Existence results for nonlinear two-parametric quantum difference equation with first-order (p, q)-derivative

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Abstract. In this paper, we study the solvability of a nonlinear two-parametric quantum difference equation Dirichlet boundary value problem. At first, we provide and prove the formula of changing the order of integration for (p, q)-double integral. Second, We obtain the existence and uniqueness criteria of solutions for this kind of boundary value problem by using Banach contraction mapping principle, Leray-Schauder nonlinear alternative theorem and Leray-Schauder continuation theorem. At last, we give two examples to illustrate our results.

Keywords. Quantum calculus; Nonlinear (p, q)-difference equation; Boundary value problems; Green function; Fixed point theorem.

1 Introduction

The history of quantum calculus dates back to the early twentieth century[1−2]. As well as known, the quantum calculus is without considering limits, which can deal with sets of non-differentiable functions. With the development of q-calculus theory, the q-difference equations had achieved great achievements[3]. After decades of research, the theory of linear q-difference equations has been continuously improved[4−6]. But the application of linear q-difference equations has its own limitations, while nonlinear q-difference equations are more widely applied, such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, the theory of relativity, hypergeometric series, complex analysis, particle physics and so on. The study of nonlinear q-difference boundary value problems (BVPs) turns to the 2010[7]. However, in recent years, the research on the existence of the solution for nonlinear q-difference equation BVPs has made great progress, see [8-14] and the references therein.

The (p, q)-calculus taken as the further development of q-calculus, originated in 1990s, see [15-17]. Recently, in [18], Sadjang P N has discussed the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas. The (p, q)-difference equation appears with the appearance of (p, q)-calculus. The study of (p, q)-

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difference equation is just beginning, especially, the theory of BVPs for (p,q)-difference equations is still in the initial stages and many aspects of this theory need to be explored. To the best of our knowledge, the study of the BVPs for nonlinear (p,q)-difference equations is few, see [18-20].

In this paper, we study the existence and uniqueness of solutions for a Dirichlet BVP with nonlinear second-order (p,q)-difference equations

$$\begin{cases} D_{p,q}^2 u(t) + f(t, u(t), D_{p,q}u(t)) = 0, & t \in I_{p,q}, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $0 < \frac{p}{q} < 1$, $0 < q < 1$, $f : I \times R^2 \longrightarrow R$ continuous, $I_{p,q} = [0,1]$ and $I = [0, \frac{1}{q}]$. The main aim of this paper is to develop some existence and uniqueness results for BVP (1). Some examples and special cases are also discussed.

Throughout this paper, we assume that:

(H) For each $r > 0$, there exists $\varphi_r(t) \in L^1(I_{p,q}, R^+)$ with $t\varphi_r(t) \in L^1(I_{p,q}, R^+)$ on $I_{p,q}$ such that $max\{|u|, |v|\} \leq r$ implies $|f(t,u,v)| \leq \varphi_r(t)$, for a.e. $t \in I_{p,q}$, where $L^1(I_{p,q}, R^+) = \{u \in C_{p,q} : \int_0^1 u(t) d_{p,q}t$ exists$\}$, and normed by $\|u\|_{L^1} = \int_0^1 |u(t)| d_{p,q}t$ for all $u \in L^1(I_{p,q}, R^+)$.

2 Preliminary results

In this section, we recall some basic definitions and results, which can be find in [21].

Definition 2.1 The (p,q)-derivative of the function f is defined as

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, x \neq 0,$$

and $(D_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0.

Definition 2.2 The so-called (p,q)-bracket or twin-basic number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}.$$

Remark 2.3 Note that for $p = 1$, the (p,q)-derivative reduces to the q-derivative given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0.$$

Proposition 2.4 The (p,q)-derivative fulfils the following product rules

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}(g(x)) + g(qx)D_{p,q}(f(x)),$$

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}(f(x)) + f(qx)D_{p,q}(g(x)).$$

Proposition 2.5 The (p,q)-derivative fulfils the following product rules

$$D_{p,q}(\frac{f(x)}{g(x)}) = \frac{g(qx)D_{p,q}(f(x)) - f(qx)D_{p,q}(g(x))}{g(px)g(qx)},$$

$$D_{p,q}(\frac{f(x)}{g(x)}) = \frac{g(px)D_{p,q}(f(x)) - f(px)D_{p,q}(g(x))}{g(px)g(qx)}.$$

Proposition 2.6 Let n be an integer $n \geq 0$, then the following formula applies

$$D^n_{p,q}[\frac{1}{x}] = (-1)^n \frac{[n]_{p,q}!}{(pq)^{\binom{n+1}{2}} x^{n+1}}.$$

Definition 2.7 Let f be an arbitrary function. We define the (p, q)-integral of f as follows:

$$\int f(x)d_{p,q}x = (p-q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f(\frac{q^k}{p^{k+1}}x) \tag{2}$$

We now apply formula (2) to define the definite (p, q)-integral.

Theorem 2.8 Suppose $0 < \frac{q}{p} < 1$. If $|f(x)x^{\alpha}|$ is bounded on the interval $(0, A]$ for some $0 \leq \alpha < 1$, then the (p, q)-integral (2) converges to a function $F(x)$ on $(0, A]$, which is a (p, q)-antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$.

