Nonexistence of Global Solutions of Systems of Time Fractional Differential equations posed on the Heisenberg group

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Abstract

We first consider the nonlinear time fractional diffusion equation

$$\mathbb{D}_{0|t}^{1+\alpha}u + \mathbb{D}_{0|t}^{\beta}u - \Delta_{\mathbb{H}}u = |u|^p$$

posed on the Heisenberg group $\mathbb{H}$, where $1 < p$ is a positive real nimber to be specified later; $\mathbb{D}_{0|t}^{\delta}$ is the Liouville-Caputo derivative of order δ. For $0 < \alpha < 1, 0 < \beta \leq 1$. This equation interpolates the heat equation and the wave equation with the linear damping $\mathbb{D}_{0|t}^{\beta}u$. We present the Fujita exponent for blow-up. Then establish sufficient conditions ensuring non-existence of local solutions. We extend the analysis to the case of the system

$$\mathbb{D}_{0|t}^{1+\alpha}u + \mathbb{D}_{0|t}^{\beta}u - \Delta_{\mathbb{H}}u = |v|^q$$

$$\mathbb{D}_{0|t}^{1+\delta}v + \mathbb{D}_{0|t}^{\gamma}v - \Delta_{\mathbb{H}}v = |u|^p.$$

Our method of proof is based on the nonlinear capacity method.

Keywords. Nonlinear fractional differential equations, Heisenberg's group, Fujita's exponent

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1 Introduction

This paper is devoted to proving nonexistence results first for the equation

$$\mathbb{D}_{0|t}^{1+\alpha}u + \mathbb{D}_{0|t}^{\beta}u - \Delta_{\mathbb{H}}u = |u|^p \tag{1}$$

posed on the Heisenberg group $\mathbb{H}$, supplemented with the initial data

$$u(\eta, 0) = u_0(\eta), \qquad u_t(\eta, 0) = u_1(\eta). \tag{2}$$

Then we consider the system

$$\mathbb{D}_{0|t}^{1+\alpha}u + \mathbb{D}_{0|t}^{\beta}u - \Delta_{\mathbb{H}}u = |v|^q, \tag{3}$$

$$\mathbb{D}_{0|t}^{1+\delta}v + \mathbb{D}_{0|t}^{\gamma}v - \Delta_{\mathbb{H}}v = |u|^p, \tag{4}$$

under the initial data

$$u(\eta, 0) = u_0(\eta),\ u_t(\eta, 0) = u_1(\eta),\ v(\eta, 0) = v_0(\eta),\ \text{ and }\ v_t(\eta, 0) = v_1(\eta). \tag{5}$$

Where $0 < \alpha, \gamma, \beta, \delta < 1$ and $p, q > 1$. We will present a "threshold" exponent depending on the data.

Let us mention from the beginning that, in practice, the exponents α, β in (1) are in general of the form $\alpha = 1 \pm \varepsilon_1, \beta = 1 \pm \varepsilon_2$ with $0 < \varepsilon_1, \varepsilon_2$ small; here, we consider the case where $\alpha = 1 - \varepsilon_1, \beta = 1 - \varepsilon_2$. Of course a comparaison of our results with those of the wave equation with fractional damping corresponding to $\alpha = \beta = 1$ will be of great interest and will shed light on the modeling.

Before describing our results in details, let us dwell on existing references on the subject. It took more than twenty years and great efforts of many researchers to obtaining the critical exponent for the wave equation in the case the problem is posed on the euclidean space; the final result for any dimension of the space is due to Yordanov and Zhang [16]. The Fujita exponent for the wave equation with linear damping has been obtained by Zhang [17] and Kirane and Qafsaoui [9].

For the wave equation posed on the Heisenberg group, very few articles appeared tille now. To describe them, let us precise the framework [13]: Let $\eta = (x, y, \tau) = (x_1, ..., x_N, y_1, ..., y_N, \tau) \in \mathbb{R}^{2N+1}$ with $N \geq 1$. The Heisenberg group $\mathbb{H}$, whose points are denoted by η, is the set $\mathbb{R}^{2N+1}$ endowed with the group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(< x, \tilde{y} > - < \tilde{x}, y >)),$$

where $< ., . >$ is the usual inner product in $\mathbb{R}^N$. The laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is defined, via the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} + 2x_i\frac{\partial}{\partial \tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N}(X_i^2 + Y_i^2),$$

