Moments of a Non-homogenous Bivariate Fragmentation Process Using Integral Equations Tools

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Abstract

In this paper we are interested in two-dimensions fragmentation process that describes the evolution of an object having a rectangular shape. We focus on the non-homogenous fragmentation process where we break a rectangle according to a distribution that depends on its dimensions. Via Integral equations tools, we compute the mean and the variance of the distribution of the total number of the sub-rectangles obtained at the end of the process.

1 Introduction

Random fragmentation applies in several fields, such as biology [17], physics [3], computer sciences [7, 18], etc. The fragmentation process has been the interest of many authors since the works of Brennan [6] and Sibuya [18]. Afterwards, Janson [20] focused on a non-homogenous fragmentation of an interval of length x, i.e the fragmentation of the interval depends on x, he studied the case when the fragmentation probability is $p(x) = \mathbf{1}_{\{x\geq 1\}}$, he gave the asymptotic behavior of the total number of fragments obtained at the end of the process. More recently, Aguech [1] studied the fragmentation of an interval, he considered the case when the fragmentation probability is given by $p(x) = 1 - e^{-x}$ where x is length of the interval, he described the asymptotic distribution of the total number of the obtained fragments.

In the literature, it is always common to consider an interval of length x at the beginning. In this paper we study a fragmentation process in two dimensions. We start with a rectangle of dimensions x and y. We suppose that with probability

$p(x,\ y) = 1_{\{x\geq 1,\ y\geq 1\}}$ we decide to cut, independently and uniformly, x into b slides and y into b' slides where b and b' are two nonnegative integers, with complementary probability we decide to let them definitively stable. Let $\mathbf{U} = (U_1,\ \cdots,\ U_b)$ and $\mathbf{V} = (V_1,\ \cdots,\ V_b')$ be two independent random vectors such that the lengths of the sub-pieces obtained by cutting x are respectively: $U_1 x, \cdots, U_b x$ and the lengths of the sub-pieces obtained by cutting y are respectively: $V_1 x, \cdots, V_{b'} x$. We repeat recursively and independently this procedure on all the sub-rectangles with new and independent copies of $\mathbf{U}$ and $\mathbf{V}$. The figure below illustrates an example of fragmentation of a rectangle when $b = 3$ and $b' = 2$.

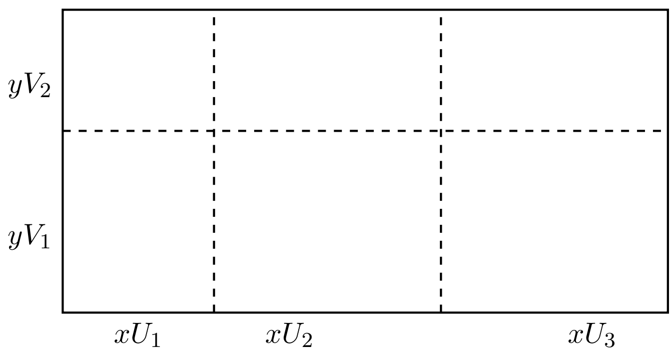

Figure 1: A cut rectangle at step 1 when $b = 3,\ b' = 2$.

Note that the process stops almost surely after a finite number of steps and it leaves a finite number of rectangles all stable, we note them by $N(x, y)$. A fundamental method to study the behavior of $N(x, y)$ is the multivariate renewal theory. Such a model has been studied by numerous authors, namely Mode, Hunter [10, 11], Mallora, Omey and Santos [12] and Omey [16]. Smith [19] developed the renewal theory, in particular the renewal density, for one dimension. Afterwards, these results have been extended by Mode [13] who studied the case of a bi-dimensional renewal process. Unfortunately, the previous results are not sufficient to compute the second moment of a non-homogenous fragmentation process. In this note, we prove a renewal theorem which is efficient for computing the variance.
The paper is organized as follows: In Section (2), we give the notations and the assumptions that we need in all the paper, we prove that the mean of $N(x, y)$ satisfies a bi-renewal equation for all integers $b,\ b' \geq 2$. In Section (3), we prove Theorem (1) which gives under some assumptions the asymptotic of the bi-renewal density function. In Section (4), we take $b = b' = 2$ and we show that the variance of $N(x, y)$ satisfies a bi-renewal equation, via Theorem (1) we determine the approximations of the mean and the variance of $N(x, y)$.

2 Preliminaries

In this section, we describe the model of the fragmentation in the general case i.e. b and b' are two arbitrary nonnegative integers. We define the notations and we prove that the mean of the total number of the rectangles obtained at the end of the process is solution of a bi-renewal equation.

2.1 Description of the model:

We consider a rectangle of sides of lengths x and y. We fix b and b' two integers and let $\mathbf{U} = (U_1 \cdots, U_b)$ and $\mathbf{V} = (V_1 \cdots, V_{b'})$ be two independent random vectors such that $\sum_{i=1}^{b} U_i = 1$ and $\sum_{j=1}^{b'} V_j = 1$. The fragmentation process is described as follows:

- If $x \geq 1$ and $y \geq 1$, we cut the rectangle according to the random vectors $\mathbf{U} = (U_1, \cdots, U_b)$ for the dimension x and $\mathbf{V} = (V_1, \cdots, V_{b'})$ for the dimension y.

- If $x < 1$ or $y < 1$, we decide to leave the rectangle definitively stable.

- Recursively, we repeat independently at each step this procedure on all sub-rectangles with independent copies of $\mathbf{U}$ and $\mathbf{V}$.

Obviously, our fragmentation process can be considered as a random tree where the root is the first rectangle, the internal nodes are the unstable rectangles and the leaves are the stable rectangles. Let $N(x,\ y)$ be the total number of pieces in the fragmentation tree. Note that if $x < 1$ or $y < 1$, $N(x, y) = 1$. We assume that we start with a rectangle with dimensions greater than 1, then $N(x,\ y)$ satisfies the following equation:

$$N(x,\ y) \stackrel{\mathcal{D}}{=} 1 + \sum_{i=1}^{b} \sum_{j=1}^{b'} N_{i,\,j}(xU_i,\ yV_j),$$

where for $1 \leq i \leq b$ and $1 \leq j \leq b'$, $N_{i,\,j}(.,.)$ are independent copies of $N(.,.)$.
Let us define

$$m(x, y) = \mathbb{E}[N(x,\ y)],\ m_*(x,\ y) = m(e^x,\ e^y),\ X_i = -\ln(U_i),\ Y_j = -\ln(V_j),$$

$\mu_{i,\,j}$ is the joint distribution of the random vector (X_i, Y_j) and $\mu = \sum_{i=1}^{b} \sum_{j=1}^{b'} \mu_{i,\,j}$.

The bivariate function $m_*(t_1, t_2)$ satisfies the following equation:

$$m_*(t_1,\ t_2) = 1 + (m_* * \mu)(t_1, t_2).$$

Note that μ is not a probability measure, we define so the probability measure ν by

$$d\nu(t_1, t_2) = e^{-(t_1+t_2)} d\mu(t_1, t_2). \tag{1}$$

The function $M_*(t_1, t_2) = e^{-(t_1+t_2)} m_*(t_1, t_2)$ satisfies immediately the bivariate renewal equation:

$$M_*(t_1, t_2) = f(t_1,\ t_2) + (M_* * \nu)(t_1,\ t_2) \text{ where } f(t_1,\ t_2) = e^{-(t_1+t_2)}. \tag{2}$$

2.2 The class of distributions $\mathbf{J_2}$

Let $M > 0$ and $k \in \{1,\ 2\}$ we define

$$A_k = \{t_k : |t_k| < M\} \quad \text{and } A_k^{\mathcal{C}} \text{ its complementary set.}$$

Let $\mathbf{X}$ be a random vector following some distribution ω and let Ψ be its associated characteristic function i.e

$$\Psi(t_1,\ t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{it_1 s_1 + it_2 s_2} d\omega(s_1, s_2),$$

we shall called that ω belongs to the class J_2 if Ψ satisfies: for some nonnegative reals $\alpha_1,\ \alpha_2$ and c, the following conditions

- $|\Psi(t_1,t_2)| \leq \frac{c}{|t_1|^{\alpha_1}}$ for all $(t_1,t_2) \in A_1^{\mathcal{C}} \times A_2$,
- $|\Psi(t_1,t_2)| \leq \frac{c}{|t_2|^{\alpha_2}}$ for all $(t_1,t_2) \in A_1 \times A_2^{\mathcal{C}}$,
- $|\Psi(t_1,t_2)| \leq \frac{c}{|t_1|^{\alpha_1}|t_2|^{\alpha_2}}$ for all $(t_1,t_2) \in A_1^{\mathcal{C}} \times A_2^{\mathcal{C}}$.