Definition 2.9 If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a formal power series, then among formal power series, $f(x)$ has aunique (p, q)-antiderivative up to a constant term, which is

$$\int f(x)d_{p,q}x = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{[n+1]_{p,q}} + C$$

Definition 2.10 Let f be an arbitrary function and a be a real number, we set

$$\int_0^a f(x)d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f(\frac{p^k}{q^{k+1}}a), |\frac{p}{q}| < 1,$$
$$\int_0^a f(x)d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f(\frac{q^k}{p^{k+1}}a), |\frac{p}{q}| > 1.$$

Definition 2.11 Let f be an arbitrary function, a and b are two nonnegative numbers such that $a < b$, then we get

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x.$$

Theorem 2.12 (Fundamental theorem of (p, q)-calculus) If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, we have

$$\int_a^b f(x)d_{p,q}x = F(b) - F(a),$$

where $0 \leq a \leq b \leq \infty$.

Theorem 2.13([22])(Leray-Schauder continuation theorem) Let X be a Banach space, $T : X \times [0, 1] \longrightarrow X$ be a completely continuous operator, and for any $x \in X$, there is $T(x, 0) = 0$. Suppose any solution x of $x = T(x, \sigma)$, $(x, \sigma) \in X \times [0, 1]$ satisfies the priori bound $\|x\|_X \leq M$ for some $M > 0$, and T_1 is a mapping from X to X such that $T_1 x = T(x, 1)$, then T_1 has a fixed point.

Theorem 2.14 (Leray-Schauder Nonlinear alternative for single-valued maps) Let E be a Banach's space, let C be a closed and convex subset of E, and let U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$(the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

3 Main results

In this section, we will study the existence and uniqueness of solutions for BVP (1). At first, we prove some lemmas required for the main results.

Lemma 3.1 Let function $f : I \to R$ is continuous, then

$$(i) \int_0^t \int_0^r f(s) d_{p,q}s d_{p,q}r = \frac{1}{q} \int_0^t \int_{ps}^t f(\frac{1}{q}s) d_{p,q}r d_{p,q}s, |\frac{p}{q}| < 1, \tag{3}$$

$$(ii) \int_0^t \int_0^r f(s) d_{p,q}s d_{p,q}r = \frac{1}{p} \int_0^t \int_{qs}^t f(\frac{1}{p}s) d_{p,q}r d_{p,q}s, |\frac{p}{q}| > 1. \tag{4}$$

Proof. (i) When $|\frac{p}{q}| < 1$, from definition 2.10, we can get

$$\begin{aligned}
\int_0^t \int_0^r f(s) d_{p,q}s d_{p,q}r &= \int_0^t (q-p) r \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f(\frac{p^k}{q^{k+1}} r) d_{p,q}r \\
&= \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} (q-p) \int_0^t r f(\frac{p^k}{q^{k+1}} r) d_{p,q}r \\
&= \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} (q-p) [(q-p) t \sum_{h=0}^{\infty} \frac{p^h}{q^{h+1}} \frac{p^h}{q^{h+1}} t f(\frac{p^k}{q^{k+1}} \frac{p^h}{q^{h+1}} t)] \\
&= (q-p)^2 t^2 \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{p^{k+2h}}{q^{k+2h+3}} f(\frac{p^{k+h}}{q^{k+h+2}} t) \\
&= (q-p)^2 t^2 \Big\{ \frac{1}{q^3} f(\frac{1}{q^2} t) + \frac{p}{q^5} [2]_{p,q} f(\frac{p}{q^3} t) + \frac{p^2}{q^7} [3]_{p,q} f(\frac{p^2}{q^4} t) \\
&\qquad + \frac{p^3}{q^9} [4]_{p,q} f(\frac{p^3}{q^5} t) + \cdots \Big\} \\
&= (q-p)^2 t^2 \sum_{k=0}^{\infty} \frac{p^k}{q^{2k+3}} [k+1]_{p,q} f(\frac{p^k}{q^{k+2}} t) \\
&= (q-p)^2 t^2 \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} \frac{1}{q^{k+2}} \frac{p^{k+1} - q^{k+1}}{p-q} f(\frac{p^k}{q^{k+1}} \frac{1}{q} t) \\
&= (q-p) t \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} \frac{1}{q^{k+2}} (q^{k+1} - p^{k+1}) t f(\frac{p^k}{q^{k+1}} \frac{1}{q} t) \\
&= (q-p) t \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} \frac{1}{q} (t - p \frac{p^k}{q^{k+1}} t) f(\frac{p^k}{q^{k+1}} \frac{1}{q} t) \\
&= \frac{1}{q} \int_0^t [(t - ps) f(\frac{1}{q} s)] d_{p,q}s \\
&= \frac{1}{q} \int_0^t \int_{ps}^t f(\frac{1}{q} s) d_{p,q}r d_{p,q}s.
\end{aligned}$$

(ii) When $|\frac{p}{q}| > 1$, it is similar to the proof of case (i). This completes the proof.