which is explicitly

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

The homogeneous dimension of $\mathbb{H}$ is equal to $Q = 2N + 2$.
Pohozaev and Véron [13] obtained, among other results, a nonexistence result for the equation

$$u_{tt} - \Delta_{\mathbb{H}}(au) \geq |u|^p.$$

Concerning local existence and global existence results for small initial data, the results obtained by Zuily [18] are worth to be mentioned.
For the following problem

$$u_{tt} - \sum_{j=1}^{2N} X_j^2 u + \sum_{j,k=1}^{2N} X_j^2 \gamma_{jk}(u, Xu) X_j X_k u + \lambda u_t + \mu u = F(u, Xu), \quad t > 0, \tag{6}$$

subject to the initial conditions

$$u|_{t=0} = \varepsilon u_0, \qquad u_t|_{t=0} = \varepsilon u_1, \tag{7}$$

where the nonlinearities $F(u, Xu)$ and $\gamma_{jk}(u, Xu)$ are C^∞ functions of their arguments $u, X = (\partial_t, X_1, \ldots \cdots X_{2N}), X^2 u = (X_i X_j u)_{0 \leq i,j \leq 2n}, \lambda > 0, \mu \geq 0$; the functions F and γ_{jk} satisfy, for $1 \leq j, k \leq 2N$:

$$|\gamma_{jk}(u, \xi)| \leq C(\mu |u| + |\xi|), \quad |F(u, \xi)| \leq C(\mu |u|^2 + |\xi|^2),$$

for $|u| + |\xi| \leq 1$, he obtained the following results:

- If $u_0, u_1 \in C^\infty$, then there exist a finite time $T > 0$ such that problem (6)-(7) admits a unique solution $u \in C^\infty([0, T] \times \mathbb{R}^{2n+1})$ (of course without the restriction on F and γ_{jk})
- . If $u_0, u_1 \in C^\infty$ and the restrictions on F and γ_{jk} here above are assumed, then there exists $\varepsilon_0 > 0$ but small enough such that, if $\varepsilon \leq \varepsilon_0$, the problem (6)-(7) admits a unique global solution $u \in C^\infty(\mathbb{R}^{2n+1} \times \mathbb{R}^+)$.

Of course, following the argument in [18], one can obtain local and global (though for small initial data) existence results for the equations considered in this paper modulo some modifications inherent to the fractional in time setting.

More recently, Georgiev and Palmieri [6] treated the problem

$$u_{tt} - \Delta_{\mathbb{H}} u + u_t = |u|^p, \quad p > 1, \, t > 0,$$

with given initial data

$$u|_{t=0} = \varepsilon u_0, \qquad u_t|_{t=0} = \varepsilon u_1.$$

They find, as expected, that the Fujita exponent is $p_{Fuj} = 1 + \frac{2}{Q}$. Let us mention here in passing that the fact that solutions blow-up for $1 < p \leq p_{Fuj}$ has already been decided by Kirane et al. [3]. The new points in [6] is the local existence result and the global existence result for suitable initial data when $p > p_{Fuj}$.
An other remark is that in Theorem 2.3 in [6] the condition

$$\lim_{R\to\infty} \int_{D_R} (u_0(\eta) + u_1(\eta))\, d\eta > 0$$

can be relaxed into

$$\lim_{R\to\infty} \int_{D_R} u_0(\eta)\, d\eta > 0$$

by taking $\beta\left(\frac{t^2}{R^2}\right)$ instead of $\beta\left(\frac{t}{R^2}\right)$ in the proof as it was noticed by Pohozaev and Véron [13].

2 Preliminaries

In this section, we present some preliminaries that will be used in the sequel and we announce the main results. At first, let us recall some definitions and properties concerning fractional integrals and derivatives.

