Non-homogenous bivariate fragmentation process: asymptotic distribution via contraction method

Rafik Aguech.
Department of Statistics and Operations Research
King Saud University KSA
rafik.aguech@ipeit.rnu.tn

Samia Ilji
Département des Mathematiques
Faculté des Sciences de Monastir
Monastir, Tunisia
samia.ilji@yahoo.com

February 11, 2020

Abstract

In this paper we investigate the size of a bi-dimensional fragmentation process. A rectangle of dimensions x and y is considered, it is split into four sub-rectangles with some probability that depends on x and y, we iterate until the stop of the process. The total number of the all the obtained rectangles at the end of the process satisfies some equality in distribution which is resolved , using some tools on integral equations, via the contraction method.

Keywords: Fragmentation process, contraction method, Zolotarev metric, Integral equations.

1 Introduction

Fragmentation process is a process that describes the evolution of an object which is break over the time according some distribution, it applies in a wide range of fields such that biology [10], physics [3], computer sciences [5, 9] etc. There are two major types of fragmentation process, the first is the homogenous fragmentation where a piece is cut independently on its mass, the second is the inhomogeneous fragmentation where the fragmentation of the object depends on its mass. Several authors studied the fragmentation process of an interval x, namely Janson [8] who gave when the probability of dislocation is $p(x) = \mathbf{1}_{\{x\geq 1\}}$ the mean, the variance and the asymptotic distribution of the total number of the intervals. Afterwards, Aguech [2] studied the fragmentation of the interval x when the fragmentation probability is $p(x) = 1 - e^{-x}$, he obtained the behavior of the size of the fragmentation process. A generalization of this model in the bi-dimensional case is introduced by Aguech and Ilji [1], they consider a rectangle of dimensions x and y at the beginning. At time $n = 1$, with probability $p(x, y) = \mathbf{1}_{\{x \geq 1, y \geq 1\}}$ the rectangle is cut into four sub-rectangles according to a vector (U_1, U_2) for the side x and (V_1, V_2) for the side y in such a way we obtain four sub-rectangles with dimensions (xU_1, yV_1), (xU_1, yV_2), (xU_2, yV_1) and (xU_2, yV_2), if $x < 1$ or $y < 1$ the rectangle remains stable for ever. This procedure is repeated in all the sub-rectangles with new and independent copies of (U_1, U_2) and (V_1, V_2). The process stops almost surely after a finite number of steps, it leaves a finite number of rectangles denoted by $N(x, y)$. Basing on the bivariate renewal theory, Aguech and Ilji [1] give the expectation and the variance of $N(x, y)$. The purpose of this paper is to give the asymptotic distribution of $N(x, y)$ via a contraction Theorem that underlines such equality in distribution. The contraction Theorem can also be proved using Integral equations techniques. Integral equations tools and applications can be found on [4]

2 Description of the model, notations and assumptions

Let $\mathbf{U} = (U_1, U_2)$ and $\mathbf{V} = (V_1, V_2)$ be two independent random vectors such that $U_1 + U_2 = 1$ and $V_1 + V_2 = 1$ almost surely. We start with a rectangle with dimensions x and y larger than one ($x \geq 1$ and $y \geq 1$). At discrete time $n = 1$, we break the rectangle into four rectangles as the following: x (respectively y) is divided according to $\mathbf{U}$ (respectively $\mathbf{V}$) into two intervals of lengths xU_1 and xU_2 (respectively yV_1 and yV_2). We obtain four sub-rectangles of dimensions (xU_1, yV_1), (xU_1, yV_2), (xU_2, yV_1) and (xU_2, yV_2). We repeat on each rectangle with dimensions larger than one (each of the dimensions is larger than one) this procedure with independent copies of $(\mathbf{U}, \mathbf{V})$. The process stops almost surely leaving a finite number of rectangles denoted by $N(x, y)$. By definition we have the following equality in distribution ,

$$N(x, y) \stackrel{\mathcal{D}}{=} \begin{cases} 1, & \text{if } x < 1 \text{ or } y < 1; \\ 1 + \sum_{i=1}^{2} \sum_{j=1}^{2} N_{i,j}(xU_i, yV_j), & \text{if not,} \end{cases}$$

where $N_{i,j}(.,.)$, $i, j \in \{1, 2\}$ are independent copies of $N(.,.)$.
For all $(i,\ j) \in \{1, 2\}^2$, let $X_i = -\ln(U_i)$, $Y_j = -\ln(V_j)$, μ_{ij} the joint distribution of (X_i, Y_j) and we define on $[0, +\infty[\times[0, +\infty[$ the probability measure ν by

$$d\nu(t_1, t_2) = e^{-(t_1+t_2)} \sum_{i=1}^{2} \sum_{j=1}^{2} d\mu_{ij}. \tag{1}$$

Let M be a positif number, for $i = 1, 2$ let $A_i = \{t_i,\ |t_i| < M\}$ and $A_i^{\mathcal{C}}$ its complementary set on $\mathbb{R}_+ \times \mathbb{R}_+$. The measure ν is said to belong to the class J_2 if its characteristic function Ψ satisfies the following conditions: There exist some nonnegative numbers α_1, α_2 and c such that

- $|\Psi(t_1, t_2)| \leq \frac{c}{|t_1|^{\alpha_1}}$ for all $(t_1, t_2) \in A_1^{\mathcal{C}} \times A_2$,
- $|\Psi(t_1, t_2)| \leq \frac{c}{|t_2|^{\alpha_2}}$ for all $(t_1, t_2) \in A_1 \times A_2^{\mathcal{C}}$,

- $|\Psi(t_1,t_2)| \leq \frac{c}{|t_1|^{\alpha_1}|t_2|^{\alpha_2}}$ for all $(t_1,t_2) \in A_1^{\mathcal{C}} \times A_2^{\mathcal{C}}$.