Lemma 3.2 Let $y \in C[0,1]$, then the BVP

$$\begin{cases} D_{p,q}^2 u(t) + y(t) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{5}$$

has a unique solution

$$u(t) = -\frac{1}{q}\int_0^t (t-ps)y(\frac{1}{q}s)d_{p,q}s + \int_0^1 \frac{t}{q}(1-ps)y(\frac{1}{q}s)d_{p,q}s$$
$$= \int_0^1 G(t,s;p,q)y(\frac{1}{q}s)d_{p,q}s, \qquad |\frac{p}{q}| < 1,$$

or

$$u(t) = -\frac{1}{p}\int_0^t (t-qs)y(\frac{1}{p}s)d_{p,q}s + \int_0^1 \frac{t}{p}(1-qs)y(\frac{1}{p}s)d_{p,q}s$$
$$= \int_0^1 G(t,s;q,p)y(\frac{1}{p}s)d_{p,q}s, \qquad |\frac{p}{q}| > 1,$$

where

$$G(t,s;m,n) = \frac{1}{n}\begin{cases} ms(1-t), s \le t, \\ t(1-ms), s \ge t \end{cases} \tag{6}$$

is called Green's fountion of (p,q)-difference boundary value problem.

Proof. (i) When $|\frac{p}{q}| < 1$, we integrate the (p,q)-difference equation from 0 to t, and get

$$D_{p,q}u(t) = \int_0^t -y(s)d_{p,q}s + a_0. \tag{7}$$

Integrating (7) from 0 to t and changing the order of integration, we have

$$u(t) = -\frac{1}{q}\int_0^t (t-ps)y(\frac{1}{q}s)d_{p,q}s + a_0t + a_1, \tag{8}$$

where a_0, a_1 are arbitrary constants.

Using the boundary conditions $u(0) = 0, u(1) = 0$, we have that

$$a_1 = 0,$$

and

$$a_0 = \frac{1}{q}\int_0^1 (1-ps)y(\frac{1}{q}s)d_{p,q}s.$$

Hence, we obtain

$$u(t) = -\frac{1}{q}\int_0^t (t-ps)y(\frac{1}{q}s)d_{p,q}s + \int_0^1 \frac{t}{q}(1-ps)y(\frac{1}{q}s)d_{p,q}s$$
$$= \int_0^1 G(t,s;p,q)y(\frac{1}{q}s)d_{p,q}s.$$

(ii) When $|\frac{p}{q}| > 1$, the proof is similar to the case (i). The proof is completed.

Remark 3.3 For $p \to 1$, it reduces to the Green's function of second-order Dirichlet q-difference boundary values problem

$$G(t,s;q) = \frac{1}{q}\begin{cases} s(1-t), s \le t, \\ t(1-s), s \ge t. \end{cases}$$

Remark 3.4 For $(p,q)\to(1,1)$, it reduces to the Green's function of classical Dirichlet boundary values problem

$$G(t,s)=\begin{cases} s(1-t), s\le t,\\ t(1-s), s\ge t.\end{cases}$$

Lemma 3.5 Let $0<\frac{p}{q}<1$, $0<q<1$ and $G(t,s;p,q)$ be Green's function given in (6). Then

$$G(t,s;p,q)\le G(s,s;p,q)\le N, \tag{9}$$

where

$$N=\frac{1}{4pq}.$$

Proof. (1) When $s\le t$, $G(t,s;p,q)=\frac{ps(1-t)}{q}$ is a decreasing function of t. Thus,

$G(t,s;p,q)\le G(s,s;p,q)=\frac{ps(1-s)}{q}\le\frac{p}{4q}$ for $0\le s,t\le 1$.

(2) when $s\ge t$, $G(t,s;p,q)=\frac{t(1-ps)}{q}$ is an increasing function of t. Thus,

$G(t,s;p,q)\le G(s,s;p,q)=\frac{s(1-ps)}{q}\le\frac{1}{4pq}$ for $0\le s,t\le 1$.

We set $N=\max\{\frac{p}{4q},\frac{1}{4pq}\}=\frac{1}{4pq}$, therefore, $G(t,s;p,q)\le G(s,s;p,q)\le N$. This completes the proof of lemma 3.5.

We consider the Banach space $C_{p,q}=C(I_{p,q},R)$ equipped with the standard norm $\|u\|=\max\{\|u\|_\infty,\|D_{p,q}u\|_\infty\}$ and $\|\cdot\|_\infty=\sup\{\|\cdot\|,t\in I\}, u\in C_{p,q}$.

Define an integral operator $T:C_{p,q}\to C_{p,q}$ by

$$Tu(t)=\int_0^1 G(t,s;p,q)f(\frac{1}{q}s,u(\frac{1}{q}s),D_{p,q}u(\frac{1}{q}s))d_{p,q}s,$$

where $t\in I_{p,q}, u\in C_{p,q}$. Obviously, T is well defined and $u\in C_{p,q}$ is a solution of BVP (1) if and only if u is a fxed point of T.