Let $n \in \mathbb{N}$ and $\theta \in (n-1, n)$. For a function f belongs to $C^n([0,T])$, the left –handed Liouville-Caputo derivative is given by

$$\left(\mathbb{D}^{\theta}_{0|t} f\right)(t) = \mathbb{I}^{n-\theta}_{0|t}(f^{(n)})(t).$$

Where, for all $f \in L^q(0,T)$, $1 \leq q \leq \infty$ and $\alpha \in (0,1)$ the left–handed and right–handed fractional integral of order α are the following

$$\left(\mathbb{I}^{\alpha}_{0|t} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \text{ and } \quad \left(\mathbb{I}^{\alpha}_{t|T} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)\, ds.$$

Lemma 2.1. *Let f, $g \in C([0,T])$, we have the formula of integration by parts (see[14])*

$$\int_0^T \left(\mathbb{I}^{\alpha}_{0|t} f\right)(t) g(t)\, dt = \int_0^T f(t) \left(\mathbb{I}^{\alpha}_{t|T} g\right)(t)\, dt.$$

Lemma 2.2. *For $t \geq 0$, $T > 0$ and $\sigma \gg 1$, let*

$$f(t) = \left(1 - \frac{t}{T}\right)_+^{\sigma}.$$

Then for all $\alpha \in (0,1)$ and $k \in \{0,1,2\}$, we have

$$\left(\mathbb{I}_{t|T}^{1-\alpha} f\right)^{(k)}(t) = \frac{(-1)^k \Gamma(\sigma+1)}{\Gamma(2+\sigma-\alpha-k)} T^{1-\alpha-k} \left(1-\frac{t}{T}\right)_+^{1+\sigma-\alpha-k}$$

This yields

$$\mathbb{I}_{t|T}^{1-\alpha} f(T) = (\mathbb{I}_{t|T}^{1-\alpha} f)_t(T) = 0,$$

and

$$\mathbb{I}_{t|T}^{1-\alpha} f(0) = \frac{\Gamma(\sigma+1)}{\Gamma(2+\sigma-\alpha)} T^{1-\alpha}, \text{ and } (\mathbb{I}_{t|T}^{1-\alpha} f)_t(0) = -\frac{\Gamma(\sigma+1)}{\Gamma(1+\sigma-\alpha)} T^{-\alpha}.$$

3 Results

First let

$$Q_T = \mathbb{H} \times [0,T].$$

Then, for the case of problem(1)-(2), we start with

Definition 3.1. *A weak solution of problem (1)-(2) is a function $u \in C([0,T]; L^p_{loc}(\mathbb{R}^{2N+1}) \cap C([0,T]; L^1_{loc}(\mathbb{R}^{2N+1})$ such that*

$$\int_{Q_T} \varphi |u|^p + \int_{\mathbb{H}} u_0(\eta)\Big(\mathbb{I}_{t|T}^{1-\beta}\varphi - (\mathbb{I}_{t|T}^{1-\alpha}\varphi)_t\Big)(\eta,0)\,d\eta + \int_{\mathbb{H}} u_1(\eta)(\mathbb{I}_{t|T}^{1-\alpha}\varphi)(\eta,0)\,d\eta$$
$$= \int_{Q_T} u(\mathbb{I}_{t|T}^{1-\alpha}\varphi)_{tt} - \int_{Q_T} u(\mathbb{I}_{t|T}^{1-\beta}\varphi)_t - \int_{Q_T} u\,\Delta_{\mathbb{H}}\varphi, \tag{8}$$

for any nonnegative function $\varphi \in C^2(Q_T)$, such that

$$\mathbb{I}_{t|T}^{1-\alpha}\varphi(\eta,T) = \mathbb{I}_{t|T}^{1-\beta}\varphi(\eta,T) = (\mathbb{I}_{t|T}^{1-\alpha}\varphi)_t(\eta,T) = 0.$$

Theorem 3.2. *Let $1 < p$. Assume that $\alpha, \beta \in (0,1)$ and $u_0, u_1 \in L^1(\mathbb{H})$. If*

$$\limsup_{T\to+\infty} T^{(\frac{1}{p-1}-\frac{Q}{2})\beta} \int_{\mathbb{H}} \Big(u_0(\eta) + T^{\beta-\alpha}u_1(\eta)\Big)\,d\eta = +\infty, \tag{9}$$

then problem (1)-(2) does not admit global in time solutions.

Corollary 3.3. *Let $1 < p$. Assume that $\alpha, \beta \in (0,1)$ and $u_0, u_1 \in L^1(\mathbb{H})$. Then, in the all following situation, problem (1)-(2) does not admit global in time solutions.*

(i) $0 < \beta < \alpha$, $\int_{\mathbb{H}} u_0(\eta)\,d\eta > 0$, *and* $1 < p < 1 + \frac{2}{Q}$.