The sub-class J_2^* of J_2 consists on all the distributions that belong to J_2 and having finite mean vectors and definite-positive correlation matrices.

2.3 Notations:

For the rest the we will need the following notations

- $||.||$ is an arbitrary norm on $\mathbb{R}_+^2$,
- for all $t_1,\ t_2 \in \mathbb{R}_+,\ \Phi(t_1,t_2) = \sum_{i=1}^{b}\sum_{j=1}^{b'} \mathbb{E}(U_i^{t_1} V_j^{t_2})$,
- $\theta_1 = -\sum_{i=1}^{b} \mathbb{E}\big[U_i \ln(U_i)\big],\ \theta_2 = -\sum_{j=1}^{b'} \mathbb{E}\big[V_j \ln(V_j)\big]$
- $\gamma = \sum_{i=1}^{b}\sum_{j=1}^{b'} \mathbb{E}\Big[U_i^2 V_j^2 \Big(\ln(V_j) - \frac{\theta_2}{\theta_1}\ln(U_i)\Big)\Big]$,
- $\rho = \sum_{i=1}^{b}\sum_{j=1}^{b'} \mathbb{E}\Big[U_i^2 V_j^2 \Big(\ln(V_j) - \frac{\theta_2}{\theta_1}\ln(U_i)\Big)^2\Big]$.

For a random vector $\mathbf{X} = (X^{(1)},\ X^{(2)})$ with mean $\boldsymbol{\lambda} = \Big(\lambda_1, \lambda_2\Big)$ and with definite positive covariance matrix Σ and with finite moment of order 3, we denote by:

- $|\Sigma|$ the determinant of Σ and Σ^{-1} its inverse matrix,
- $\sigma_k^2 = Var(X^{(k)}),\ k = 1,\ 2,\ K = \frac{\lambda_1}{\sqrt{2\pi|\Sigma|(\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda})}}$,

- for $k=1,2,\ a_k=\frac{\mathbb{E}\left[(X^{(k)}-\lambda_k)^3\right]}{\sigma_k^4}$,

- $c_0=\frac{-1-2(a_1\lambda_1+a_2\lambda_2)}{4\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}+\frac{1}{2(\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda})^2}\left(\frac{a_1\lambda_1^3}{\sigma_1^2}+\frac{a_2\lambda_2^3}{\sigma_2^2}\right)$,

- $c_1=\lambda_2 a_2-1+\frac{\lambda_2^2-a_1\lambda_1\lambda_2^2-2a_2\lambda_2^3}{\sigma_2^2\boldsymbol{\lambda}\Sigma^{-1}\boldsymbol{\lambda}}+\frac{\lambda_2^2}{\sigma_2^2(\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda})^2}\left(\frac{a_1\lambda_1^3}{\sigma_1^2}+\frac{a_2\lambda_2^3}{\sigma_2^2}\right)$,

- $c_2=\frac{\lambda_2^2}{\sigma_2^2}\left(-1+\frac{\lambda_2^2}{\sigma_2^2\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}\right)$.

In particular, if the distribution of $\mathbf{X}$ is given by ν (1), then $\lambda_1=\theta_1$ and $\lambda_2=\theta_2$.

- For all $i,\ j\in\{1,2\},\ \upsilon_{ij}=-\ln(V_j)+\frac{\theta_2}{\theta_1}\ln(U_i)$,

  $$L_1(U_i,V_j)=\frac{K}{2\theta_2^{\frac{3}{2}}}\left[(1+c_1-2\frac{c_2}{\theta_1}+2\frac{c_2}{\theta_2})\upsilon_{ij}+\frac{c_2}{\theta_2}\upsilon_{ij}^2\right] \text{ and } L_2(U_i,V_j)=\frac{c_2K\upsilon_{ij}}{\theta_2^{\frac{5}{2}}},$$

- $\widetilde{L}_1=\frac{K\sum\limits_{i=1}^{2}U_i\ln(U_i)}{2\theta_1\sqrt{\theta_2}}-\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2}U_iV_jL_1(U_i,V_j),\ \widetilde{L}_2=\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2}U_iV_jL_2(U_i,V_j)$,

- $A_1=\mathbb{E}\left[\widetilde{L}_1^2\right],\ A_2=\mathbb{E}\left[\widetilde{L}_1\widetilde{L}_2\right]$ and $A_3=\mathbb{E}\left[\widetilde{L}_2^2\right]$.

2.4 Remarks

- if $\min(t_1,t_2)>1$ we have $\Phi(t_1,t_2)<1$,
- the characteristic function Ψ of a random vector with distribution ν (1) can be written in terms of Φ as follows: $\Psi(t_1,t_2)=\Phi(1-it_1,\ 1-it_2)$,
- the random variables U_i and V_j belong to the interval $[0,1]$ for all $i\in\{1,\cdots,b\},\ j\in\{1,\cdots,b'\}$, in other words ν (1) is defined on $[0,\infty[\times[0,\infty[$.

2.5 Assumptions

We will need these assumptions later:

- **(A):** Each U_i and V_j is an absolutely continuous random variable such that $\sum\limits_{i=1}^{b}\mathbb{E}\left[U_i|\ln(U_i)|^3\right]<\infty$ and $\sum\limits_{i=1}^{b'}\mathbb{E}\left[V_j|\ln(V_j)|^3\right]<\infty$,

- **(B):** the probability measure ν given by (1) belongs to the set J_2^*.

3 Bi-renewal Theory

The one dimension renewal theory is well studied by Feller [9], Blackwell [4, 5] and Asmussen [2]. The next Lemma (1) gives an extension of the renewal Theorem (Theorem 2.4 of [2]) for two dimensions. For a measure ω we denote by ω^{*n} the n-fold convolution of ω with itself. A very useful tool to study renewal theory can be found on [8].

Lemma 1. *Let ω be a finite measure and g be a bounded function on the compacts of $\mathbb{R}_+ \times \mathbb{R}_+$. Consider the bi-dimensional renewal equation*

$$F(t_1, t_2) = g(t_1, t_2) + (F * \omega)(t_1, t_2). \tag{3}$$

Then the function $h(t_1, t_2) := \sum_{n=0}^{\infty} \omega^{*n}(t_1, t_2)$ is well defined, furthermore Equation (3) admits a unique solution, bounded on the compacts of $\mathbb{R}_+^2$ given by $(h * g)$.