Theorem 3.6 Let $0<\frac{p}{q}<1$, $0<q<1$, $f:I\times R^2\to R$ be a continuous function, and there exist $L_1(t),L_2(t)\in C(I_{p,q},R^+)$ such that

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le L_1(t)|u_1-u_2|+L_2(t)|v_1-v_2|,\forall t\in I_{p,q},(u_1,v_1),(u_2,v_2)\in R^2.$$

In addition, suppose $\Gamma<1$ holds, where

$$\Gamma=\Lambda M_1,\quad \Lambda=\max_{t\in[0,1]}\{L_1(t)+L_2(t)\},\quad M_1=1+\frac{1}{q+p}.$$

Then BVP (1) has a unique solution.

Proof. Let us set $\sup_{t\in I}|f(t,0,0)|=M_0$, and choose $r>\frac{M_0\cdot M_1}{1-\delta}$, and $\Lambda M_1\le\delta\le 1$. Now we show that $TB_r\subset B_r$, where $B_r=\{u\in C_{p,q}:\|u\|\le r\}$. For each $u\in B_r$, we have

$$|Tu(t)|\le\sup_{t\in I_{p,q}}\left|\int_0^t\frac{1}{q}(ps-t)f(\frac{1}{q}s,u(\frac{1}{q}s),D_{p,q}u(\frac{1}{q}s))d_{p,q}s\right.$$

$$
\begin{aligned}
&\quad + \int_0^1 \frac{t}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \Big| \\
&\leq \sup_{t\in I_{p,q}} \Big| \int_0^t \frac{1}{q}(ps-t)(|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) - f(\frac{1}{q}s, 0, 0)\Big| \\
&\quad + \Big| f(\frac{1}{q}s, 0, 0)\Big|) d_{p,q}s + \int_0^1 \frac{t}{q}(1-ps)(\Big| f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) \\
&\quad - f(\frac{1}{q}s, 0, 0)\Big| + \Big| f(\frac{1}{q}s, 0, 0)\Big|) d_{p,q}s \Big| \\
&\leq \sup_{t\in I_{p,q}} \Big| \int_0^t \frac{1}{q}(ps-t)(L_1(s)\Big|u(\frac{1}{q}s)\Big| + L_2(s)\Big|D_{p,q}u(\frac{1}{q}s)\Big| + \Big|f(\frac{1}{q}s, 0, 0)\Big|) d_{p,q}s \\
&\quad + \int_0^1 \frac{t}{q}(1-ps)(L_1(s)\Big|u(\frac{1}{q}s)\Big| + L_2(s)\Big|D_{p,q}u(\frac{1}{q}s)\Big| + \Big|f(\frac{1}{q}s, 0, 0)\Big|) d_{p,q}s \Big| \\
&\leq \sup_{t\in I_{p,q}} (\Lambda\|u\| + M_0) \Big| \int_0^t \frac{1}{q}(ps-t) d_{p,q}s + \int_0^1 \frac{t}{q}(1-ps) d_{p,q}s \Big| \\
&= \sup_{t\in I_{p,q}} (\Lambda\|u\| + M_0) \Big| \frac{-t^2}{p+q} + \frac{t}{q+p} \Big| \\
&\leq (\Lambda\|u\| + M_0) M_1 \\
&\leq \Lambda M_1 r + M_0 M_1 \\
&\leq \Lambda M_1 r + (1-\delta) r = (\Lambda M_1 + 1 - \delta) r \leq r,
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_{p,q}Tu(t)| \\
&\leq \sup_{t\in I_{p,q}} \Big| \int_0^t -f(s, u(s), D_{p,q}u(s)) d_{p,q}s + \int_0^1 \frac{1}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \Big| \\
&\leq \sup_{t\in I_{p,q}} \Big| \int_0^t (\Big| f(s, u(s), D_{p,q}u(s)) - f(s, 0, 0)\Big| + \Big| f(s, 0, 0)\Big|) d_{p,q}s \\
&\quad + \int_0^1 \frac{1}{q}(1-ps)(\Big| f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) - f(\frac{1}{q}s, 0, 0)\Big| + \Big| f(\frac{1}{q}s, 0, 0)\Big|) d_{p,q}s \Big| \\
&\leq \sup_{t\in I_{p,q}} \Big| \int_0^t (L_1(s)\Big|u(s)\Big| + L_2(s)\Big|D_{p,q}u(s)\Big| + \Big|f(s, 0, 0)\Big|) d_{p,q}s \\
&\quad + \int_0^1 \frac{1}{q}(1-ps)(L_1(s)\Big|u(\frac{1}{q}s)| + L_2(s)\Big|D_{p,q}u(\frac{1}{q}s)\Big| + \Big|f(\frac{1}{q}s, 0, 0)\Big|) d_{p,q}s \Big| \\
&\leq \sup_{t\in I_{p,q}} (\Lambda\|u\| + M_0) \Big| t + \int_0^1 \frac{1}{q}(1-ps) d_{p,q}s \Big| \\
&= \sup_{t\in I_{p,q}} (\Lambda\|u\| + M_0) |t + \frac{1}{p+q}| \\
&\leq (\Lambda\|u\| + M_0)(1 + \frac{1}{p+q}) = (\Lambda\|u\| + M_0) M_1 \\
&\leq \Lambda M_1 r + (1-\delta) r \\
&= (\Lambda M_1 + 1 - \delta) r \leq r.
\end{aligned}
$$

Hence, we obtain that $\|Tu\| \leq r$. So $TB_r \subset B_r$.