If furthermore, ν has a finite mean and a definite positive matrix, ν is called to belong to the set J_2^*.
Let us define the function Φ on $\mathbb{R}_+ \times \mathbb{R}_+$ as follows:

$$\Phi(t_1,t_2) = \sum_{i=1}^{2}\sum_{j=1}^{2} \mathbb{E}(U_i^{t_1} V_j^{t_2}),$$

and let

$$\begin{aligned} \theta_1 &= -\sum_{i=1}^{2} \mathbb{E}(U_i \ln(U_i)) = \frac{\partial \Phi}{\partial t_1}(t_1,t_2)_{|(t_1,t_2)=(1,1)} \\ &\text{and} \\ \theta_2 &= -\sum_{j=1}^{2} \mathbb{E}(V_j \ln(V_j)) = \frac{\partial \Phi}{\partial t_2}(t_1,t_2)_{|(t_1,t_2)=(1,1)}. \end{aligned}$$

Remark: It is not difficult to see that the mean of the measure ν is given by $\boldsymbol{\theta} = (\theta_1, \theta_2)$.

Notations:

These notations will be useful in the paper: Let $\mathbf{X} = \left(X^{(1)},\ X^{(2)}\right)$ be a random vector with distribution ν (1) that belongs to J_2^*, we denote by

- Σ the covariance matrix of $\mathbf{X}$, $|\Sigma|$ the determinant of Σ and Σ^{-1} its inverse matrix,
- $\sigma^2 = (\sigma_1^2,\ \sigma_2^2) = \Big(\mathbb{V}ar(X^{(1)}), \mathbb{V}ar(X^{(2)})\Big),\ K = \frac{\theta_1}{\sqrt{2\pi|\Sigma|(\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta})}}$
- For $k = 1,2,\ a_k = \frac{\mathbb{E}\left[(X^{(k)}-\theta_k)^3\right]}{\sigma_k^4}$,
- $c_0 = \frac{-1-2(a_1\theta_1+a_2\theta_2)}{4\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta}} + \frac{1}{2(\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta})^2}\Big(\frac{a_1\theta_1^3}{\sigma_1^2} + \frac{a_2\theta_2^3}{\sigma_2^2}\Big)$,
- $c_1 = \theta_2 a_2 - 1 + \frac{\theta_2^2 - a_1\theta_1\theta_2^2 - 2a_2\theta_2^3}{\sigma_2^2\boldsymbol{\theta}\Sigma^{-1}\boldsymbol{\theta}} + \frac{\theta_2^2}{\sigma_2^2(\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta})^2}\big(\frac{a_1\theta_1^3}{\sigma_1^2} + \frac{a_2\lambda_2^3}{\sigma_2^2}\big)$,
- $c_2 = \frac{\theta_2^2}{\sigma_2^2}\big(-1 + \frac{\theta_2^2}{\sigma_2^2\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta}}\big)$,
- $\gamma = \sum_{i=1}^{b}\sum_{j=1}^{b'} \mathbb{E}\Big[U_i^2 V_j^2\Big(\ln(V_j) - \frac{\theta_2}{\theta_1}\ln(U_i)\Big)\Big]$,
- $\rho = \sum_{i=1}^{b}\sum_{j=1}^{b'} \mathbb{E}\Big[U_i^2 V_j^2\Big(\ln(V_j) - \frac{\theta_2}{\theta_1}\ln(U_i)\Big)^2\Big]$,

- For all $(i,\ j) \in \{1,2\}^2$ we denote by

$$\begin{array}{rcl}
\omega_{ij} & = & -\ln(V_j) + \frac{\theta_2}{\theta_1}\ln(U_i), \\
L_1(U_i,V_j) & = & \frac{K}{2\theta_2^{\frac{3}{2}}}\Big[(1+c_1-2\frac{c_2}{\theta_1}+2\frac{c_2}{\theta_2})\omega_{ij}+\frac{c_2}{\theta_2}\omega_{ij}^2\Big] \\
 & \text{and} & \\
L_2(U_i,V_j) & = & \frac{c_2K\omega_{ij}}{\theta_2^{\frac{5}{2}}},
\end{array}$$

- $\widetilde{L}_1 = \frac{K\sum_{i=1}^{2}U_i\ln(U_i)}{2\theta_1\sqrt{\theta_2}} - \sum_{i=1}^{2}\sum_{j=1}^{2}U_iV_jL_1(U_i,V_j),\ \widetilde{L}_2 = \sum_{i=1}^{2}\sum_{j=1}^{2}U_iV_jL_2(U_i,V_j)$

- $A_1 = \mathbb{E}\Big[\widetilde{L}_1^2\Big],\ A_2 = \mathbb{E}\Big[\widetilde{L}_1\widetilde{L}_2\Big]$ and $A_3 = \mathbb{E}\Big[\widetilde{L}_2^2\Big]$,

- $\|x\|$ the Euclidean norm of x,
- for a number x such that $n < x \leq n+1$ where $n \in \mathbb{N}$ we denote by $\lceil x \rceil = n+1$.