Proof. Let $\widehat{\omega}$ be the Laplace transform of ω i.e

$$\widehat{\omega}(t_1, t_2) = \int_0^{+\infty} \int_0^{+\infty} e^{-(t_1 s_1 + t_2 s_2)} d\omega(s_1, s_2).$$

As $\lim_{||(t_1,t_2)|| \to \infty} \widehat{\omega}(t_1, t_2) = 0$, there exist $\alpha \in]0, 1[$ and $a, \ b \in \mathbb{R}_+$ such that $\widehat{\omega}(a, b) < \alpha$. Then for all $t_1 > 0$ and $t_2 > 0$ we have

$$\begin{aligned} \omega^{*n}(t_1, t_2) &= \int_0^{t_1} \int_0^{t_2} d\omega^{*n}(s_1, s_2) \le \int_0^{t_1} \int_0^{t_2} e^{a(t_1 - s_1) + b(t_2 - s_2)} d\omega^{*n}(s_1, s_2) \\ &= e^{at_1 + bt_2} \widehat{\omega^{*n}}(a, b) = e^{at_1 + bt_2} \widehat{\omega}^n(a, b) \le e^{at_1 + bt_2} \alpha^n, \end{aligned}$$

therefore $h(t_1, t_2) \le \frac{e^{at_1 + bt_2}}{1 - \alpha} < \infty$. Since the function g is bounded by some positive constant M we conclude

$$\begin{aligned} |(g * h)(t_1, t_2)| &\le M \sum_{n=0}^{\infty} \int_0^{t_1} \int_0^{t_2} d\omega^{*n}(s_1, s_2) \\ &\le M \frac{e^{at_1 + bt_2}}{1 - \alpha}. \end{aligned}$$

This means that $g * h$ is well defined and bounded on all the compacts of $\mathbb{R}_+^2$. On the other hand, $g * h$ satisfies Equation (3), therefore to prove that is the unique solution of Equation (3) we assume that there exist such two solutions F_1 and F_2 bounded on the compacts of $\mathbb{R}_+^2$. In other words, we suppose that

$$F_1(t_1, t_2) = g(t_1, t_2) + (F_1 * \omega)(t_1, t_2) \tag{4}$$

and

$$F_2(t_1, t_2) = g(t_1, t_2) + (F_2 * \omega)(t_1, t_2). \tag{5}$$

The difference function $G = F_1 - F_2$ satisfies

$$G = (G * \omega).$$

Then for all $n \geq 1$, we have $G = (G * \omega^{*n})$. This implies that, for all $n \geq 1$,

$$
\begin{aligned}
|G(t_1, t_2)| &= \Big| \int_0^{t_1} \int_0^{t_2} G(t_1 - s_1, t_2 - s_2) d\omega^{*n}(s_1, s_2) \Big| \\
&\leq (|\sup G|_{[0,t_1]\times[0,t_2]}) \omega^{*n}(t_1, t_2) \ (G \text{ is bounded on } [0, t_1] \times [0, t_2]) \\
&\leq \Big(|\sup G|_{[0,t_1]\times[0,t_2]} \Big) e^{at_1 + bt_2} \alpha^n, \quad \alpha \in]0, 1[.
\end{aligned}
$$

As a consequence, we have $G(t_1, t_2) = \lim_{n\to\infty} (G * \omega^{*n})(t_1, t_2) = 0$. □

Remarks:

1. If ω is a probability measure, Equation (3) is called proper bi-renewal equation.
2. By Lemma (1), the solution of the proper renewal Equation (2) is given by

$$M_*(t_1,\ t_2) = \sum_{n=0}^{\infty} (f * \nu^{*n})(t_1,\ t_2) = \sum_{n=0}^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-(t_1 - s_1 + t_2 - s_2)} d\nu^{*n}(s_1, s_2),$$

   where ν is the probability measure given by Equation (1).

Under different assumptions, Hunter [11], Mode [13], Mallor, Omey and Santos [16] described the asymptotic behavior of $H := \sum_{n=0}^{\infty} \nu^{*n}$. Namely, there principal results are:

- $\lim_{t\to+\infty} \frac{H(tx,\ tx)}{t} = \min(\frac{x}{\theta_1},\ \frac{y}{\theta_2})$ for all $x,\ y \in \mathbb{R}$ (Theorem (4.1) of [16]).
- If we consider a vector $\mathbf{X}$ following the bivariate exponential distribution, in other words the probability density function of $\mathbf{X}$ is given by

$$u(t_1, t_2) = \frac{1}{\theta_1 \theta_2 (1 - \varrho)} \exp\Big(- \frac{\theta_1^{-1} t_1 + \theta_2^{-1} t_2}{1 - \varrho} \Big) I_0\Big(\frac{2(\varrho \theta_1^{-1} \theta_2^{-1} t_1 t_2)^{1/2}}{1 - \varrho} \Big)$$

  where $\varrho \in [0, 1[$ and I_0 is the modified Bessel function of the first kind of order zero, then as t goes to infinity [11]

$$H(\theta_1 t,\ \theta_2 t) = t - \sqrt{\frac{t(1 - \varrho)}{\pi}} + o(\sqrt{t}).$$

- Hunter [11] proved that in the case of the bivariate exponential distribution we have as t_1 and t_2 tend to $+\infty$ with $\frac{t_2}{t_1}$ tends to some constant K:

$$\frac{H(t_1,\ t_2)}{\sqrt{t_1 t_2}} \longrightarrow \min\Big(\frac{\sqrt{K}}{\theta_1},\ \frac{1}{\theta_2 \sqrt{K}} \Big).$$

Unfortunately, all the previous results are insufficient to give the behavior of $N(x, y)$ for our model. In fact, they give the approximation of H along the line $\{(tx, ty),\ t \in \mathbb{R}\}$ where x and y are two constants, but they don't give any information about the asymptotic behavior of the renewal density. For this reason, we give in the next Lemma a refined version of Mode's Theorem [13].

Lemma 2. *Let* $(\boldsymbol{X}_n)_{n\in\mathbb{N}} = \left(X_n^{(1)}, X_n^{(2)}\right)_{n\in\mathbb{N}}$ *be a sequence of independent and identically distributed (i.i.d) centered random vectors whose distribution function belongs to* J_2^* *such that* $Cov(\boldsymbol{X}) = Id$, $\mathbb{E}[||\mathbf{X}_1||^3] < +\infty$. *Let* h_n *be the probability density function of the random vector* $Z_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{X}_i$, *then we have for all* $(x_1, x_2) \in \mathbb{R}_+^2$, *as* n *goes to infinity*

$$\sqrt{n}\Big[h_n(x_1,x_2) - \frac{e^{-\frac{1}{2}(x_1^2+x_2^2)}}{2\pi}\Big] = \frac{e^{-\frac{1}{2}(x_1^2+x_2^2)}}{12\pi}\Big[x_1(x_1^2-3)\mathbb{E}(X^{(1)^3}) + x_2(x_2^2-3) \times \mathbb{E}(X^{(2)^3})\Big] + o(\frac{1}{\sqrt{n}}).$$

Proof. The techniques used to prove this lemma are similar used in the case of one dimension, for the convenience of the reader we refer to Theorem 1 of [19] and Lemma 2.1 of [13]. □

Theorem 1. *Let* $(\boldsymbol{X}_n)_{n\in\mathbb{N}} = \left(X_n^{(1)}, X_n^{(2)}\right)_{n\in\mathbb{N}}$ *be a sequence of i.i.d and absolutely continuous random vectors in* $\mathbb{R}_2^+$ *with mean* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, *covariance matrix* $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$ *and with common distribution* ω. *Let* h_n *be the probability density function of the random vector* $T_n = \sum_{i=1}^{n}\boldsymbol{X}_i$ *and let* $h(x_1,x_2) = \sum_{n=1}^{\infty} h_n(x_1,x_2)$. *Assume that* $\omega \in J_2^*$ *and that*

$$\lim_{x_k\to+\infty} x_k^{\frac{3}{2}} h_n(x_1,x_2) = 0 \quad \textit{for all } n \geq 1 \textit{ and } k = 1,2. \tag{6}$$

Then for an arbitrary nonnegative constant B *we have as* $x_1,\ x_2 \to \infty$ *such that* $\mid \frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2} \mid \leq B$:

$$\sqrt{x_2}h(x_1,x_2) = K + \frac{C(x_1,x_2)}{x_1} + o(\frac{1}{x_1})$$

where

$$C(x_1,x_2) = \frac{K\lambda_1}{2}\Big[c_0 + c_1(\frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2}) + c_2(\frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2})^2\Big] \tag{7}$$

and K *and* c_i, $i = 0,1,2$ *are given in paragraph (2.3).*

Proof. We denote by $\mathbf{x} = (x_1, x_2)$ and by

$$F(\mathbf{x}) = \frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2}. \tag{8}$$

Let $\mathbf{Y} = (Y^{(1)}, Y^{(2)})$ where $Y^{(k)} = \frac{X^{(k)}-\lambda_k}{\sigma_k}$ for $k = 1,\ 2$ and let $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ be a sequence of i.i.d random vectors following the same distribution as $\mathbf{Y}$. The random vector $\mathbf{Y}$ is centered and its covariance matrix $Cov(\mathbf{Y}) = Id$, then by Lemma (2) the probability density function f_n of the random vector