Now, for $u, v \in C_{p,q}$ and for each $t \in I_{p,q}$, we have

$$
|Tu(t) - Tv(t)|
$$

$$
\begin{aligned}
&\le \sup_{t\in I_{p,q}} \Big| \int_0^t \frac{1}{q}(ps-t)\Big|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) - f(\frac{1}{q}s, v(\frac{1}{q}s), D_{p,q}v(\frac{1}{q}s))\Big| d_{p,q}s \\
&\qquad + \int_0^1 \frac{t}{q}(1-ps)\Big|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) - f(\frac{1}{q}s, v(\frac{1}{q}s), D_{p,q}v(\frac{1}{q}s))\Big| d_{p,q}s\Big| \\
&\le \sup_{t\in I_{p,q}} \Big| \int_0^t \frac{1}{q}(ps-t)(L_1(s)|u(\frac{1}{q}s) - v(\frac{1}{q}s)\Big| + L_2(s)\Big|D_{p,q}u(\frac{1}{q}s) - D_{p,q}v(\frac{1}{q}s)\Big|)d_{p,q}s \\
&\qquad + \int_0^1 \frac{t}{q}(1-ps)(L_1(s)\Big|u(\frac{1}{q}s) - v(\frac{1}{q}s)\Big| + L_2(s)\Big|D_{p,q}u(\frac{1}{q}s) - D_{p,q}v(\frac{1}{q}s)\Big|)d_{p,q}s\Big| \\
&\le \sup_{t\in I_{p,q}} \Lambda\|u-v\|\Big| \int_0^t \frac{1}{q}(ps-t)d_{p,q}s + \int_0^1 \frac{t}{q}(1-ps)d_{p,q}s\Big| \\
&= \sup_{t\in I_{p,q}} \Lambda\|u-v\|\Big|\frac{-t^2}{p+q} + \frac{t}{q+p}\Big| \\
&\le \Lambda\|u-v\|M_1 = \Gamma\|u-v\| < \|u-v\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_{p,q}Tu(t) - D_{p,q}Tv(t)| \\
&\le \sup_{t\in I_{p,q}} \Big| \int_0^t \Big|f(s, u(s), D_{p,q}u(s)) - f(s, v(s), D_{p,q}v(s))\Big| d_{p,q}s \\
&\qquad + \int_0^1 \frac{1}{q}(1-ps)\Big|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) - f(\frac{1}{q}s, v(\frac{1}{q}s), D_{p,q}v(\frac{1}{q}s))\Big| d_{p,q}s\Big| \\
&\le \sup_{t\in I_{p,q}} \Big| \int_0^t (L_1(s)\Big|u(s) - v(s)\Big| + L_2(s)\Big|D_{p,q}u(s) - D_{p,q}v(s)\Big|)d_{p,q}s \\
&\qquad + \int_0^1 \frac{1}{q}(1-ps)(L_1(s)\Big|u(\frac{1}{q}s) - v(\frac{1}{q}s)\Big| + L_2(s)\Big|D_{p,q}u(\frac{1}{q}s) - D_{p,q}v(\frac{1}{q}s)\Big|)d_{p,q}s\Big| \\
&\le \sup_{t\in I_{p,q}} \Big| \int_0^t (L_1(s) + L_2(s))d_{p,q}s + \int_0^1 \frac{1}{q}(1-ps)(L_1(s)+L_2(s))d_{p,q}s\Big| \|u-v\| \\
&\le \sup_{t\in I_{p,q}} \Lambda\|u-v\|\Big|t + \int_0^1 \frac{1}{q}(1-ps)d_{p,q}s\Big| \\
&\le \sup_{t\in I_{p,q}} \Lambda\|u-v\|\Big|t + \frac{1}{p+q}\Big| \\
&\le \Lambda\|u-v\|(1 + \frac{1}{p+q}) \\
&\le \Lambda\|u-v\|M_1 = \Gamma\|u-v\| < \|u-v\|.
\end{aligned}
$$

Therefore, we obtain that $\|Tu - Tv\| < \|u - v\|$, so T is a contraction. Then existence of at least one fixed point for T follows by Banach's contraction mapping principle. This completes the proof.

Corollary 3.7 Assume that $0 < \frac{p}{q} < 1$, $0 < q < 1$, $f : I \times R^2 \to R$ is a continuous function, and there exist two positive constants L_1, L_2 such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le L_1|u_1 - u_2| + L_2|v_1 - v_2|, \forall t \in I_{p,q}, (u_1, v_1), (u_2, v_2) \in R^2.$$

In addition, suppose $\Gamma_1 < 1$ holds, where

$$\Gamma_1 = (L_1 + L_2)M_1, \quad M_1 = 1 + \frac{1}{q+p}.$$

Then BVP (1) has a unique solution.