Assumptions:

We need these assumptions in the paper

- **(A)**: The random variables U_1 and V_1 are two absolutely continuous, such that

$$\sum_{i=1}^{2}\mathbb{E}\Big[U_i|\ln(U_i)|^3 + V_i|\ln(V_i)|^3\Big] < \infty.$$

- **(B):** The probability measure ν (1) belongs to the set J_2^*,

3 Contraction Theorem

We prove in this section a contraction Theorem that allows to obtain the size of our fragmentation process. Let $\mathcal{M}^d$ be the space of the measures on $\mathbb{R}^d$ and let ℓ_s be the metric defined on the sub-space $\mathcal{M}_s^d := \{\mu \in \mathcal{M}^d : \|\mu\|_s = \mathbb{E}(\|\mu\|^s)^{\frac{1}{s}} < \infty\}$ by

$$\ell_s(\mu,\nu) = \inf\{\|X-Y\|_s^{s\wedge 1} : \ell(X)=\mu,\ \ell(Y)=\nu\} \text{ for all } \mu,\nu \in \mathcal{M}_s^d.$$

We define the sub-spaces of $\mathcal{M}_s^d$ by:

$$\mathcal{M}_s^d(0,Id) = \begin{cases} \mathcal{M}_s^d, & \text{if } 0 < s \leq 1; \\ \{\mu \in \mathcal{M}_s^d : \mathbb{E}(\mu) = 0\}, & \text{if } 1 < s \leq 2; \\ \{\mu \in \mathcal{M}_s^d : \mathbb{E}(\mu) = 0,\ \texttt{Cov}(\mu) = Id\}, & \text{if } 2 < s \leq 3. \end{cases}$$

3.1 Zolotarev metric: Definition and properties

Definition: For $s > 0$, the Zolotarev metric ξ_s, is defined by

$$\xi_s(X,Y) = \sup_{f \in \mathcal{F}_s} \{| \mathbb{E}(f(X) - F(Y)) |\}$$

where

$$\mathcal{F}_s = \{f \in \mathcal{C}^m(\mathbb{R}^d, \mathbb{R}) : \|f^{(m)}(x) - f^{(m)}(y)\| \leq \| x - y \|^{\alpha}\}$$

with $m = \lceil s \rceil - 1 \geq 0,\ \alpha = s - m,\ f^{(m)}$ the m derivative of f and $\mathcal{C}^m(\mathbb{R}^d, \mathbb{R})$ denotes the set of functions m times continuously differentiable.

Properties: Let's recall some properties of ξ_s (more details can be found in [6, 7, 11, 12]).

- The convergence according to ξ_s implies the weak convergence.

- For all $c \neq 0,\ \xi_s(cX, cY) =| c |^s\ \xi_s(X,Y)$ (we say that ξ_s is homogeneous of order s)

- For X and Y linear combinations of independent random vectors $(X_i)_{1 \leq i \leq p}$ and $(Y_j)_{1 \leq j \leq p}$,that is $X = \sum\limits_{i=1}^{p} c_i X_i$ and $Y = \sum\limits_{i=1}^{p} c_i Y_i$ we have

$$\xi_s(X,Y) \leq \sum_{i=1}^{p} | c_i |^s\ \xi_s(X_i, Y_i)$$

- $\xi_s(X+Z, Y+Z) \leq \xi_s(X,Y)$ for all Z independent of (X,Y) and

$$\xi_s(AX, AY) \leq ||A||_{op}^s \xi_s(X,Y) \text{ where } ||A||_{op} = \sup_{||u||=1} ||Au||.$$

- $\xi_s(X,Y) \leq \frac{\Gamma(1+\alpha)}{\Gamma(1+s)} \Big(\mathbb{E}(||X||^s) + \mathbb{E}(||Y||^s)\Big).$

- $\pi^{1+s}(||X||,\ ||Y||) \leq C\xi_s(X,Y)$ where $C > 0$ and π is the Prohorov metric

- If all the mixed moments of X and Y up to order m are zero and the moments of order s are finite, then we have:

$$\xi_s(X,Y) \leq \frac{\Gamma(1+\alpha)}{\Gamma(1+s)} \Big[2m\kappa_s(X,Y) + (2\kappa_s(X,Y))^{\alpha} \Big(\min\big(\mathbb{E}(||X||^s)^{\frac{1}{s}},\ \mathbb{E}(||Y||^s)^{\frac{1}{s}}\big)\Big)^{1-\alpha}\Big]$$

where,

$$\kappa_s(X,Y) = \sup\Big\{|\mathbb{E}(f(X) - f(Y))| : |f(x) - f(y)| \leq \Big|\Big| ||x||^{s-1}x - ||y||^{s-1}x \Big|\Big|\Big\}.$$

3.2 Contraction Theorem:

Let $\tau_0 > 0,\ B_+(0,\tau_0) = \{(t_1,t_2) \in \mathbb{R}_+^2 : \|(t_1,t_2)\| \le \tau_0\}$ and $B_+^c(0,\tau_0) = \mathbb{R}_+^2 \backslash B_+(0,\tau_0)$. Let $(Y(t_1,t_2))_{t_1,t_2 \ge 0}$ be a $d-$dimensional process satisfying the recursion

$$Y(t_1,t_2) \overset{\mathcal{D}}{=} \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} A_{ij}(t_1,t_2) Y_{ij}(T_{ij}(t_1,t_2)) + b(t_1,t_2),\ (t_1,t_2) \in B_+^c(0,\tau_0), \tag{2}$$