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbf{Y}_k = \Big(\frac{\sum_{k=1}^{n} X_k^{(1)} - n\lambda_1}{\sqrt{n}\sigma_1}, \frac{\sum_{k=1}^{n} X_k^{(2)} - n\lambda_2}{\sqrt{n}\sigma_2}\Big)$$

satisfies:

$$\lim_{n\to\infty} \sqrt{n}\Big[f_n(\mathbf{x}) - \frac{e^{-\frac{1}{2}(x_1^2+x_2^2)}}{2\pi}\Big] = \frac{e^{-\frac{1}{2}(x_1^2+x_2^2)}}{12\pi}\Big[x_1(x_1^2-3)E({Y^{(1)}}^3) + x_2(x_2^2-3)E({Y^{(2)}}^3)\Big].$$

Moreover, we have:

$$n\sigma_1\sigma_2 h_n(\mathbf{x}) = f_n\Big(\frac{x_1 - n\lambda_1}{\sqrt{n}\sigma_1}, \frac{x_2 - n\lambda_2}{\sqrt{n}\sigma_2}\Big). \tag{9}$$

Let $K_n(\mathbf{x}) = nh_n(\mathbf{x}) - \frac{e^{-\frac{1}{2n}(\mathbf{x}-n\boldsymbol{\lambda})'\Sigma^{-1}(\mathbf{x}-n\boldsymbol{\lambda})}}{2\pi|\Sigma|^{\frac{1}{2}}} - \frac{e^{-\frac{1}{2n}(\mathbf{x}-n\boldsymbol{\lambda})'\Sigma^{-1}(\mathbf{x}-n\boldsymbol{\lambda})}\sum_{k=1}^{2} a_k(x_k - n\lambda_k)\Big[(\frac{x_k-n\lambda_k}{\sqrt{n}\sigma_k})^2 - 3\Big]}{12\pi n|\Sigma|^{\frac{1}{2}}}$,
we conclude then that

$$\lim_{n\to+\infty} K_n(\mathbf{x}) = 0.$$

For $k = 1,\ 2$ and $r \in [0, 2]$, define the functions

$$V_n(\mathbf{x}) \quad = \quad n\Big(\frac{x_k - n\lambda_k}{\sqrt{n}\sigma_k}\Big)^r h_n(\mathbf{x}), \tag{10}$$

$$W_n(\mathbf{x}) \quad = \quad \Big(\frac{x_k - n\lambda_k}{\sqrt{n}\sigma_k}\Big)^r \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\Big[-\frac{1}{2n}(\mathbf{x} - n\boldsymbol{\lambda})'\Sigma^{-1}(\mathbf{x} - n\boldsymbol{\lambda})\Big]. \tag{11}$$

By Theorem 2.1 of Mode [13], we have for all $r \in [0, 2]$ and $k = 1, 2$

$$\lim_{n\to+\infty} \Big[V_n(\mathbf{x}) - W_n(\mathbf{x})\Big] = 0.$$

We conclude that for $r = 2,\ k = 1$ there exist a positive constant D and a nonnegative integer $n_0 \geq 1$ such that for all $n \geq n_0$

$$h_n(\mathbf{x}) \leq \frac{D}{(x_1 - n\lambda_1)^2}. \tag{12}$$

Under assumption (6), for all $\epsilon = \epsilon(n) > 0$ there exists a constant $A > 0$ such that for all $|x_1| \geq A$, we have: $x_1^{\frac{3}{2}} h_n(\mathbf{x}) \leq \epsilon$. Furthemore, as $|F(\mathbf{x})| \leq B$ (where $F(\mathbf{x})$ is given by (8)), $\sqrt{\frac{x_2}{x_1}}$ is increased by some constant M, accordingly

$$\sqrt{x_2}\sum_{n=1}^{n_0-1} h_n(\mathbf{x}) \leq \frac{1}{x_1}\sqrt{\frac{x_2}{x_1}}\sum_{n=1}^{n_0-1}\epsilon \leq \frac{M}{x_1}\sum_{n=1}^{n_0-1}\epsilon := \frac{\epsilon'(n_0)}{x_1}.$$

Hence, when $||x||$ tends to infinity with the condition $|F(\mathbf{x})| \leq B$, we have

$$\sqrt{x_2}\sum_{n=1}^{n_0-1} h_n(\mathbf{x}) = o(\frac{1}{x_1}). \tag{13}$$

For x_1 positif and x_2 positif, we define two real numbers z_1 and z_2 by:

$$z_k = \sqrt{\frac{\lambda_k}{x_k}}\Big(n\lambda_k - x_k\Big), \quad k = 1,\ 2. \tag{14}$$

Equation (14) implies that

$$z_2 = (\frac{\lambda_2}{\lambda_1})^{\frac{3}{2}} \sqrt{\frac{x_1}{x_2}} z_1 + \frac{\lambda_2{}^{\frac{3}{2}}}{\sqrt{x_2}} F(\mathbf{x}),$$

$$n = \frac{x_1}{\lambda_1} + z_1 \frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}} = \frac{x_2}{\lambda_2} + z_2 \frac{\sqrt{x_2}}{\lambda_2^{\frac{3}{2}}} = \frac{x_2}{\lambda_2} + \frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}} z_1 + F(\mathbf{x}). \tag{15}$$

Take $r = 0$ in (10) and (11) and denote, respectively by $V(z_1, \mathbf{x})$ and $W(z_1, \mathbf{x})$ the functions obtained by replacing n by (15) in (10)and (11), we obtain

$$\begin{aligned} V(z_1, \mathbf{x}) &= \sqrt{x_1 \lambda_1^{-1} + z_1 \sqrt{x_1} \lambda_1^{-\frac{3}{2}}} \sqrt{x_2 \lambda_2^{-1} + \sqrt{x_1} \lambda_1^{-\frac{3}{2}} z_1 + F(\mathbf{x})} h_n(\mathbf{x}) \\ & \quad \times \mathbf{1}_{\{n = \frac{x_1}{\lambda_1} + z_1 \frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}}\}}, \\ W(z_1, \mathbf{x}) &= \frac{1}{2\pi \sqrt{|\Sigma|}} \exp\Big[- \frac{1}{2(x_1 \lambda_1^{-1} + z_1 \sqrt{x_1} \lambda_1^{-\frac{3}{2}})} \{ \frac{z_1^2 x_1 \boldsymbol{\lambda}' \Sigma^{-1} \boldsymbol{\lambda}}{\lambda_1^3} + \lambda_2^2 F(\mathbf{x}) \\ & \quad \times (\frac{2\sqrt{x_1} z_1}{\lambda_1^{\frac{3}{2}}} + F(\mathbf{x}) \} \Big]. \end{aligned}$$

We define the following functions

$$\begin{aligned} V^*(z_1, \mathbf{x}) &= [V(z_1, \mathbf{x}) - W(z_1, \mathbf{x})] \prod_{k=1}^{2} \sqrt{\frac{x_k \lambda_k^{-1}}{x_k \lambda_k^{-1} + z_k \sqrt{x_k} \lambda_k^{-\frac{3}{2}}}}, \\ W^*(z_1, \mathbf{x}) &= W(z_1, \mathbf{x}) \prod_{k=1}^{2} \sqrt{\frac{x_k \lambda_k^{-1}}{x_k \lambda_k^{-1} + z_k \sqrt{x_k} \lambda_k^{-\frac{3}{2}}}}, \end{aligned}$$

Using Equations (14), V^* and W^* can be expressed as:

$$\begin{aligned} V^*(z_1, \mathbf{x}) &= \big[V(z_1, \mathbf{x}) - W(z_1, \mathbf{x})\big] \sqrt{\frac{x_1 \lambda_1^{-1}}{x_1 \lambda_1^{-1} + z_1 \sqrt{x_1} \lambda_1^{-\frac{3}{2}}}} \\ & \quad \times \sqrt{\frac{x_2 \lambda_2^{-1}}{x_2 \lambda_2^{-1} + z_1 \sqrt{x_1} \lambda_1^{-\frac{3}{2}} + F(\mathbf{x})}}, \end{aligned} \tag{16}$$

$$\begin{aligned} W^*(z_1, \mathbf{x}) &= W(z_1, \mathbf{x}) \sqrt{\frac{x_1 \lambda_1^{-1}}{x_1 \lambda_1^{-1} + z_1 \sqrt{x_1} \lambda_1^{-\frac{3}{2}}}} \times \\ & \quad \sqrt{\frac{x_2 \lambda_2^{-1}}{x_2 \lambda_2^{-1} + z_1 \sqrt{x_1} \lambda_1^{-\frac{3}{2}} + F(\mathbf{x})}}. \end{aligned} \tag{17}$$