Corollary 3.8 Assume that $0 < \frac{p}{q} < 1$, $0 < q < 1$, $f : I \times R^2 \to R$ is a continuous function, and there exist $L_1(t), L_2(t) \in L^1(I_{p,q}, R^+)$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le L_1(t)|u_1 - u_2| + L_2(t)|v_1 - v_2|, \forall t \in I_{p,q}, (u_1, v_1), (u_2, v_2) \in R^2.$$

In addition, suppose $A < 1$ holds, where

$$A = \int_0^1 (\frac{1}{q} + 1 - \frac{p}{q}s)(L_1(s) + L_2(s))d_{p,q}s.$$

Then BVP (1) has a unique solution.

The next existence result is based on the Leray-Schauder nonlinear alternative theorem.

Lemma 3.9 Assume that $0 < \frac{p}{q} < 1$, $0 < q < 1$, and (H) holds. Then $T : C_{p,q} \to C_{p,q}$ is completely continuous.

Proof. The proof consists of several steps.

(i) T maps bounded sets into bounded sets in $C_{p,q}$.

Let $B_r = \{u \in C_{p,q} : \|u\| \le r\}$ be a bounded set in $C_{p,q}$ and $u \in B_r$. Then we have

$$\begin{aligned}
|Tu(t)| &\le \int_0^t \left|\frac{1}{q}(ps - t)\right|\left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&\quad + \int_0^1 \left|\frac{t}{q}(1 - ps)\right|\left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&= \int_0^t \frac{1}{q}(t - ps)\left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&\quad + \int_0^1 \frac{t}{q}(1 - ps)\left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&\le \int_0^1 [\frac{1}{q}(1 - ps) + \frac{1}{q}(1 - ps)]\left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&\le \frac{2}{q}\int_0^1 \left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&\le \frac{2}{q}\int_0^1 \varphi_r(s)d_{p,q}s = \frac{2}{q}\|\varphi_r\|_{L^1},
\end{aligned}$$

and

$$\begin{aligned}
|D_{p,q}Tu(t)| &\le \int_0^t \left|f(s, u(s), D_{p,q}u(s))\right|d_{p,q}s + \int_0^1 \left|\frac{1}{q}(1 - ps)\right|\left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&\le \int_0^1 \left|f(s, u(s), D_{p,q}u(s))\right|d_{p,q}s + \int_0^1 \left|\frac{1}{q}(1 - ps)\right|\left|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\right|d_{p,q}s \\
&\le \int_0^1 \varphi_r(s)d_{p,q}s + \frac{1}{q}\int_0^1 \varphi_r(\frac{1}{q}s)d_{p,q}s \\
&= \|\varphi_r\|_{L^1} + \frac{1}{q}\|\varphi_r\|_{L^1} = (1 + \frac{1}{q})\|\varphi_r\|_{L^1} \le \frac{2}{q}\|\varphi_r\|_{L^1}.
\end{aligned}$$

Thus, $\|Tu\| = \max\{\|Tu\|_\infty, \|D_{p,q}Tu\|_\infty\} \le \frac{2}{q}\|\varphi_r\|_{L^1}$.

(ii) T maps bounded sets into equicontinuous sets of $C_{p,q}$.

Let $r_1, r_2 \in I_{p,q}, r_1 < r_2$, and let B_r be a bounded set of $C_{p,q}$ as before, then for each $u \in B_r$, we have

$$\begin{aligned}
|Tu(r_2) - Tu(r_1)| = \Big| &\frac{1}{q}\int_0^{r_2} (ps - r_2) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \\
&+ \int_0^1 \frac{r_2}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \\
&- \frac{1}{q}\int_0^{r_1} (ps - r_1) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \\
&- \int_0^1 \frac{r_1}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \Big|
\end{aligned}$$

$$\begin{aligned}
= \Big| &\frac{1}{q}\int_0^{r_2} (ps - r_2) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s - \frac{1}{q}\int_0^{r_1} (ps - r_1) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \\
&+ \int_0^1 \frac{1}{q}(1-ps)(r_2 - r_1) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \Big| \\
= \Big| &- \frac{1}{q}[\int_0^{r_1} (r_2 - r_1) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s + \int_{r_1}^{r_2} (ps - r_2) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s] \\
&+ \int_0^1 \frac{1}{q}(1-ps)(r_2 - r_1) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \Big| \\
\le &\frac{1}{q}\int_0^{r_1} |r_2 - r_1| \varphi_r(s) d_{p,q}s + \int_{r_1}^{r_2} |ps - r_2| \varphi_r(s) d_{p,q}s + \int_0^1 \frac{1}{q}(1-ps)|r_2 - r_1| \varphi_r(s) d_{p,q}s \to 0, \\
&((r_2 - r_1) \to 0),
\end{aligned}$$

and

$$\begin{aligned}
|D_{p,q}Tu(r_2) - D_{p,q}Tu(r_1)| &= \Big| \int_0^{r_2} -f(s, u(s), D_{p,q}u(s)) d_{p,q}s \\
&\quad + \int_0^1 \frac{1}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \\
&\quad - \int_0^{r_1} -f(s, u(s), D_{p,q}u(s)) d_{p,q}s \\
&\quad - \int_0^1 \frac{1}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \Big| \\
&= | - \int_{r_1}^{r_2} f(s, u(s), D_{p,q}u(s)) d_{p,q}s | \\
&\le \int_{r_1}^{r_2} \varphi_r(s) d_{p,q}s \to 0, \qquad ((r_2 - r_1) \to 0).
\end{aligned}$$

As a consequence of the Arzela-Ascoli theorem, we can conclude that $T : C_{p,q} \to C_{p,q}$ is completely continuous. This proof is completed.