where

- K_1 and K_2 are two nonnegative integers,
- $\left(Y_{11}(t_1,t_2)\right)_{t_1,t_2\ge 0}, \cdots, \left(Y_{K_1K_2}(t_1,t_2)\right)_{t_1,t_2\ge 0}, \Big(A_{11}(t_1,t_2), \cdots, A_{K_1K_2}(t_1,t_2), b(t_1,t_2), T(t_1,t_2)\Big)_{t_1,t_2\ge 0}$ are independent,
- $A_{ij}(t_1,t_2)$ is a random $d \times d$ matrix for all $(i,j) \in \{1,\cdots,K_1\} \times \{1\cdots,K_2\}$ and $T(t_1,t_2) = \Big(T_{11}(t_1,t_2), \cdots, T_{K_1K_2}(t_1,t_2)\Big)$ is a vector of random indices such that $T_{ij}(t_1,t_2) \in [0,t_1] \times [0,t_2]$,
- for $(u,v) \in [0,t_1] \times [0,t_2]$ and $(i,j) \in \{1,\cdots K_1\} \times \{1,\cdots K_2\},\ Y_{ij}(u,v)$ is an independent copy of $Y(u,v)$.

Assume that there exists $\tau_1 \ge \tau_0$ such that the covariance matrix of $Y(t_1,t_2)$ denoted by $\mathtt{Cov}(Y(t_1,t_2))$ is a definite positive matrix for all $(t_1,t_2) \in B_+^c(0,\tau_1)$. Consider the rescaled process defining by:

$$X(t_1,t_2) = C^{-\frac{1}{2}}(t_1,t_2)\Big(Y(t_1,t_2) - M(t_1,t_2)\Big),\ (t_1,t_2) \in \mathbb{R}_+^2, \tag{3}$$

where $C(t_1,t_2)$ is a definite symmetric positive matrix and $M(t_1,t_2) \in \mathbb{R}^d$ satisfying:

$$M(t_1,t_2) = \mathbb{E}\Big[Y(t_1,t_2)\Big] \quad \text{for} \quad 1 < s \le 3,$$

$$\text{and} \quad C(t_1,t_2) = \begin{cases} Id & \text{if } (t_1,t_2) \in B_+(0,\tau_1) \\ \\ \mathtt{Cov}\big(Y(t_1,t_2)\big) & \text{if } (t_1,t_2) \in B_+^c(0,\tau_1) \end{cases} \qquad \text{for} \quad 2 < s \le 3.$$

Thus

$$\begin{aligned} \mathbb{E}\big(X(t_1,t_2)\big) &= 0 \quad \text{for} \quad 1 < s \le 3 \quad \text{ and for } \quad 2 < s \le 3, \\ \mathtt{Cov}\big(X(t_1,t_2)\big) &= \begin{cases} \mathtt{Cov}\big(Y(t_1,t_2)\big) & \text{if } (t_1,t_2) \in B_+(0,\tau_1) \\ \\ Id & \text{if } (t_1,t_2) \in B_+^c(0,\tau_1). \end{cases} \end{aligned}$$

Furthermore, $\Big(X(t_1,t_2)\Big)_{t_1,t_2\geq 0}$ satisfies the following recursion:

$$X(t_1,t_2) \overset{\mathcal{D}}{=} \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} \widetilde{A}_{ij}(t_1,t_2)X_{ij}(T_{ij}(t_1,t_2)) + \widetilde{b}(t_1,t_2) \quad \text{for } (t_1,t_2)\in B_+^c(0,\tau_1) \tag{4}$$

where $X_{ij}(u,v)$ is an independent copy of $X(u,v)$ for all $i=1,\cdots,K_1$

and $j=1,\cdots,K_2$, $\widetilde{A}_{ij}(t_1,t_2) = C^{-\frac{1}{2}}(t_1,t_2)A_{ij}(t_1,t_2)C^{\frac{1}{2}}(T_{ij}(t_1,t_2))$
and $\widetilde{b}(t_1,t_2) = C^{-\frac{1}{2}}(t_1,t_2)\Big[b(t_1,t_2) - M(t_1,t_2) + \sum\limits_{i=1}^{K_1}\sum\limits_{j=1}^{K_2} A_{ij}(t_1,t_2)M_{ij}(T_{ij}(t_1,t_2))\Big]$.
Clearly the two sequences $\Big(\widetilde{A}_{11}(t_1,t_2),...,\widetilde{A}_{K_1K_2}(t_1,t_2),\widetilde{b}(t_1,t_2),T(t_1,t_2)\Big)_{t_1,t_2\geq 0}$
and $\big(X_{11}(t_1,t_2)\big)_{t_1,t_2\geq 0}, \cdots, \big(X_{K_1K_2}(t_1,t_2)\big)_{t_1,t_2\geq 0}$ are independent.

Proposition 1. *Let T the map defined on $\mathcal{M}_s^d(0,Id)$ by*

$$T:\mu \longmapsto \mathcal{L}\Big(\sum_{i=1}^{K_1}\sum_{j=1}^{K_2} A_{ij}^* Z_{ij} + b^*\Big)$$

such that $(A_{11}^*,\cdots,A_{K_1K_2}^*,b^*), Z_{11},\cdots,Z_{K_1K_2}$ are i.i.d and $\mathcal{L}(Z_{ij}) = \mu$ for all $(i,j)\in\{1,\cdots,K_1\}\times\{1,\cdots,K_2\}$. If the following assumptions are satisfied

- *for $0<s\leq 3$, $\sum\limits_{i=1}^{K_1}\sum\limits_{j=1}^{K_2}\mathbb{E}(\parallel A_{ij}^*\parallel_{op}^s) < 1$ (the Lipschitz property),*
- *$\mathbb{E}(b^*) = 0$ for all $1<s\leq 2$,*
- *for $2<s\leq 3$*
$$\mathbb{E}(bb^{*Tr}) + \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}(A_{ij}^*A_{ij}^{*Tr}) = Id, \tag{5}$$

then T admits a unique fixed point in $\mathcal{M}_s^d(0,Id)$.