Let a be a nonnegative number such that $N := 2a\sqrt{\frac{\lambda_1 \lambda_2}{x_1}} + 1 \in \mathbb{N}$, define the regular subdivision of the interval $[-a, a)$ by

$$[-a, a) = \bigcup_{j=1}^{N-1} [w_j,\ w_{j+1}) \text{ such that} \, \forall\, 1 \le j \le N-1,\ w_{j+1} - w_j = \sqrt{\frac{\lambda_1 \lambda_2}{x_1}}.$$

Let f and g be two functions defined on $[-a,a)\times\mathbb{R}_+^2$ by:

$$f(y,\mathbf{x}) = \sum_{j=1}^{N-1} V^*(y,\mathbf{x})\mathbf{1}_{[w_j,w_{j+1}]}(y), \tag{18}$$

$$g(y,\mathbf{x}) = \sum_{j=1}^{N-1} W^*(y,\mathbf{x})\mathbf{1}_{[w_j,w_{j+1}]}(y). \tag{19}$$

where V^* and W^* are respectively given by (16) and (17). Recall that for all $j\in\{1,\cdots,N-1\}$,

$$y\in[w_j,w_{j+1}) \text{ implies that } n=\frac{x_1}{\lambda_1}+y\frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}}\in\Big[\alpha_j(x_1),\beta_j(x_1)\Big)\cap\mathbb{N}^*,$$

$$\text{where}\quad \alpha_j(x_1)=\frac{x_1}{\lambda_1}+w_j\frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}}\quad\text{and}\quad \beta_j(x_1)=\frac{x_1}{\lambda_1}+w_{j+1}\frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}}.$$

Let $n_1^j(x_1)$ and $n_2^j(x_1)$ be respectively the smallest and the largest nonnegative integer in the interval $\Big[\alpha_j(x_1),\beta_j(x_1)\Big)$, then we conclude that for $j=1,\cdots,N-1$,

$$y\in[w_j,w_{j+1}) \text{ if and only if } n=\frac{x_1}{\lambda_1}+y\frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}}\in\{n_1^j(x_1),\cdots,n_2^j(x_1))\}. \tag{20}$$

Summing (18) and (19) we obtain

$$\begin{aligned} f(y,\mathbf{x})+g(y,\mathbf{x}) &= \sqrt{\frac{x_1x_2}{\lambda_1\lambda_2}}\sum_{j=1}^{N-1} h_n(\mathbf{x})\mathbf{1}_{\{n=x_1\lambda_1^{-1}+y\sqrt{x_1}\lambda_1^{-\frac{3}{2}}\}}\mathbf{1}_{[w_j,w_{j+1}[}(y)\\ &= \sqrt{\frac{x_1x_2}{\lambda_1\lambda_2}}\sum_{j=1}^{N-1}\sum_{n=n_1^j(x_1)}^{n_2^j(x_1)} h_n(\mathbf{x})\mathbf{1}_{[w_j,w_{j+1}[}(y), \end{aligned}$$

with the convention that the empty sum is equal to zero. By integrating we get

$$\begin{aligned} \int_{-a}^{a} f(y,\mathbf{x})dy+\int_{-a}^{a} g(y,\mathbf{x})dy &= \sqrt{\frac{x_1x_2}{\lambda_1\lambda_2}}\sum_{j=1}^{N-1}\sum_{n=n_1^j(x_1)}^{n_2^j(x_1)} h_n(\mathbf{x})\int_{w_j}^{w_{j+1}}dy\\ &= \sqrt{\frac{x_1x_2}{\lambda_1\lambda_2}}\sum_{j=1}^{N-1}\sum_{n=n_1^j(x_1)}^{n_2^j(x_1)} h_n(\mathbf{x})(w_{j+1}-w_j)\\ &= \sqrt{x_2}\sum_{n=n_1^1(x_1)}^{n_2^{N-1}(x_1)} h_n(x). \end{aligned}$$

In the rest of the proof we denote by $n_1(x_1,a)=n_1^1(x_1)$ and by $n_2(x_1,a)=n_2^{N-1}(x_1)$ and we conclude that

$$\sqrt{x_2}\sum_{n=n_1(x_1,a)}^{n_2(x_1,a)} h_n(\mathbf{x})=\int_{-a}^{a} f(y,\mathbf{x})dy+\int_{-a}^{a} g(y,\mathbf{x})dy. \tag{21}$$

Moreover, as $||\mathbf{x}||$ tends to infinity, such that $|F(\mathbf{x})| < B$ where F is given by (8), we have :

$$\frac{\lambda_2}{x_2} = \frac{\lambda_1}{x_1} + o(x_1^{-\frac{3}{2}}).$$

Let $R(y, \mathbf{x}) = \exp\Big[- \frac{1}{2\big(x_1\lambda_1^{-1} + y\sqrt{x_1}\lambda_1^{-\frac{3}{2}}\big)} \big\{ \frac{y^2 x_1 \boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{\lambda_1^3} + \lambda_2^2 F(\mathbf{x})(\frac{2\sqrt{x_1}y}{\lambda_1^{\frac{3}{2}}} + F(\mathbf{x}))\big\}\Big]$,

$$\begin{aligned} R(y,\mathbf{x}) &= e^{-y^2\frac{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2}}\Bigg[1 - \frac{y}{\sqrt{x_1}}\Big(\frac{\lambda_2^2 F(\mathbf{x})}{\sigma_2^2\sqrt{\lambda_1}} - \frac{y^2\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^{\frac{5}{2}}}\Big) + \frac{1}{x_1}\Big\{ - \frac{\lambda_1\lambda_2^2F^2(\mathbf{x})}{2\sigma_2^2} + \\ &\quad \frac{\lambda_2^2 y^2 F(\mathbf{x})}{\sigma_2^2\lambda_1}\Big(1 + \frac{\lambda_2^2F(\mathbf{x})}{2\sigma_2^2}\Big) - \frac{y^4\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^3}\Big(1 + \frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2}\Big) + \frac{y^6(\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda})^2}{8\lambda_1^5}\Big\}\Bigg] \\ &\quad + o(\frac{1}{x_1}), \end{aligned} \tag{22}$$

$$\begin{aligned} \sqrt{\frac{x_1\lambda_1^{-1}}{x_1\lambda_1^{-1} + y\sqrt{x_1}\lambda_1^{-\frac{3}{2}}}}\sqrt{\frac{x_2\lambda_2^{-1}}{x_2\lambda_2^{-1} + y\sqrt{x_1}\lambda_1^{-\frac{3}{2}} + F(\mathbf{x})}} &= 1 - \frac{y}{\sqrt{x_1\lambda_1}} + \frac{1}{x_1}\Big[\frac{y^2}{\lambda_1} - \frac{\lambda_1}{2}F(\mathbf{x})\Big] \\ &\quad + o(\frac{1}{x_1}). \end{aligned} \tag{23}$$

Multiplying Equation (22) by Equation (23), we get

$$f(y,\mathbf{x}) = \frac{e^{-y^2\frac{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2}}}{2\pi\sqrt{|\Sigma|}}\Bigg[1 + \frac{P_1(y,\mathbf{x})}{\sqrt{x_1}} + \frac{P_2(y,\mathbf{x})}{x_1}\Bigg] + o(\frac{1}{x_1}),$$

where

$$P_1(y,\mathbf{x}) = \frac{y}{\sqrt{\lambda_1}}\Big(\frac{y^2\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2} - \frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2} - 1\Big),$$

$$\begin{aligned} P_2(y,\mathbf{x}) &= -\frac{\lambda_1F(\mathbf{x})}{2} - \frac{\lambda_1\lambda_2^2F^2(\mathbf{x})}{2\sigma_2^2} + \frac{y^2}{\lambda_1}\Big(1 + \frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2}(2 + \frac{\lambda_2^2F(\mathbf{x})}{2\sigma_2^2})\Big) \\ &\quad - \frac{y^4(\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda})}{2\lambda_1^3}(2 + \frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2}) + \frac{y^6(\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda})^2}{8\lambda_1^5}. \end{aligned}$$