Theorem 3.10 If the assumption (H) holds and there exists a real number $M > 0$ such that

$$\frac{qM}{2\|\varphi_r\|_{L^1}} > 1,$$

then BVP (1) has at least an solution, where $\|\varphi_r\|_{L^1} = \int_0^1 \varphi_r(s) d_{p,q}s \neq 0$.

Proof. In view of lemma 3.9, we obtain that $T : C_{p,q} \to C_{p,q}$ is completely continuous. Let $\lambda \in (0, 1)$ and $u = \lambda Tu$, then for each $t \in I_{p,q}$, we have

$$|u(t)| = |\lambda Tu(t)|$$

$$
\begin{aligned}
&\le \Big| \int_0^t \frac{1}{q}(ps-t) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s + \int_0^1 \frac{t}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) d_{p,q}s \Big| \\
&\le \int_0^t \Big|\frac{1}{q}(ps-t)\Big| \Big| f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) | d_{p,q}s + \int_0^1 |\frac{t}{q}(1-ps)| | f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) \Big| d_{p,q}s \\
&= \int_0^t \frac{1}{q}(t-ps) \Big| f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) \Big| d_{p,q}s + \int_0^1 \frac{t}{q}(1-ps) \Big| f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) \Big| d_{p,q}s \\
&\le \frac{2}{q} \int_0^1 \Big| f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) \Big| d_{p,q}s \\
&\le \frac{2}{q} \int_0^1 \varphi_r(s) d_{p,q}s = \frac{2}{q} \|\varphi_r\|_{L^1},
\end{aligned}
$$

and

$$
\begin{aligned}
|D_{p,q}u(t)| &= |D_{p,q}\lambda Tu(t)| \\
&\le \int_0^t \Big| f(s, u(s), D_{p,q}u(s)) \Big| d_{p,q}s + \int_0^1 \Big|\frac{1}{q}(1-ps)\Big| \Big| f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) \Big| d_{p,q}s \\
&\le (1+\frac{1}{q}) \|\varphi_r\|_{L^1} \le \frac{2}{q} \|\varphi_r\|_{L^1}.
\end{aligned}
$$

Hence,

$$
\frac{q\|u\|}{2\|\varphi_r\|_{L^1}} \le 1.
$$

Therefore, there exists $M > 0$ such that $\|u\| \ne M$. Let us set $U = u \in C_{p,q} : \|u\| < M$. Note that the operator $T : \overline{U} \to C_{p,q}$ is completely continuous. From the choice of U, there is no $u \in \overline{U}$ which is a solution of problem (1). This completes the proof.

The next existence result is based on the Leray-Schauder continuation theorem.

Theorem 3.11 Let $f : I \times R^2 \to R$ be a continuous function that satisfies the assumption (H). Suppose futher that there exist functions $p(t), q(t), r(t) \in L^1(I_{p,q}, R^+)$ with $tp(t) \in L^1(I_{p,q}, R^+)$ such that

$|f(t,u,v)| \le p(t)|u| + q(t)|v| + r(t)$ for a.e. $t \in I_{p,q}$ and $(u,v) \in R^2$.

Then BVP (1) has at least one solution provided

$$
\max\{(1+\frac{1}{q})(P + P_1 + Q), NP\} < 1,
$$

where

$$
P = \int_0^1 p(s) d_{p,q}s, \quad P_1 = \int_0^1 sp(s) d_{p,q}s, \quad Q = \int_0^1 q(s) d_{p,q}s, \quad R = \int_0^1 r(s) d_{p,q}s.
$$

Proof. We consider the space $P = \{u \in C_{p,q} : u(0) = 0, u(1) = 0\}$ and define the operator $T_1 : P \times [0,1] \longrightarrow P$ by $T_1(u,\lambda) = \lambda Tu = \lambda \int_0^1 G(t,s,p,q) f(s, u(s), D_{p,q}u(s)) d_{p,q}s$. Obviously, we can see that $P \subset C_{p,q}$. It is easy to know that for each $\lambda \in [0,1]$, $T_1(u,\lambda)$ is completely continuous in P. It is clear that $u \in P$ is a solution of BVP(1), if and only if u is a fixed point of $T_1(\cdot,1)$. Clearly, $T_1(u,0) = 0$ for each $u \in P$. If for each $\lambda \in [0,1]$, the fixed points of $T_1(\cdot,1)$ in P belongs to a closed ball of P independent of λ, then the leray-Schauder continuation theorem completes the proof. Next, we show that the fixed point of $T_1(\cdot,1)$ has a priori bound M, which is independent of λ.