Proof. For all $\mu\in\mathcal{M}_s^d(0,Id)$, $T\mu\in\mathcal{M}_s^d(0,Id)$ for $0<s\leq 3$, in addition

$$\begin{aligned}\xi_s(T\mu,T\nu) &= \xi_s\Big(\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}A_{ij}^*Z_{ij}+b^*,\ \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}A_{ij}^*W_{ij}+b^*\Big)\\ &\leq \Big(\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}(\parallel A_{ij}^*\parallel_{op}^s)\Big)\xi_s(\mu,\nu).\end{aligned}$$

Then T is a strict contraction in $\mathcal{M}_s^d(0,Id)$ who is a complete space (by Svante Janson [8]) so by the fixed point Theorem , T has a unique solution in $\mathcal{M}_s^d(0,Id)$. $\square$

Theorem 1. *Let $(Y(t_1,t_2))_{t_1,t_2\geq 0}$ be an $s-$integrable ($0<s\leq 3$) process satisfying (2) and $(X(t_1,t_2))_{t_1,t_2\geq 0}$ its rescaled process defined by (3) and satisfying*

(4). Assume that $\widetilde{A}_{11}(t_1,t_2),\cdots,\widetilde{A}_{K_1K_2}(t_1,t_2),\ \widetilde{b}(t_1,t_2)$ are $s-$integrables and that $\sup_{(u,v)\in[0,t_1]\times[0,t_2]}||X(u,v)||_s<\infty$ for all $(t_1,t_2)\in\mathbb{R}_+^2$ and as $\|(t_1,t_2)\|$ goes to infinity

$$\Big(\widetilde{A}_{11}(t_1,t_2),\cdots,\widetilde{A}_{11}(t_1,t_2),\widetilde{b}(t_1,t_2)\Big)\xrightarrow{\ell_s}\Big(A_{11}^*,\cdots,A_{K_1K_2}^*,\ b^*\Big) \tag{6}$$

$$\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}(||A_{ij}^*||_{op}^s)<1 \tag{7}$$

$$\mathbb{E}\Big(||\widetilde{A}_{ij}(t_1,t_2)||_{op}^s\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+(0,\tau)\}}\Big)\underset{\|(t_1,t_2)\|\to+\infty}{\longrightarrow}0 \text{ for all } \tau>0,\ 1\le i\le K_1,\ 1\le j\le K_2. \tag{8}$$

Then $(X(t_1,t_2))_{(t_1,t_2)}$ converges in distribution in $\mathcal{M}_s^d(0,Id)$ to a random variable X where $\mathcal{L}(X)$ is the unique solution of the map

$$T:\mu\longmapsto\mathcal{L}\Big(\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}A_{ij}^*Z_{ij}+b^*\Big) \tag{9}$$

and $(A_{11}^*,\cdots,A_{K_1K_2}^*,b^*),Z_{11},\cdots,Z_{K_1K_2}$ are independent and $\mathcal{L}(Z_{ij})=\mu$ for all $1\le i\le K_1$ and $1\le j\le K_2$. Furthermore, we have,

$$\begin{cases}\mathbb{E}(X)=0 & \text{for } 1<s\le3\\ \mathtt{Cov}(X)=Id & \text{for } 2<s\le3.\end{cases}$$

Proof. As A_{ij}^* and $\widetilde{b^*}$ are s-integrable then $||X||_s$ is finite. Moreover $\mathbb{E}\big(X(t_1,t_2)\big)=0$ for $1<s\le3$ then $\mathbb{E}\big(\widetilde{b}(t_1,t_2)\big)=0$ but $\widetilde{b}(t_1,t_2)\xrightarrow{\ell_s}b^*$ thus $\mathbb{E}(b^*)=0$. In addition, for $2<s\le3$ and $(t_1,t_2)\in B_+^c(0,\tau_1)$

$$\begin{aligned}Id &= \mathtt{Cov}(X(t_1,t_2))=\mathbb{E}\Big(X(t_1,t_2)X(t_1,t_2)^{Tr}\Big)\\ &= \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}\big[\widetilde{A}_{ij}(t_1,t_2)\widetilde{A}_{ij}(t_1,t_2)^{Tr}\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+^c(0,\tau_1)\}}\big]+\mathbb{E}\big[\widetilde{b}(t_1,t_2)\widetilde{b}(t_1,t_2)^{Tr}\big]\\ &\quad+\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}\Big[\widetilde{A}_{ij}(t_1,t_2)\mathtt{Cov}\big[Y_{ij}(T_{ij}(t_1,t_2))\big]\widetilde{A}_{ij}(t_1,t_2)^{Tr}\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+(0,\tau_1)\}}\Big].\end{aligned}$$