This means that when $||\mathbf{x}||$ tends to infinity, we have

$$\Big\{x_1 e^{y^2\frac{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2}} f(y,\mathbf{x}) - \frac{1}{2\pi\sqrt{|\Sigma|}}\Big[x_1 + P_1(y,\mathbf{x})\sqrt{x_1} + P_2(y,\mathbf{x})\Big]\Big\}\mathbf{1}_{\{|F(\mathbf{x})| \le B\}} \to 0. \tag{24}$$

Thus, for all $\epsilon > 0$, there exists a constant $A > 0$ such that for $||\mathbf{x}|| \ge A$ we have

$$\Big|x_1 e^{y^2\frac{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2}} f(y,\mathbf{x}) - \frac{1}{2\pi\sqrt{|\Sigma|}}\Big[x_1 + P_1(y,\mathbf{x})\sqrt{x_1} + P_2(y,\mathbf{x})\Big]\Big|\mathbf{1}_{\{|F(\mathbf{x})| \le B\}} \le \epsilon. \tag{25}$$

Let $F(y,\mathbf{x})=\Big\{x_1f(y,\mathbf{x})-\frac{e^{-y^2\frac{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2}}}{2\pi\sqrt{|\Sigma|}}\Big[x_1+P_1(y,\mathbf{x})\sqrt{x_1}+P_2(y,\mathbf{x})\Big]\Big\}\mathbf{1}_{\{|F(\mathbf{x})|\leq B\}}$, by (25) we deduce that for $||\mathbf{x}||\geq A$,

$$|F(y,\mathbf{x})|\leq \epsilon e^{-y^2\frac{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2}}.$$

Denote that if $||\mathbf{x}||\leq A$, the function $F(y,\mathbf{x})$ is bounded. Then F is dominated by an integrable function on $[-a,a)$ independent of $\mathbf{x}$. Furthermore by (24) we have as $||\mathbf{x}||\to+\infty$,

$$F(y,\mathbf{x})\to 0.$$

We conclude by the dominated convergence Theorem if $||\mathbf{x}||\to+\infty$ we have, $\int_{-a}^{a}F(y,\mathbf{x})dz_1=o(1)$, which means that when x_1 and x_2 tends to infinity with $|F(\mathbf{x})|\leq B$ we have

$$\int_{-a}^{a}f(y,\mathbf{x})dy=\frac{1}{2\pi\sqrt{|\Sigma|}}\int_{-a}^{a}e^{-y^2\frac{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}{2\lambda_1^2}}\Big[1+\frac{P_1(y,\mathbf{x})}{\sqrt{x_1}}+\frac{P_2(y,\mathbf{x})}{x_1}\Big]dy+o(\frac{1}{x_1}).$$

By choosing a large enough, we obtain:

$$\begin{aligned}\int_{-a}^{a}f(y,\mathbf{x})dy=&K\Bigg[1+\frac{\lambda_1}{2x_1}\Bigg\{-F(\mathbf{x})(1+\frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2})+\frac{1}{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}\Big[-\frac{1}{4}+\frac{\lambda_2^2}{\sigma_2^2}F(\mathbf{x})\\&\times(1+\frac{\lambda_2^2}{\sigma_2^2}F(\mathbf{x}))\Big]\Bigg\}\Bigg]+o(\frac{1}{x_1}).\end{aligned}\tag{26}$$

By similar steps and using (9) , we get

$$\begin{aligned}\int_{-a}^{a}g(y,\mathbf{x})dy=&\frac{\lambda_1K}{2x_1}\Bigg\{\lambda_2a_2F(\mathbf{x})-\frac{\lambda_1}{\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda}}\Big[a_1(\frac{1}{2}+\frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2})+a_2\frac{\lambda_2}{\lambda_1}(\frac{1}{2}+\\&2\frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2})\Big]+\frac{\lambda_1^3}{(\boldsymbol{\lambda}'\Sigma^{-1}\boldsymbol{\lambda})^2}(\frac{1}{2}+\frac{\lambda_2^2F(\mathbf{x})}{\sigma_2^2})(\frac{a_1}{\sigma_1^2}+\frac{\lambda_2^3a_2}{\lambda_1^3\sigma_2^2})\Bigg\}+o(\frac{1}{x_1}),\end{aligned}\tag{27}$$

where a_1 and a_2 are given in paragraph (2.3). Summing up (26) and (27), we obtain:

$$\sqrt{x_2}\sum_{\substack{n\geq n_0\\ n_1(x_1,a)\leq n\leq n_2(x_1,a)}}h_n(\mathbf{x})=K+\frac{C(x_1,x_2)}{x_1}+o(\frac{1}{x_1}),\tag{28}$$

where K is given in paragraph (2.3) and $C(x_1,x_2)$ is given by (7).
It remains now to approximate the sum $\sqrt{x_2}\sum\limits_{\substack{n\geq n_0\\ n>n_2(x_1,a)\ or\\ n<n_1(x_1,a)}}h_n(\mathbf{x})$, for this purpose let $\alpha=\frac{x_1}{\lambda_1}+a\frac{\sqrt{x_1}}{\lambda_1^{\frac{3}{2}}}$, using (12) we get:

$$\begin{aligned}\sqrt{x_2}\sum_{\substack{n\geq n_0\\ n>n_2(x_1,a)\ or\\ n<n_1(x_1,a)}}h_n(\mathbf{x})&\leq 2D\sqrt{x_2}\sum_{\substack{n\geq n_0\\ n>n_2(x_1,a)}}\frac{1}{(n\lambda_1-x_1)^2}\\&\leq 2D\sqrt{x_2}\int_{\alpha-1}^{\infty}\frac{dv}{(n\lambda_1-x_1)^2}=\frac{2D\sqrt{x_2}}{a\sqrt{x_1}\lambda_1^{-1/2}-\lambda_1},\end{aligned}$$

as a is arbitrarily chosen we can assume that $a > x_1$. Consequently,

$$\sqrt{x_2} \sum_{\substack{n \geq n_0 \\ n > n_2(x_1,a) \text{ or } \\ n < n_1(x_1,a)}} h_n(\mathbf{x}) = o(\frac{1}{x_1}). \tag{29}$$

Furthermore, we have

$$\sum_{n=1}^{\infty} h_n(\mathbf{x}) = \sum_{n=1}^{n_0-1} h_n(\mathbf{x}) + \sum_{\substack{n \geq n_0 \\ n > n_2(x_1,a) \text{ or } \\ n < n_1(x_1,a)}} h_n(\mathbf{x}) + \sum_{\substack{n \geq n_0 \\ n_1(x_1,a) \leq n \leq n_2(x_1,a)}} h_n(\mathbf{x}),$$

thus by Equations (13), (28) and (29), we obtain the requested result. □

4 Mean and Variance of the total number of pieces

Theorem 2. *Let a be a positive number, under assumptions ($\boldsymbol{A}$) and ($\boldsymbol{B}$), the mean and the variance of $N(x^{\theta_1}, ax^{\theta_2})$ are given as $x \to +\infty$ by:*

$$m(x^{\theta_1}, ax^{\theta_2}) = \frac{Kax^{\theta_1+\theta_2}}{\sqrt{\theta_2 \ln x}} + \frac{a\eta(\ln a)x^{\theta_1+\theta_2}}{\ln^{\frac{3}{2}} x} + o(\frac{x^{\theta_1+\theta_2}}{(ln^{\frac{3}{2}} x)}) \tag{30}$$

and

$$\sigma^2(x^{\theta_1}, ax^{\theta_2}) = \frac{\tau(\ln a)a^2x^{2(\theta_1+\theta_2)}}{\ln^3 x} + o\Big(\frac{x^{2(\theta_1+\theta_2)}}{\ln^3 x}\Big) \tag{31}$$

where for all $\alpha \in \mathbb{R}$,

$$\eta(\alpha) = \frac{K}{2\sqrt{\theta_2}}\Big[c_0 - \frac{c_1}{\theta_1} + \frac{c_2}{\theta_2^2} + 2\frac{c_2}{\theta_1^2} + \frac{(1-\alpha)}{\theta_2}(1 + c_1 - 2\frac{c_2}{\theta_1}) + \frac{c_2(1-\alpha)^2}{\theta_2^2}\Big], \tag{32}$$

$$\tau(\alpha) = \frac{A_1 + 2A_2\alpha + A_3\alpha^2}{1 - \Phi(2,2)} + \frac{2(A_2 + \alpha A_3)\gamma + A_3\rho}{[1 - \Phi(2,2)]^2} + \frac{A_3\gamma^2}{[1 - \Phi(2,2)]^3} \tag{33}$$

and the constants c_i, $i = 0, 1, 2$ and A_j, $j = 1, 2, 3$ are all given in paragraph (2.3).

