t h(t-s)\Delta u(s)ds\right)\left(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right)dx \\ & + \mu_1\int_\Omega\int_0^t h(t-s)(u(t)-u(s))g(u'(t))dsdx \\ & + \mu_2\int_\Omega\int_0^t h(t-s)(u(t)-u(s))g(z(x,1,t))dsdx \\ & - \frac{1}{\rho+1}\int_0^t h'(t-s)(|u'(t)|^\rho u'(t), u(t)-u(s))ds - \int_0^t h(s)\|\nabla u'(t)\|^2ds \\ & - \frac{1}{\rho+1}\int_0^t h(s)\|u'(t)\|^{\rho+2}ds \\ =& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \\ & - \int_0^t h(s)\|\nabla u'(t)\|^2ds - \frac{1}{\rho+1}\int_0^t h(s)\|u'(t)\|^{\rho+2}ds.\end{aligned} \tag{72}$$

In what follows we will estimate $I_1, ..., I_7$ in (72).

$$|I_1| \le \eta\|\Delta u(t)\|^2 + \frac{1-\beta_1}{4\eta}(h\Box\Delta u)(t), \quad \forall\eta > 0. \tag{73}$$

$$|I_2| \le \eta\|\nabla u'(t)\|^2 - \frac{h(0)C_s^2}{4\eta}(h'\Box\Delta u)(t), \quad \forall\eta > 0. \tag{74}$$

$$
\begin{aligned}
|I_3| &\le \eta(1-\beta_1)M(\|\nabla u\|^2)\|\nabla u\|^2 + \frac{\delta C_s^2}{4\eta}\|\nabla u\|^{2\gamma}(h\Box\Delta u)(t) \\
&\le \eta(1-\beta_1)M(\|\nabla u\|^2)\|\nabla u\|^2 + \frac{\delta C_s^2}{4\eta}(2E(0)/m_0)^{\gamma}(h\Box\Delta u)(t).
\end{aligned} \tag{75}
$$

For I_4 in (72), Applying Hölder's inequality and Young's inequality, we get

$$
\begin{aligned}
|I_4| &\le \eta\int_\Omega\left(\int_0^t h(t-s)\Delta u(s)ds\right)^2 dx + \frac{1}{4\eta}\int_\Omega\left(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right)^2 dx \\
&\le 2\eta\left(\int_0^t h(s)ds\right)^2\|\Delta u(t)\|^2 + (2\eta+\frac{1}{4\eta})\left(\int_0^t h(s)ds\right)(h\Box\Delta u)(t), \quad \forall\eta>0.
\end{aligned} \tag{76}
$$

By (A3), we obtain from (76) that

$$
|I_4| \le 2\eta(1-\beta_1)^2\|\Delta u(t)\|^2 + (2\eta+\frac{1}{4\eta})(1-\beta_1)(h\Box\Delta u)(t), \quad \forall\eta>0. \tag{77}
$$

Similarly,

$$
|I_5| \le \mu_1\eta\|g(u')\|^2 + \tfrac{\mu_1(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t)c_1\mu_1\eta\int_\Omega u'(x,t)g(u'(x,t))dx + \tfrac{\mu_1(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t), \quad \forall\eta>0. \tag{78}
$$

$$
\begin{aligned}
|I_6| &\le \mu_2\eta\|g(z(.,1,t))\|^2 + \tfrac{\mu_2(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t) \\
&\le c_1\mu_2\eta\int_\Omega z(x,1,t)g(z(x,1,t))dx + \tfrac{\mu_2(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t), \quad \forall\eta>0.
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
|I_7| &\le \frac{\eta}{\rho+1}\|u'(t)\|_{2(\rho+1)}^{2(\rho+1)} - \frac{h(0)C_s^2}{4\eta(\rho+1)}(h'\Box\Delta u)(t) \\
&\le \frac{\eta C_s^{2(\rho+1)}}{\rho+1}(2E(0))^{\rho}\|\nabla u'\|^2 - \frac{h(0)C_s^2}{4\eta(\rho+1)}(h'\Box\Delta u)(t) \\
&\le \frac{a_0\eta}{\rho+1}\|\nabla u'\|^2 - \frac{h(0)C_s^2}{4\eta(\rho+1)}(h'\Box\Delta u)(t), \quad \forall\eta>0,
\end{aligned} \tag{80}
$$