Using (6) we obtain: $\mathbb{E}\Big[\widetilde{b}(t_1,t_2)\widetilde{b}(t_1,t_2)^{Tr}\Big]\underset{\|(t_1,t_2)\|\to+\infty}{\longrightarrow}\mathbb{E}(b^*b^{*Tr})$ in fact, by the Hölder inequality we have:

$$\begin{aligned}|\mathbb{E}[\widetilde{b}(t_1,t_2)\widetilde{b}(t_1,t_2)^{Tr}-b^*b^{*Tr}]| &= |\mathbb{E}[\widetilde{b}(t_1,t_2)(\widetilde{b}(t_1,t_2)^{Tr}-b^{*Tr})+(\widetilde{b}(t_1,t_2)-b^*)b^{*Tr}]|\\ &\le \mathbb{E}\Big[\Big|(\widetilde{b}(t_1,t_2)-b^*)\Big|^s\Big]^{\frac{1}{s}}\mathbb{E}\Big[\Big|b^{*Tr}\Big|^{\frac{s}{s-1}}\Big]^{\frac{s-1}{s}}\\ &\quad+\mathbb{E}\Big[\Big|(\widetilde{b}(t_1,t_2)-b^*)^{Tr}\Big|^s\Big]^{\frac{1}{s}}\mathbb{E}\Big[|\widetilde{b}(t_1,t_2)^{Tr}|^{\frac{s}{s-1}}\Big]^{\frac{s-1}{s}},\end{aligned}$$

therefore

$$\lim_{\|(t_1,t_2)\|\to+\infty}|\mathbb{E}[\widetilde{b}(t_1,t_2)\widetilde{b}(t_1,t_2)^{Tr}-b^*b^{*Tr}]|\to0.$$

Similarly, we prove that $\mathbb{E}\Big(\widetilde{A}_{ij}(t_1,t_2)\widetilde{A}_{ij}(t_1,t_2)^{Tr}\Big) \longrightarrow \mathbb{E}\Big(A^*_{ij}A^{*Tr}_{ij}\Big)$ then

$$Id = \mathbb{E}(bb^{*Tr}) + \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}(A^*_{ij}A^{*Tr}_{ij}) \text{ for } 2 < s \leq 3,$$

then by proposition (1), T admits a unique fixed point X. Proving now that $\xi_s(X(t_1,t_2),X) \longrightarrow 0$, for this define the process Q by: for $(t_1,t_2) \in B^c_+(0,\tau_1)$

$$\begin{aligned} Q(t_1,t_2) \quad &\overset{\mathcal{D}}{=} \quad \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\widetilde{A}_{ij}(t_1,t_2)\Big[X_{ij}(T_{ij}(t_1,t_2))\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+(0,\tau_1)\}} \\ &\qquad + X_{ij} \times \mathbf{1}_{\{T_{ij}(t_1,t_2)\in B^C_+(0,\tau_1)\}}\Big] + \widetilde{b}(t_1,t_2) \end{aligned}$$

where $\Big(\widetilde{A}_{11}(t_1,t_2),\cdots,\widetilde{A}_{K_1K_2}(t_1,t_2),\widetilde{b}(t_1,t_2),T(t_1,t_2)\Big)_{t_1,t_2\geq 0}, X_{11}, \cdots, X_{K_1K_2}$, $\Big(X_{11}(t_1,t_2)\Big)_{t_1,t_2\geq 0}, \cdots, \Big(X_{K_1K_2}(t_1,t_2)\Big)_{t_1,t_2\geq 0}$ are independent, $X_{ij} \overset{\mathcal{D}}{=} X$, $X_{ij}(u,v) \overset{\mathcal{D}}{=} X(u,v)$, for $(i,j) \in \{1,\cdots,K_1\}\times\{1,\cdots,K_2\}$ and for $(u,v) \in B_+(0,\tau_1)$.

Clearly, $||Q(t_1,t_2)||_s < \infty$ for all $0 < s \leq 3$ and $\mathbb{E}[Q(t_1,t_2)] = 0$ for $1 < s \leq 3$. For $2 < s \leq 3$

$$\begin{aligned} \texttt{Cov}(Q(t_1,t_2)) \quad &= \quad \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}\Big[\widetilde{A}_{ij}(t_1,t_2)\texttt{Cov}[X_{ij}(T_{ij}(t_1,t_2))]\widetilde{A}_{ij}(t_1,t_2)^{Tr}\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+(0,\tau_1)\}}\Big] \\ &\quad + \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}\Big[\widetilde{A}_{ij}(t_1,t_2)\texttt{Cov}[X_{ij}]\widetilde{A}_{ij}(t_1,t_2)^{Tr}\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B^c_+(0,\tau_1)\}}\Big] \\ &\quad + \mathbb{E}[\widetilde{b}(t_1,t_2)\widetilde{b}(t_1,t_2)^{Tr}] \\ &= \quad \texttt{Cov}(X_{ij}(t_1,t_2)) \qquad \Big(\text{since } \texttt{Cov}(X_{ij}) = Id \text{ for } 2 < s \leq 3\Big) \\ &= \quad Id. \end{aligned}$$

By the triangular inequality we have

$$\xi_s(X(t_1,t_2),X) \leq \xi_s(X(t_1,t_2),Q(t_1,t_2)) + \xi_s(Q(t_1,t_2),X).$$

Furthermore, for $1 < s \leq 3$, $\mathbb{E}\Big[Q(t_1,t_2)\Big] = \mathbb{E}(X) = 0$, then

$$\xi_s(Q(t_1,t_2),X) \leq \tfrac{\Gamma(1+\alpha)}{\Gamma(1+s)}\Big[2m\kappa_s(Q(t_1,t_2),X)+(2\kappa_s(Q(t_1,t_2),X))^{\alpha}\Big(\min\left(||Q(t_1,t_2)||_s, ||X||_s\right)\Big)^{1-\alpha}\Big].$$