(ii) $0 < \alpha < \beta$, $\int_{\mathbb{H}} u_1(\eta)\,d\eta > 0$, *and* $1 < p < 1 + \frac{2\beta}{\beta(Q-2)+2\alpha}$.

(iii) $\alpha = \beta$, $\int_{\mathbb{H}}(u_0(\eta) + u_1(\eta))\,d\eta > 0$, *and* $1 < p < 1 + \frac{2}{Q}$.

(iv) $\int_{\mathbb{H}} u_0(\eta)d\eta > 0$, $\int_{\mathbb{H}} u_1(\eta)\,d\eta > 0$, *and* $1 < p < 1 + \frac{2\beta}{\beta Q - 2\max\{0,\beta-\alpha\}}$.

Now for the problem(3)-(4)-(5), the weak solution is defined as follow

Definition 3.4. *A weak solution of system* (3)-(4)-(5) *is a couple of functions*

$u \in C([0,T];L^p_{loc}(\mathbb{R}^{2N+1})) \cap C([0,T];L^1_{loc}(\mathbb{R}^{2N+1})), v \in C([0,T];L^q_{loc}(\mathbb{R}^{2N+1})) \cap C([0,T];L^1_{loc}(\mathbb{R}^{2N+1})),$

such that

$$\int_{Q_T} \varphi|v|^q + \int_{\mathbb{H}} u_0(\eta)\Big(\mathbb{I}^{1-\beta}_{t|T}\varphi - (\mathbb{I}^{1-\alpha}_{t|T}\varphi)_t\Big)(\eta,0)\,d\eta + \int_{\mathbb{H}} u_1(\eta)(\mathbb{I}^{1-\alpha}_{t|T}\varphi)(\eta,0)\,d\eta$$
$$= \int_{Q_T} u(\mathbb{I}^{1-\alpha}_{t|T}\varphi)_{tt} - \int_{Q_T} u(\mathbb{I}^{1-\beta}_{t|T}\varphi)_t - \int_{Q_T} u\,\Delta_{\mathbb{H}}\varphi \tag{10}$$

$$\int_{Q_T} \varphi|u|^p + \int_{\mathbb{H}} v_0(\eta)\Big(\mathbb{I}^{1-\gamma}_{t|T}\varphi - (\mathbb{I}^{1-\delta}_{t|T}\varphi)_t\Big)(\eta,0)\,d\eta + \int_{\mathbb{H}} v_1(\eta)(\mathbb{I}^{1-\delta}_{t|T}\varphi)(\eta,0)\,d\eta$$
$$= \int_{Q_T} v(\mathbb{I}^{1-\delta}_{t|T}\varphi)_{tt} - \int_{Q_T} v(\mathbb{I}^{1-\gamma}_{t|T}\varphi)_t - \int_{Q_T} v\,\Delta_{\mathbb{H}}\varphi, \tag{11}$$

for any nonnegative function $\varphi \in C^2(Q_T)$*, such that at*

$\mathbb{I}^{1-\alpha}_{t|T}\varphi(\eta,T) = \mathbb{I}^{1-\beta}_{t|T}\varphi(\eta,T) = (\mathbb{I}^{1-\alpha}_{t|T}\varphi)_t(\eta,T) = \mathbb{I}^{1-\delta}_{t|T}\varphi(\eta,T) = \mathbb{I}^{1-\gamma}_{t|T}\varphi(\eta,T) = (\mathbb{I}^{1-\alpha}_{t|T}\delta)_t(\eta,T) = 0.$

Theorem 3.5. *Let* $p,q>1$*, assume that* $\alpha,\beta \in (0,1)$ *and* $u_0,u_1,v_0,v_1 \in L^1(\mathbb{H})$*. If*

$$\limsup_{T\to+\infty} T^{\frac{2(\beta+q\gamma)-Q(q(p-1)\beta+(q-1)\gamma)}{2(pq-1)}} \int_{\mathbb{H}}\Big(u_0(\eta)+T^{\beta-\alpha}u_1(\eta)\Big)\,d\eta = +\infty, \tag{12}$$

or

$$\limsup_{T\to+\infty} T^{\frac{2(\gamma+p\beta)-Q(p(q-1)\gamma+(p-1)\beta)}{2(pq-1)}} \int_{\mathbb{H}}\Big(v_0(\eta)+T^{\gamma-\delta}v_1(\eta)\Big)\,d\eta = +\infty. \tag{13}$$

Then problem (3)-(4)-(5) *does not admit global in time solutions.*

Depending on the initial data (5), we give in the following some particular cases when the blow up occurs.