where $a_0 = C_s^{2(\rho+1)}(2E(0))^{\rho}$. Combining (72)-(75) and (77)-(80) together, we arrive at

$$
\begin{aligned}
\Psi'(t) \;\le\; & -\tfrac{\int_0^t h(s)ds}{\rho+1}\|u'\|_{\rho+2}^{\rho+2} + \left(\eta + \tfrac{a_0\eta}{\rho+1} - \int_0^t h(s)ds\right)\|\nabla u'(t)\|^2 + \eta\left(1+2(1-\beta_1)^2\right)\|\Delta u(t)\|^2 \\
& + c_1\mu_1\eta\int_\Omega u'(x,t)g(u'(x,t))dx \\
& + c_1\mu_2\eta\int_\Omega z(x,1,t)g(z(x,1,t))dx\left[\left(2\eta+\tfrac{1}{2\eta}+\tfrac{(\mu_1+\mu_2)C_s^2}{4\eta}\right)(1-\beta_1) + \tfrac{\delta C_s^2}{4\eta}(2E(0)/m_0)^{\gamma}\right](h\Box\Delta u)(t) \\
& - \tfrac{(\rho+2)h(0)C_s^2}{4(\rho+1)\eta}(h'\Box\Delta u)(t), \quad \forall\eta>0
\end{aligned} \tag{81}
$$

Taking also the derivative of $\Upsilon'(t)$ it follows from (6) and (A4) that

$$
\begin{aligned}
\Upsilon'(t) &= \int_\Omega\int_0^1 e^{-2\tau\varrho}z_\varrho(x,\varrho,t)g(z(x,\varrho,t)) \\
&= -\frac{1}{\tau}\int_\Omega\int_0^1\left[\frac{\partial}{\partial\varrho}(e^{-2\tau\varrho}G(z(x,\varrho,t))) + 2\tau e^{-2\tau\varrho}G(x,\varrho,t)\right]d\varrho dx \\
&= -\frac{1}{\tau}\int_\Omega\left[e^{-2\tau}G(z(x,\varrho,t)) - G(u'(x,t))\right]dx - 2\Upsilon(t) \\
&\le -2\Upsilon(t) + \frac{\alpha_2}{\tau}\int_\Omega u'(x,t)g(u(x,t))dx - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g(z(x,1,t))dx.
\end{aligned} \tag{82}
$$

Then we conclude that from (61),(71), (81) and (82) that for any $t \geq t_0 > 0$,

$$\begin{aligned}
F'(t) &= NE'(t) + \epsilon\Phi'(t) + \Psi(t) + \epsilon_2\Upsilon'(t) \\
&\leq -\tfrac{h_0-\epsilon}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+2} - \left[h_0 - \epsilon - \eta\left(1+\tfrac{a_0}{\rho+1}\right)\right]\|\nabla u'(t)\|^2 - \left[\epsilon - \eta(1-\beta_1)\right]\widehat{M}(\|\nabla u\|^2) \\
&\quad - \left[\epsilon\left(1-(1-\beta_1)(1+\eta) - (\mu_1+\mu_2)\eta C_s^2\right) - \eta(1+2(1-\beta_1)^2)\right]\|\Delta u\|^2 \\
&\quad + \left[\tfrac{\epsilon}{4\eta} + \left(2\eta + \tfrac{1}{2\eta} + \tfrac{(\mu_1+\mu_2)C_s^2}{4\eta}\right)(1-\beta_1) + \tfrac{\delta C_s^2}{4\eta}(2E(0)/m_0)^\gamma\right](h\Box\Delta u)(t) \\
&\quad + \left[\tfrac{N}{2} - \tfrac{(\rho+2)h(0)C_s^2}{4(\rho+1)\eta}\right](h'\Box\Delta u)(t) - \left(N\theta_1 - \tfrac{\epsilon_2\alpha_2}{\tau} - \tfrac{\epsilon\mu_1 c_1}{4\eta} - \mu_1 c_1\eta\right)\int_\Omega u'(x,t)g(u'(x,t))dx \\
&\quad - \left(N\theta_2 + \tfrac{\epsilon_2\alpha_1 e^{-2\tau}}{\tau} - \tfrac{\epsilon c_1\mu_2}{4\eta} - c_1\mu_2\eta\right)\int_\Omega z(x,1,t)g(z(x,1,t))dx - 2\epsilon_2\Upsilon(t), \quad \forall\eta > 0,
\end{aligned} \tag{83}$$