As κ_s and ℓ_s are topological equivalent, then to get $\kappa_s(Q(t_1,t_2),X) \to 0$, it is sufficient to prove that $\Lambda_s(t_1,t_2) := \ell_s(Q(t_1,t_2),X) \to 0$. On the other hand

$$\begin{aligned} \Lambda_s(t_1,t_2) \quad &\leq \quad \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}||\widetilde{A}_{ij}(t_1,t_2) - A^*_{ij}||_s||X||_s + \sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\Big|\Big|\mathbf{1}_{\{(t_1,t_2)\in B(0,\tau_1)\}}||\widetilde{A}_{ij}(t_1,t_2)||_{op}\Big|\Big|_s \\ &\quad \times(\sup_{(u,v)\in[0,t_1]\times[0,t_2]}||X(u,v)|| + ||X||_s) + ||\widetilde{b}(t_1,t_2) - b^*||_s. \end{aligned}$$

Thus by (6) and (8), $\Lambda_s(t_1,t_2) \to 0$, it follows that $\xi_s(Q(t_1,t_2),X) \to 0$. On the other hand, there exists some positive real c such that

$$\xi_s(X(t_1,t_2),X) \le c\Big(||X(t_1,t_2)||_s^s + ||X||_s^s\Big) \le c\Big(\sup_{(u,v)\in[0,t_1]\times[0,t_2]} ||X(u,v)||_s^s + ||X||_s^s\Big) < \infty.$$

The last inequality means that $\Big(\xi_s(X(t_1,t_2),X)\Big)_{t_1,t_2}$ is bounded.
Let $a(t_1,t_2) = \xi_s(X(t_1,t_2),Q(t_1,t_2))$ and $L = \limsup \xi_s(X(t_1,t_2),X)$ thus for all $\epsilon > 0$ there exists $\tau_2 > 0$, such that:

$$\forall\,(t_1,t_2) \in B_+^c(0,\tau_2),\ \xi_s(X(t_1,t_2),X) \le L + \epsilon,$$

we have

$$\begin{aligned}
a(t_1,t_2) &\le \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} \mathbb{E}\Big[||\widetilde{A}_{ij}(t_1,t_2)||_{op}^s \xi_s(X_{ij}(T_{ij}(t_1,t_2),X)\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+^c(0,\tau_1)\}}\Big] \\
&\le \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} \mathbb{E}\Big[||\widetilde{A}_{ij}(t_1,t_2)||_{op}^s \xi_s(X_{ij}(T_{ij}(t_1,t_2),X)\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+^c(0,\tau_1)\cap B_+(0,\tau_2)\}}\Big] \\
&\quad + \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} E\Big[||\widetilde{A}_{ij}(t_1,t_2)||_{op}^s \xi_s(X_{ij}(T_{ij}(t_1,t_2),X)\mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+^c(0,\tau_1)\cap B_+^c(0,\tau_2)\}}\Big] \\
&\le \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} \mathbb{E}[||\widetilde{A}_{ij}(t_1,t_2)||_{op}^s \mathbf{1}_{\{T_{ij}(t_1,t_2)\in B_+^c(0,\tau_1)\cap B_+(0,\tau_2)\}}] \sup_{B_+^c(0,\tau_1)} \xi_s(X(u,v),X) \\
&\quad + (L+\epsilon)\sum_{i=1}^{K_1}\sum_{j=1}^{K_2} \mathbb{E}\Big[||\widetilde{A}_{ij}(t_1,t_2)||_{op}^s\Big].
\end{aligned}$$

When $||(t_1,t_2)|| \to \infty$, we get $L \le (L+\epsilon)\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}\Big(||A_{ij}^*||_{op}^s\Big)$ for all $\epsilon > 0$. In particular, if we choose $\epsilon = \dfrac{L\Big(1-\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}\Big(||A_{ij}^*||_{op}^s\Big)}{2\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\mathbb{E}\Big(||A_{ij}^*||_{op}^s\Big)}$, necessarily $L = 0$. $\square$

4 Limit in distribution

Let $m(x,y)$ and $\sigma^2(x,y)$ be respectively the mean and the variance of the total number $N(x,y)$ of the rectangles. Aguech and Ilji [1] prove that for all positif number a we have

$$\begin{aligned}
m(x^{\theta_1}, ax^{\theta_2}) &= \frac{Kax^{\theta_1+\theta_2}}{\sqrt{\theta_2 \ln x}} + \frac{a\eta(\ln a)x^{\theta_1+\theta_2}}{\ln^{\frac{3}{2}} x} + o(\frac{x^{\theta_1+\theta_2}}{(ln^{\frac{3}{2}} x)}), \\
\sigma^2(x^{\theta_1}, ax^{\theta_2}) &= \frac{\tau(\ln a)a^2x^{2(\theta_1+\theta_2)}}{\ln^3 x} + o\Big(\frac{x^{2(\theta_1+\theta_2)}}{\ln^3 x}\Big),
\end{aligned}$$

where η and τ are given for all $\alpha \in \mathbb{R}$ by:

$$\eta(\alpha) = \frac{K}{2\sqrt{\theta_2}}\Big[c_0 - \frac{c_1}{\theta_1} + \frac{c_2}{\theta_2^2} + 2\frac{c_2}{\theta_1^2} + \frac{(1-\alpha)}{\theta_2}(1 + c_1 - 2\frac{c_2}{\theta_1}) + \frac{c_2(1-\alpha)^2}{\theta_2^2}\Big],$$

$$\tau(\alpha) = \frac{A_1 + 2A_2\alpha + A_3\alpha^2}{1-\Phi(2,2)} + \frac{2(A_2+\alpha A_3)\gamma + A_3\rho}{[1-\Phi(2,2)]^2} + \frac{A_3\gamma^2}{[1-\Phi(2,2)]^3}.$$