Assume that $u = T_1(u, \lambda)$. It is clear that $|G(t, s, p, q)| \le N$. For any $u \in P$, we have

$$|u(t)| = |u(1) - \int_t^1 D_{p,q}u(s)d_{p,q}s| \le |\int_t^1 D_{p,q}u(s)d_{p,q}s| \le (1-t)\|D_{p,q}u\|_\infty \le (1+t)\|D_{p,q}u\|_\infty,$$

and so it holds that

$$\begin{aligned}
|D_{p,q}Tu| &= |\lambda D_{p,q}Tu| \\
&\le \int_0^1 \Big| f(s, u(s), D_{p,q}u(s)) \Big| d_{p,q}s + \int_0^1 \Big| \frac{1}{q}(1-ps) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s)) \Big| d_{p,q}s \\
&\le \int_0^1 [[p(s)|u(s)| + q(s)|D_{p,q}u(s)| + r(s)]d_{p,q}s \\
&\quad + \frac{1}{q}\int_0^1 [[p(s)|u(\frac{1}{q}s)| + q(s)|D_{p,q}u(\frac{1}{q}s)| + r(s)]d_{p,q}s \\
&\le \|D_{p,q}u\|_\infty \int_0^1 p(s)(1+s)d_{p,q}s + \|D_{p,q}u\|_\infty \int_0^1 q(s)d_{p,q}s + R \\
&\quad + \|D_{p,q}u\|_\infty \int_0^1 \frac{1}{q}p(s)(1+s)d_{p,q}s + \|D_{p,q}u\|_\infty \int_0^1 \frac{1}{q}q(s)d_{p,q}s + \frac{1}{q}R \\
&\le (1+\frac{1}{q})(P + P_1 + Q)\|D_{p,q}u\|_\infty + (1+\frac{1}{q})R.
\end{aligned}$$

Therefore,

$$\|D_{p,q}u\|_\infty \le \frac{(1+\frac{1}{q})R}{1-(1+\frac{1}{q})(P+P_1+Q)} := M_1.$$

At the same time, we have

$$\begin{aligned}
\|u\|_\infty &\le \|\lambda G(t, s, p, q) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\|_{L^1} \\
&\le \|G(t, s, p, q) f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\|_{L^1} \\
&\le N\|f(\frac{1}{q}s, u(\frac{1}{q}s), D_{p,q}u(\frac{1}{q}s))\|_{L^1} \\
&\le N\|p(s)|u| + q(s)|D_{p,q}u| + r(s)\|_{L^1} \\
&\le NP\|u\|_\infty + NQ\|D_{p,q}u\|_\infty + NR,
\end{aligned}$$

and also

$$\|u\|_\infty \le \frac{NQM_1 + NR}{1-NP} := M_2.$$

Set $M = \max\{M_1, M_2\}$, which is independent of λ. Therefore, BVP (1) has least one solution. This completes the proof.

Remark 3.12 Using the same method, we can obtain the existence and uniqueness theorems of solutions for BVP (1), when $-1 < \frac{p}{q} < 0$ and $|\frac{p}{q}| > 1$.

4 Application

In this section, we give two examples to illustrate our main results.

Example 4.1 Consider the following BVP:

$$\begin{cases} (D_{p,q}^2 u(t) + t + \frac{1}{8}\sin(u(t)) + \frac{1}{7}\arctan(D_{p,q}u(t)) = 0, & t \in I_{p,q}, \\ u(0) = u(1) = 0, \end{cases} \tag{10}$$

Here, $f(t, u(t), D_{p,q}u(t)) = t + \frac{1}{8}\sin(u(t)) + \frac{1}{7}\arctan(D_{p,q}u(t))$, $p = \frac{1}{4}, q = \frac{1}{2}$. Clearly, $|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{8}|u_1 - u_2| + \frac{1}{7}|v_1 - v_2|$. Then $L_1 = \frac{1}{8}, L_2 = \frac{1}{7}$ and $(L_1 + L_2) \cdot M_1 = \frac{5}{8} < 1$. Therefore, by Corollary 3.7, we obtain that BVP (10) has a unique solution.

Example 4.2 Consider the following BVP:

$$\begin{cases} (D_{p,q}^2 u(t) + t + 3\sin(u(t)) + \frac{1}{3}\sin(D_{p,q}u(t)) = 0, & t \in I_{p,q}, \\ u(0) = u(1) = 0, \end{cases} \tag{11}$$

Here, $p = \frac{1}{4}, q = \frac{1}{2}, M = 56$. It is obvious that$|f(t, u, v)| \leq t + \frac{10}{3}$, where $\varphi_r(t) = t + \frac{10}{3}$. Then $\int_0^1 \varphi_r(t) d_{p,q}t = \frac{14}{3}$, so$\frac{qM}{2\|\varphi_r\|_{L^1}} = 3 > 1$. By Theorem 3.10, we obtain that BVP (11) has at least an solution.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

The author contributed to each part of this work equally and read and approved the final version of the manuscript.

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