Corollary 3.6. *Let* $p,q>1$*, assume that* $\alpha,\beta \in (0,1)$ *and* $u_0,u_1,v_0,v_1 \in L^1(\mathbb{H})$*. Thanks to* (12)*, the problem* (3)-(4)-(5) *does not admit global in time solutions in the follwing cases:*

(i) $0<\beta<\alpha$, $\int_{\mathbb{H}} u_0(\eta)\,d\eta > 0$, *and* $\frac{q(p-1)\beta+(q-1)\gamma}{\beta+q\gamma} < \frac{2}{Q}$.

(ii) $0<\alpha<\beta$, $\int_{\mathbb{H}} u_1(\eta)\,d\eta > 0$, *and* $\frac{q(p-1)\beta+(q-1)\gamma}{(pq-1)(\beta-\alpha)\beta+q\gamma} < \frac{2}{Q}$.

(iii) $\alpha=\beta$, $\int_{\mathbb{H}}(u_0(\eta)+u_1(\eta))\,d\eta > 0$, *and* $\frac{q(p-1)\beta+(q-1)\gamma}{\beta+q\gamma} < \frac{2}{Q}$.

(iv) $\int_{\mathbb{H}} u_0(\eta)d\eta > 0$, $\int_{\mathbb{H}} u_1(\eta)\,d\eta > 0$, *and* $\frac{q(p-1)\beta+(q-1)\gamma}{(pq-1)\max\{0,\beta-\alpha\}+\beta+q\gamma} < \frac{2}{Q}$.

Using (13)*, similar contraints can be obtained to ensure the nonexistence of global in time solutions to the problem* (3)-(4)-(5).

4 The proof of main results

Let f be the function introduced in Lemma2.2. We define

$$\varphi(t,\eta)=f(t)g(\eta),$$

with

$$g(\eta)=\psi^{\lambda}\left(\frac{|x|^4+|y|^4+\tau^2}{T^{2\beta}}\right),\ \lambda\gg 1,$$

and $\psi\in C^{\infty}([0,\infty))$ be defined by

$$\psi(\xi)=\begin{cases}1, & 0\le\xi\le 1,\\ \searrow, & 1\le\xi\le 2,\\ 0, & \xi\ge 2.\end{cases}$$

It is easy to verify that φ can be chosen as a test function in Definition3.1 and Definition3.4.

Proof. Starting with the proof of Theorem3.2 and assume that we have a global solution non identically equal to zero to (1)-(2). Then from (8), we deduce that

$$\int_{Q_T}\varphi|u|^p+\mathcal{A}(u_0,u_1,\alpha,\beta,T)\le\int_{Q_T}|u(\mathbb{I}^{1-\alpha}_{t|T}\varphi)_{tt}|+\int_{Q_T}|u(\mathbb{I}^{1-\beta}_{t|T}\varphi)_t|+\int_{Q_T}|u\,\Delta_{\mathbb{H}}\varphi|,\quad(14)$$

with

$$\mathcal{A}(u_0,u_1,\alpha,\beta,T)=\int_{\mathbb{H}}u_0(\eta)\Big(\mathbb{I}^{1-\beta}_{t|T}\varphi-(\mathbb{I}^{1-\alpha}_{t|T}\varphi)_t\Big)(\eta,0)\,d\eta+\int_{\mathbb{H}}u_1(\eta)(\mathbb{I}^{1-\alpha}_{t|T}\varphi)(\eta,0)\,d\eta.\quad(15)$$

Using the ε-Young inequality

$$ab\le\varepsilon a^p+C_\varepsilon b^{p'},\quad a,b,\varepsilon,C_\varepsilon>0,\ \ 1<p,p',\ p+p'=pp',$$

we obtain the following inequalities

$$\int_{Q_T}|u(\mathbb{I}^{1-\alpha}_{t|T}\varphi)_{tt}|=\int_{Q_T}|u\varphi^{\frac{1}{p}}\varphi^{-\frac{1}{p}}(\mathbb{I}^{1-\alpha}_{t|T}\varphi)_{tt}|\le\varepsilon\int_{Q_T}|u|^p\varphi+C(\varepsilon)\int_{Q_T}\varphi^{-\frac{p'}{p}}|(\mathbb{I}^{1-\alpha}_{t|T}\varphi)_{tt}|^{p'},\quad(16)$$