where $h_0 = \int_0^{t_0} h(s)ds > 0$, guaranteed by (A3). At this stage, we take $\epsilon < h_0$ and η sufficiently small that

$$a_2 \triangleq h_0 - \epsilon_1 - \eta\left(1 + \frac{a_0}{\rho+1}\right) > 0, \quad a_3 \triangleq \epsilon_1 - \eta(1-\beta_1) > 0,$$

$$\text{and} \quad a_4 \triangleq \epsilon_1\left(1-(1-\beta_1)(1+\eta) - (\mu_1+\mu_2)\eta C_s^2\right) - \eta\left(1+2(1-\beta_1)^2\right) > 0.$$

Choosing $\epsilon_2 > \dfrac{\xi e^{2\tau}}{2}$ for which

$$-2\epsilon_2\Upsilon(t) \leq -\xi\int_\Omega\int_0^1 G(x,\varrho,t)dxd\varrho.$$

As long as ϵ_1, ϵ_2 and η are fixed, we choose N large enough that

$$N\theta_1 - \frac{\epsilon_2\alpha_2}{\tau} - \frac{\epsilon_1\mu_1 c_1}{4\eta} - \mu_1 c_1\eta > 0, \quad N\theta_2 + \frac{\epsilon_2\alpha_1 e^{-2\tau}}{\tau} - \frac{\epsilon_1 c_1\mu_2}{4\eta} - c_1\mu_2\eta > 0,$$

and

$$a_5 \triangleq \xi\left[\frac{N}{2} - \frac{(\rho+2)h(0)C_s^2}{4(\rho+1)\eta}\right] - \left[\frac{\epsilon}{4\eta} + \left(2\eta + \frac{1}{2\eta} + \frac{(\mu_1+\mu_2)C_s^2}{4\eta}\right)(1-\beta_1) + \frac{\delta C_s^2}{4\eta}(2E(0)/m_0)^\gamma\right] > 0.$$

This applying the assumption (A3) and (83), we deduce

$$\begin{aligned}
F'(t) &\leq -a_1\|u'(t)\|_{\rho+2}^{\rho+2} - a_2\|\nabla u'(t)\|^2 - a_3\widehat{M}(\|\nabla u\|^2) - a_4\|\Delta u\|^2 \\
&\quad - a_5(h\Box\Delta u)(t) - \xi\int_\Omega\int_0^1 G(x,\varrho,t)dxd\varrho, \quad \forall t \geq t_0,
\end{aligned} \tag{84}$$

where $a_1 = \frac{h_0-\epsilon}{\rho+1}$. Then (11) and (83) imply that there exists a positive constant M such that

$$F'(t) \leq -ME(t), \quad \forall t \geq t_0. \tag{85}$$

Combining (62) and (85), we infer

$$F'(t) \leq -\frac{M}{k_1}F(t), \quad \forall t \geq t_0. \tag{86}$$

Integrating (86) over (t_0, t), it follows that

$$F(t) \leq F(t_0)e^{-\frac{M}{k_1}}, \quad \forall t \geq t_0. \tag{87}$$

Consequently, (12) can be obtained from (62) and (87). The proof is complete.

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