Theorem 2. *Let $N(x,y)$ be the total number of rectangles obtained at the end of the fragmentation process such that*

$$A_3\gamma^2 \leq A_1[1-\Phi(2,2)]^2. \tag{10}$$

*Then under conditions (**A**)and (**B**), the normalized random variable $N^*(x^{\theta_1}, x^{\theta_2}) = \frac{N(x^{\theta_1},x^{\theta_2}) - m(x^{\theta_1},x^{\theta_2})}{\sqrt{\sigma^2(x^{\theta_1},x^{\theta_2})}}$ converges in distribution when $x \to \infty$ to the only solution of the equation*

$$N^* \overset{\mathcal{D}}{=} \sum_{i=1}^{2}\sum_{j=1}^{2} A^*_{ij} N^*_{i,j} + B^* \quad \textit{where}$$

$$A^*_{ij} = U_i V_j \sqrt{\frac{\tau(\ln(V_j) - \frac{\theta_2}{\theta_1}\ln(U_i))}{\tau(0)}} \quad \textit{and} \quad B^* = \frac{-\widetilde{L_1}}{\sqrt{\tau(0)}}.$$

Proof. $N^*(x^{\theta_1}, x^{\theta_2})$ satisfies the following equation in distribution

$$N^*(x^{\theta_1}, x^{\theta_2}) \overset{\mathcal{D}}{=} \sum_{i=1}^{2}\sum_{j=1}^{2} A_{ij}(x^{\theta_1}, x^{\theta_2}) N^*(x^{\theta_1}U_i, x^{\theta_2}V_j) + B(x^{\theta_1}, x^{\theta_2})$$

where $A_{ij}(x,y) = \sqrt{\frac{\sigma^2(xU_i, yV_j)}{\sigma^2(x,y)}}$ and $B(x,y) = \frac{1 - m(x,y) + \sum_{i=1}^{2}\sum_{j=1}^{2} m(xU_i, yV_j)}{\sqrt{\sigma^2(x,y)}}$.

Using Theorem 2 of [1], we have:

$$A_{ij}(x^{\theta_1}, x^{\theta_2}) = A^*_{ij} + o(1) \text{ and } B(x^{\theta_1}, x^{\theta_2}) = B^* + o(1).$$

The random vector

$$\Big(A_{11}(x^{\theta_1}, x^{\theta_2}),\ A_{12}(x^{\theta_1}, x^{\theta_2}),\ A_{21}(x^{\theta_1}, x^{\theta_2}),\ A_{22}(x^{\theta_1}, x^{\theta_2}),\ B(x^{\theta_1}, x^{\theta_2})\Big)$$

converges in ℓ_2 to the random vector $\Big(A^*_{11},\ A^*_{12},\ A^*_{21},\ A^*_{22},\ B^*\Big)$. Furthermore, the assumption (10) ensures that $\sum_{i=1}^{2}\sum_{j=1}^{2} \mathbb{E}(||A^*_{ij}||^2_{op}) < 1$, thus it suffices to apply Theorem (1) we conclude the requested result. □

References

[1] Aguech, R. and Ilji, S. (2019). Non-homogenous bivariate fragmentation process: Mean and variance, *https://arxiv.org/submit/2697617/view.*

[2] Aguech, R. (2008). The size of random fragmentation intervals, *Discrete Mathematics & Theoretical Computer Science,* AI, 519-530.

[3] Beysens, D., Campi, X. and Pefferkorn, E. (1995). Fragmentation Phenomena, Les Houches Series, *World Scientific Publishing Company*, Singapore.

[4] Corduneanu, C. (1991). Integral Equations and Applications. *Cambridge, England: Cambridge University Press.*

[5] Dean, D.S. and Majumdar, S.N. (2002). Phase transition in a random fragmentation problem with applications to computer science, *Journal of Physics A: Mathematical and General* , 35 (32), L501-L507.

[6] Neininger, R. and Rüschendorf, L. (2004). A general limit theorem for recursive algorithms and combinatorial structures, *Annals of Applied Probability*, 14(1), 378-418.

[7] Neininger, R. (2001). On a Multivariate Contraction Method for Random Recursive Structures with Applications to Quicksort, *Random Structures Algorithms*, 19, 498-524.

[8] Svante, J. and Neininger, R. (2008). The size of random fragmentation trees, *Probability Theory and Related Fields*, 142, 399-442.

[9] Sibuya, M. and Itoh, Y. (1987). Random sequential bisection and its associated binary tree, *Annals of the Institute of Statistical Mathematics*, 39, 69-84.

[10] Powell, E.O. (1955). Some features of generation times of individual bacteria, *Biometrika*, 42, 16-44.

[11] Zolotarev, V. M. (1976). Approximation of the distributions of sums of independent random variables with values in infinite dimensional spaces, *Theory of Probability and Its Applications*, 22(4), 721-733.

[12] Zolotarev, V. M. (1977). Ideal metrics in the problem of approximating distributions of sums of independent random variables, *Theory of Probability and Its Applications*, 22(3), 433-449.