$$\int_{Q_T}|u(\mathbb{I}^{1-\beta}_{t|T}\varphi)_t|\le\varepsilon\int_{Q_T}|u|^p\varphi+C(\varepsilon)\int_{Q_T}\varphi^{-\frac{p'}{p}}|(\mathbb{I}^{1-\beta}_{t|T}\varphi)_t|^{p'}\quad(17)$$

and

$$\int_{Q_T}|u\Delta_{\mathbb{H}}\varphi|\le\varepsilon\int_{Q_T}|u|^p\varphi+C(\varepsilon)\int_{Q_T}\varphi^{-\frac{p'}{p}}|\Delta_{\mathbb{H}}\varphi|^{p'}.\quad(18)$$

Collecting (16)-(17)-(18) and choosing ε small, the inequality (14) becomes

$$\begin{aligned}\int_{Q_T}\varphi|u|^p\ &+\ \mathcal{A}(u_0,u_1,\alpha,\beta,T)\\ &\le\ C\left(\int_{Q_T}\varphi^{-\frac{p'}{p}}|(\mathbb{I}^{1-\alpha}_{t|T}\varphi)_{tt}|^{p'}+\int_{Q_T}\varphi^{-\frac{p'}{p}}|(\mathbb{I}^{1-\beta}_{t|T}\varphi)_t|^{p'}+\int_{Q_T}\varphi^{-\frac{p'}{p}}|\Delta_{\mathbb{H}}\varphi|^{p'}\right).(19)\end{aligned}$$

First, using Lemma2.2, there exist $c_1, c_2, c_3 > 0$ such that

$$\mathcal{A}(u_0, u_1, \alpha, \beta, T) = \int_{\mathbb{H}} \Big((c_1 T^{1-\beta} + c_2 T^{-\alpha}) u_0(\eta) + c_3 T^{1-\alpha} u_1(\eta) \Big) g(\eta)\, d\eta. \tag{20}$$

Then, using again Lemma2.2, and passing to the new variables

$$\tilde{\tau} = T^{-\beta}\tau, \quad \tilde{x} = T^{-\frac{\beta}{2}} x, \quad \tilde{y} = T^{-\frac{\beta}{2}} y, \quad \tilde{\eta} = (\tilde{x}, \tilde{y}, \tilde{\tau}),$$

we otain that

$$\begin{aligned} \int_{Q_T} \varphi^{-\frac{p'}{p}} |(\mathbb{I}_{t|T}^{1-\alpha}\varphi)_{tt}|^{p'} &= \int_0^T f^{-\frac{p'}{p}} |(\mathbb{I}_{t|T}^{1-\alpha} f)_{tt}|^{p'}\, dt \int_{\mathbb{H}} g(\eta)\, d\eta \\ &= T^{Q-(\alpha+1)p'} \int_0^T (1 - \frac{t}{T})^{\sigma-(\alpha+1)p'}\, dt \int_{\mathbb{H}} \psi^{\lambda}(\tilde{\eta})\, d\tilde{\eta} \\ &\leq c\, T^{\frac{\beta Q}{2}+1-(\alpha+1)p'}, \end{aligned} \tag{21}$$

$$\int_{Q_T} \varphi^{-\frac{p'}{p}} |(\mathbb{I}_{t|T}^{1-\beta}\varphi)_{t}|^{p'} \leq c\, T^{(Q-2p')\frac{\beta}{2}+1}, \tag{22}$$

and

$$\begin{aligned} \int_{Q_T} \varphi^{-\frac{p'}{p}} |\Delta_{\mathbb{H}}\varphi|^{p'} &= \int_0^T f(t)\, dt \int_{\mathbb{H}} g^{-\frac{p'}{p}} |\Delta_{\mathbb{H}} g|^{p'}\, d\eta \\ &\leq c\, T^{(Q-2p')\frac{\beta}{2}+1}. \end{aligned} \tag{23}$$

Collecting estimations (20)–(23). We deduce from (19) that

$$\int_{Q_T} \varphi |u|^p + \int_{\mathbb{H}} \Big((c_1 T^{1-\beta} + c_2 T^{-\alpha}) u_0(\eta) + c_3 T^{1-\alpha} u_1(\eta) \Big) g(\eta)\, d\eta \leq c\, T^{(Q-2p')\frac{\beta}{2}+1}.$$

This yields

$$\int_{\mathbb{H}} \Big(T^{1-\beta} u_0(\eta) + T^{1-\alpha} u_1(\eta) \Big) g(\eta)\, d\eta \leq c\, T^{(Q-2p')\frac{\beta}{2}+1}.$$

Hence, (9) follows.