To prove this Theorem we need the following Lemma:

Lemma 3. *Let ξ be an integrable function on $\mathbb{R}_2^+$ and G be an uniformly bounded function such that $G(t_1, t_2) \longrightarrow g$ as t_1, t_2 go to infinity with $|t_1 - t_2| \leq B$ where B is an arbitrary positive constant. For all $t \in \mathbb{R}+$ and $\alpha \in \mathbb{R}$, we denote by*

$$L(t) = \int_0^t \int_0^{t+\alpha} \xi(s_1, s_2)G(t - s_1, t + \alpha - s_2)ds_1ds_2.$$

We have

$$\lim_{t \to +\infty} L(t) = g \int_0^{\infty} \int_0^{\infty} \xi(s_1, s_2)ds_1ds_2.$$

This Lemma is an immediate consequence of Lemma (3.1) in [13].

Proof. of Theorem (2): The function $M_*(t_1,t_2) := e^{-(t_1+t_2)}m_*(t_1,t_2)$ satisfies Equation (2), its solution is given by

$$M_*(t_1,t_2) = (f*h)(t_1,t_2), \text{ where } h(t_1,t_2) = \sum_{n=0}^{\infty} d\nu^{*n}(t_1,t_2).$$

Let $\widetilde{h}(t_1,t_2) = h(\theta_1 t_1, \theta_2 t_2)$ and $\widetilde{C}(t_1,t_2) = C(\theta_1 t_1, \theta_2 t_2)$, we have for all $\alpha \in \mathbb{R}$

$$\begin{aligned} M_*(\theta_1 t, \theta_2 t+\alpha) &= \int_0^{\theta_1 t}\int_0^{\theta_2 t+\alpha} e^{-(s_1+s_2)} h(\theta_1 t - s_1, \theta_2 t + \alpha - s_2) ds_1 ds_2 \\ &= \theta_1\theta_2 \int_0^t \int_0^{t+\frac{\alpha}{\theta_2}} e^{-(\theta_1 u + \theta_2 v)} \widetilde{h}(t-u, t+\frac{\alpha}{\theta_2} - v) du dv. \end{aligned}$$

Then we conclude that the expression of

$$\widehat{M}(t) := t\sqrt{(t+\frac{\alpha}{\theta_2})} M_*(\theta_1 t, \theta_2 t + \alpha) \tag{34}$$

can be written as the following

$$\begin{aligned} M(t) &= I_1(t) + I_2(t) + \theta_1 K t\sqrt{\theta_2 t+\alpha} \int_0^t \int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{-(\theta_1 u+\theta_2 v)}}{\sqrt{t+\frac{\alpha}{\theta_2} - v}} du dv \\ &\quad + \sqrt{\theta_2 t+\alpha} \int_0^t \int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{-(\theta_1 u+\theta_2 v)} \widetilde{C}(t-u, t+\frac{\alpha}{\theta_2}-v)}{\sqrt{t+\frac{\alpha}{\theta_2}-v}} du dv, \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= \theta_1\theta_2 \sqrt{t+\frac{\alpha}{\theta_2}} \int_0^t \int_0^{t+\frac{\alpha}{\theta}2} \Big[(t-u)\widetilde{h}(t-u, t+\frac{\alpha}{\theta_2}-v) - \frac{K(t-u)}{\sqrt{\theta_2(t+\frac{\alpha}{\theta_2}-v)}} \\ &\quad - \frac{\widetilde{C}(t-u, t+\frac{\alpha}{\theta_2}-v)}{\theta_1\sqrt{\theta_2(t+\frac{\alpha}{\theta_2}-v)}}\Big] e^{-(\theta_1 u+\theta_2 v)} du dv \end{aligned}$$

and

$$I_2(t) = \theta_1\theta_2 \sqrt{t+\frac{\alpha}{\theta_2}} \int_0^t \int_0^{t+\frac{\alpha}{\theta}2} u e^{-(\theta_1 u+\theta_2 v)} \Big[\widetilde{h}(t-u, t+\frac{\alpha}{\theta_2}-v) - \frac{K}{\sqrt{\theta_2(t+\frac{\alpha}{\theta_2}-v)}}\Big] du dv.$$

As t goes to infinity we have $I_1(t) = o(1)$, in fact let

$$G_1(t_1,t_2) = \sqrt{t_2}\, t_1 \widetilde{h}(t_1,t_2) - \frac{Kt_1}{\sqrt{\theta_2}} - \frac{\widetilde{C}(t_1,t_2)}{\theta_1\sqrt{\theta_2}}, \quad \xi_1(t_1,t_2) = e^{-(\theta_1 t_1+\theta_2 t_2)}$$

and

$$G_2(t_1,t_2) = t_1 \widetilde{h}(t_1,t_2) - \frac{Kt_1}{\sqrt{\theta_2 t_2}} - \frac{\widetilde{C}(t_1,t_2)}{\theta_1\sqrt{\theta_2 t_2}}, \quad \xi_2(t_1,t_2) = \sqrt{t_2}\, e^{-(\theta_1 t_1+\theta_2 t_2)}.$$

Using the fact that for all $v \in [0, t+\frac{\alpha}{\theta_2}]$, $\sqrt{t+\frac{\alpha}{\theta_2}} \le \sqrt{t+\frac{\alpha}{\theta_2}-v} + \sqrt{v}$ we obtain,

$$\begin{aligned} |I_1(t)| &\le \theta_1\theta_2 \int_0^t \int_0^{t+\frac{\alpha}{\theta_2}} |G_1(t-u, t+\frac{\alpha}{\theta_2}-v)| \xi_1(u,v) du dv \\ &\quad + \theta_1\theta_2 \int_0^t \int_0^{t+\frac{\alpha}{\theta_2}} |G_2(t-u, t+\frac{\alpha}{\theta_2}-v)| \xi_2(u,v) du dv. \end{aligned}$$