Now, for the proof of Theorem3.5, we proceed analogously as for the proof of Theorem3.2.

Let

$$\mathcal{I} = \int_{Q_T} \varphi |u|^p, \qquad \mathcal{J} = \int_{Q_T} \varphi |v|^q$$

and

$$\mathcal{M}(p, \alpha, \beta) = \left(\int_{Q_T} \varphi^{-\frac{p'}{p}} |(\mathbb{I}_{t|T}^{1-\alpha}\varphi)_{tt}|^{p'} \right)^{\frac{1}{p'}} + \left(\int_{Q_T} \varphi^{-\frac{p'}{p}} |(\mathbb{I}_{t|T}^{1-\beta}\varphi)_{t}|^{p'} \right)^{\frac{1}{p'}} + \left(\int_{Q_T} \varphi^{-\frac{p'}{p}} |\Delta_{\mathbb{H}}\varphi|^{p'} \right)^{\frac{1}{p'}}.$$

It is easy to deduce from (21)–(23) that

$$\mathcal{M}(p, \alpha, \beta) \leq c\, T^{\frac{(Q-2p')\beta+2}{2p'}}. \tag{24}$$

Using Hölder's inequality, we obtain from (10) and (11)

$$\mathcal{J} + \mathcal{A}(u_0, u_1, \alpha, \beta, T) \le \mathcal{I}^{\frac{1}{p}} \mathcal{M}(p, \alpha, \beta)$$

and

$$\mathcal{I} + \mathcal{A}(v_0, v_1, \delta, \gamma, T) \le \mathcal{J}^{\frac{1}{q}} \mathcal{M}(q, \delta, \gamma).$$

If we assume that $\mathcal{A}(v_0, v_1, \delta, \gamma, T) \ge 0$. Then, we obatin

$$\mathcal{J} + \mathcal{A}(u_0, u_1, \alpha, \beta, T) \le \mathcal{I}^{\frac{1}{p}} \mathcal{M}(p, \alpha, \beta), \quad \text{and} \quad \mathcal{I} \le \mathcal{J}^{\frac{1}{q}} \mathcal{M}(q, \delta, \gamma).$$

This yields

$$\mathcal{J} + \mathcal{A}(u_0, u_1, \alpha, \beta, T) \le \mathcal{J}^{\frac{1}{pq}} \mathcal{M}^{\frac{1}{p}}(q, \delta, \gamma) \mathcal{M}(p, \alpha, \beta).$$

Applying Young's inequality there holds

$$\mathcal{A}(u_0, u_1, \alpha, \beta, T) \le c \mathcal{M}^{\frac{q}{pq-1}}(q, \delta, \gamma) \mathcal{M}^{\frac{pq}{pq-1}}(p, \alpha, \beta).$$

Similarly, If we assume that $\mathcal{A}(u_0, u_1, \alpha, \beta, T) \ge 0$. Then, we obatin

$$\mathcal{A}(v_0, v_1, \delta, \gamma, T) \le c \mathcal{M}^{\frac{p}{pq-1}}(p, \alpha, \beta) \mathcal{M}^{\frac{pq}{pq-1}}(q, \delta, \gamma).$$

Finally, thanks to (15) and (24), we deduce from the last two inequalities that

$$\int_{\mathbb{H}} \Big(T^{1-\beta} u_0(\eta) + T^{1-\alpha} u_1(\eta) \Big) g(\eta)\, d\eta \le c\, T^{1 + \frac{Q(q(p-1)\beta + (q-1)\gamma) - 2q(p\beta+\gamma)}{2(pq-1)}},$$

and

$$\int_{\mathbb{H}} \Big(T^{1-\gamma} v_0(\eta) + T^{1-\delta} v_1(\eta) \Big) g(\eta)\, d\eta \le c\, T^{1 + \frac{Q(p(q-1)\gamma + (p-1)\beta) - 2p(q\gamma+\beta)}{2(pq-1)}}.$$

Therefore, the desired constraints in Theorem 3.5 follows.

$\square$

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