In view of Theorem (1), as t_1 and t_2 tend to infinity such that $|t_1 - t_2| \leq B$ we have $G_1(t_1, t_2) \to 0$ and $G_2(t_1, t_2) \to 0$. Furthermore, ξ_1 and ξ_2 are integrable on $\mathbb{R}_+ \times \mathbb{R}_+$, we conclude by Lemma (3) that as as t goes to infinity, $I_1(t) \to 0$. By similar argument, we prove that as $t \to +\infty$, $I_2(t) = o(1)$. Thus

$$\begin{aligned}
\widehat{M}(t) &= \sqrt{\theta_2 t + \alpha}\, e^{-(\theta_1+\theta_2)t-\alpha} \int_0^t \int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{\theta_1 s_1 + \theta_2 s_2} \widetilde{C}(s_1, s_2)}{\sqrt{s_2}} ds_1 ds_2 \\
&\quad + 2tK\sqrt{t + \frac{\alpha}{\theta_2}}(1 - e^{-\theta_1 t}) Daw(\sqrt{\theta_2 t + \alpha}) + o(1),
\end{aligned}$$

where Daw is the Dawson's integral given by

$$Daw(t) = e^{-t^2} \int_0^t e^{u^2} du.$$

As $t \to +\infty$, we have

$$\begin{aligned}
Daw(\sqrt{\theta_2 t + \alpha}) &= \frac{1}{2\sqrt{\theta_2 t + \alpha}} + \frac{1}{4(\theta_2 t + \alpha)^{\frac{3}{2}}} + \frac{3}{8(\theta_2 t + \alpha)^{\frac{5}{2}}} + o(\frac{1}{t^{\frac{5}{2}}}) \\
&= \frac{1}{2\sqrt{\theta_2 t}} + \frac{1 - \alpha}{4(\theta_2 t)^{\frac{3}{2}}} + \frac{3(\alpha^2 - 2\alpha + 2)}{16(\theta_2 t)^{\frac{5}{2}}} + o(\frac{1}{t^{\frac{5}{2}}}),
\end{aligned}$$

thus $2t\sqrt{t + \frac{\alpha}{\theta_2}}(1 - e^{-\theta_1 t}) Daw(\sqrt{\theta_2 t + \alpha}) = \frac{t}{\sqrt{\theta_2}} + \frac{1}{2\theta_2^{\frac{3}{2}}} + o(1)$ which leads to:

$$\begin{aligned}
\widehat{M}(t) &= \sqrt{\theta_2 t + \alpha}\, e^{-(\theta_1+\theta_2)t-\alpha} \int_0^t \int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{\theta_1 s_1 + \theta_2 s_2} \widetilde{C}(s_1, s_2)}{\sqrt{s_2}} ds_1 ds_2 \\
&\quad + \frac{tK}{\sqrt{\theta_2}} + \frac{K}{2\theta_2^{\frac{3}{2}}} + o(1),
\end{aligned}$$

we get, by writing explicitly $\widetilde{C}$ and integrating,

$$\begin{aligned}
\widehat{M}(t) &= \frac{K}{2\sqrt{\theta_2 t}} \sqrt{t + \frac{\alpha}{\theta_2}} \Big\{ c_0 - \frac{c_1}{\theta_1} + \frac{(1 - \alpha)}{\theta_2}(c_1 - 2\frac{c_2}{\theta_1}) + \frac{2c_2}{\theta_1^2} + \frac{c_2}{\theta_2^2} \\
&\quad + \frac{c_2(\alpha - 1)^2}{\theta_2^2} \Big\} + \frac{tK}{\sqrt{\theta_2}} + \frac{K}{2\theta_2^{\frac{3}{2}}} + o(1).
\end{aligned} \tag{35}$$

We conclude by (34) and (35) that as t goes to infinity

$$M_*(\theta_1 t, \theta_2 t + \alpha) = \frac{K}{\sqrt{\theta_2 t}} + \frac{\eta(\alpha)}{t^{\frac{3}{2}}} + o(\frac{1}{t^{\frac{3}{2}}}),$$

where η is given by (32).
To compute the variance, let $\mathcal{F} = \sigma(U_1, U_2, V_1, V_2)$

$$\begin{aligned}
\mathbb{E}\Big[(N(x, y) - m(x, y))^2 \mid \mathcal{F}\Big] &= \mathbb{E}\Big[\Big(1 + \sum_{i=1}^{2}\sum_{j=1}^{2} N(xU_i, yV_j) - m(x, y)\Big)^2 |\mathcal{F}\Big] \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} Var(N(xU_i, yV_j)) + \\
&\quad \Big(1 - m(x, y) + \sum_{i=1}^{2}\sum_{j=1}^{2} m(xU_i, yV_j)\Big)^2
\end{aligned}$$

By integrating we show that $\sigma_*^2(t_1,t_2) := \sigma^2(e^{t_1}, e^{t_2})$ satisfies the renewal equation given by

$$\sigma_*^2(t_1,t_2) = (\sigma_*^2 * \mu)(t_1,t_2) + k(t_1,t_2) \tag{36}$$

where $k(t_1,t_2) = \mathbb{E}\Big[\Big(1 - \sum_{i=1}^{2}\sum_{j=1}^{2} m(t_1 - X_i, t_2 - Y_j) + m_*(t_1,t_2)\Big)^2\Big]$. The function $V(t_1,t_2) := e^{-(t_1+t_2)}\sigma_*^2(t_1,t_2)$ satisfies the bivariate renewal equation

$$V(t_1,t_2) = (V * \nu)(t_1,t_2) + k_1(t_1,t_2)$$

where $k_1(t_1,t_2) = e^{-(t_1+t_2)}k(t_1,t_2)$. It follows that,

$$\begin{aligned} V(\theta_1 t, \theta_2 + \alpha) &= (h * k_1)(\theta_1 t, \theta_2 t + \alpha) \\ &= \sum_{n=0}^{\infty} \mathbb{E}\Big[k_1\Big(\theta_1 t - S_n^{(1)}, \theta_2 t + \alpha - S_n^{(2)}\Big)\mathbf{1}_{\{S_n^{(1)} \le \theta_1 t, S_n^{(2)} \le \theta_2 t + \alpha\}}\Big] \end{aligned}$$

where $\Big(S_n^{(1)},\ S_n^{(2)}\Big)$ is the sum of n-iid random vectors with common distribution ν. For i and $j \in \{1,\ 2\}$, let $T = t - \frac{X_i}{\theta_1}$ and $S = \frac{\theta_2}{\theta_1}X_i - Y_j + \alpha$, using the refined expression (30) of m_* we get:

$$\begin{aligned} m_*(\theta_1 T, \theta_2 T + S) &= m_*(\theta_1 t - X_i, \theta_2 t + \alpha - Y_j) \\ &= K\frac{e^{(\theta_1+\theta_2)T+S}}{\sqrt{S_2 T}} + \frac{\eta(S)e^{(\theta_1+\theta_2)T+S}}{T^{\frac{3}{2}}} + o\Big(\frac{e^{(\theta_1+\theta_2)T+S}}{T^{\frac{3}{2}}}\Big) \\ &= K\frac{e^{(\theta_1+\theta_2)t+\alpha}U_iV_j}{\sqrt{\theta_2 t}}[1 - \frac{\ln(U_i)}{2\theta_1 t} + o(\frac{1}{t})] + [\eta(\alpha) + L_1(U_i,V_j) \\ &\quad -\alpha L_2(U_i,V_j)]\frac{U_iV_je^{(\theta_1+\theta_2)t+\alpha}}{t^{\frac{3}{2}}} + o\Big(\frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^{\frac{3}{2}}}\Big). \end{aligned}$$

Thus,

$$\begin{aligned} m_*(\theta_1 t, \theta_2 t + \alpha) - \sum_{i=1}^{2}\sum_{j=1}^{2} m_*(\theta_1 t - X_i, \theta_2 t + \alpha - Y_j) &= (\widetilde{L}_1 + \alpha\widetilde{L}_2)\frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^{\frac{3}{2}}} \\ &\quad + o\Big(\frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^{\frac{3}{2}}}\Big). \end{aligned}$$

Consequently,

$$k_1(\theta_1 t, \theta_2 t + \alpha) = \frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^3}\Big(A_1 + 2A_2\alpha + A_3\alpha^2\Big) + o\Big(\frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^3}\Big).$$

We obtain, by a similar computation

$$\begin{aligned} k_1(\theta_1 t - S_n^{(1)}, \theta_2 t + \alpha - S_n^{(2)}) &= \frac{e^{(\theta_1+\theta_2)t+\alpha-(S_n^{(1)}+S_n^{(2)})}}{t^3}\Big[(A_1 + 2A_2\alpha + A_3\alpha^2) \\ &\quad + 2(A_2 + \alpha A_3)(\frac{\theta_2}{\theta_1}S_n^{(1)} - S_n^{(2)}) + A_3 \times \\ &\quad (\frac{\theta_2}{\theta_1}S_n^{(1)} - S_n^{(2)})^2\Big] + o\Big(\frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^3}\Big). \end{aligned}$$

Finally, we get

$$V(\theta_1 t, \theta_2 t + \alpha) = \frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^3}\tau(\alpha) + o\Big(\frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^3}\Big)$$

where τ is given by (33). $\square$

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