Stepanov-like almost automorphic functions on time scales and the application to cellular neural networks with time-varying delays

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Abstract

In this work, we first propose the concept of stepanov-like almost automorphic functions on time scales, and present some properties, including the translation invariance and completeness. Moreover, we also prove the connection between stepanov-like almost automorphic functions on time scales and on $\mathbb{R}$. Then we establish some existence and uniqueness result of almost automorphic solutions for some linear dynamic equation on time scales. As an application of the above results, we study the existence and global exponential stability of almost automorphic solutions for a class of cellular neural networks with time-varying delays on time scales.

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1 Introduction

The theory of time scales was developed by S. Hilger in his PhD thesis in 1988 (see [1]). This theory unifies continuous and discrete problems and provides a powerful tool for applications to economics, populations models, quantum physics among others, and hence has been attracting the attention of many mathematicians (see [2–5] and the references therein). As applications, the field of dynamic equations and cellular neural networks on time scales extended the classical differential equations and difference equations. In recent years, the qualitative properties of the solutions of dynamic equations and cellular neural networks on time scales have been extensively investigated.

In 2001, Li and Wang [6] firstly introduced the definition of almost periodic functions on time scales, and gave some results about almost periodic solutions for high-order Hopfield neural networks on time scales. The theory of almost periodic functions on time scales is an interesting topic. Lizama, Mesquita,

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and Ponce [7] proved the connection between almost periodic functions defined on time scales and on $\mathbb{R}$. Furthermore, some generalized forms of almost periodicity have been introduced on time scales, such as pseudo almost periodicity [8], almost automorphy [9], weighted pseudo almost periodicity [10], stepanov-like almost periodicity [11] and stepanov-like pseudo almost periodicity [12].

Motivated by the above, in this paper, we first introduce the concept of stepanov-like almost automorphic functions on time scales, which is the generalization of stepanov-like almost automorphy on $\mathbb{R}$. To show the connection between stepanov-like almost automorphic functions on time scales and $\mathbb{R}$, we apply the translation invariant of time scales to prove it, which extend the result in [7]. Then, we study the existence and uniqueness of almost automorphic solutions about some linear dynamic equation on time scales:

$$x^{\Delta}(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is invariant under translations, and $A(t)$ is almost automorphic, $f(t)$ is stepanov-like almost automorphic on $\mathbb{T}$. The almost automorphy [9], pseudo almost periodicity [12] of the equation (1.1) has been studied. Our results will extend the obtained results. As an application, we study the existence and global exponential stability of the almost automorphic solution for the following cellular neural networks with time-varying delays:

$$x_i^{\Delta}(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\gamma_{ij}(t))) + I_i(t),$$

with the initial condition

$$x_i(s) = \varphi_i(s), \quad s \in [-\gamma, 0]_{\mathbb{T}},$$

for $t \in \mathbb{T}, i = 1, 2, \cdots, n$, where $x_i(t)$ correspond to the activations of the ith neurons at the time t, $c_i(t)$ are positive functions, they denote the rate with which the cell i reset their potential to the resting state when isolated from the other cells and inputs at time t, $a_{ij}(t)$ and $b_{ij}(t)$ are the connection weights at time t, $\gamma_{ij}(t) \geq 0$ correspond to the transmission delays, $I_i(t)$ denote the external inputs at time t, f_i are the activation functions of signal transmission.

This paper is organized as follows. The second section is devoted to present the preliminary results concerning the definition and properties of stepanov-like almost automorphic functions on time scales. In the third section, we give the existence and uniqueness of almost automorphic solutions for some linear dynamic equation on time scales. Finally, the last section is devoted to show the existence and global exponential stability of the cellular neural networks with time-varying delays, and present some example.

2 Stepanov-like almost automorphic functions on time scales

Throughout this paper, we denote by $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ and $\mathbb{R}^+$ the sets of integers, real numbers, complex numbers and nonnegative real numbers, respectively. $\mathbb{E}^n$ denotes the Euclidian space $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidian norm $|\cdot|$.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T}, s < t\}, \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{E}^n$ is right-dense continuous (rd-continuous) provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$. We denote

$$C_{rd}(\mathbb{T}, \mathbb{E}^n) = \{f : \mathbb{T} \to \mathbb{E}^n : f \text{ is right-dense continuous }\},$$
$$L^p_{loc}(\mathbb{T}, \mathbb{E}^n) = \{f : \mathbb{T} \to \mathbb{E}^n : f \text{ is locally } L^p\ \Delta\text{-integrable}\}.$$

Given a pair of numbers $a, b \in \mathbb{T}$ and $a \le b$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}; a \le t \le b\}$. On the other hand, $[a, b]$ is the usual closed interval on the real line.

Definition 2.1 ([2]) *For $f : \mathbb{T} \to \mathbb{E}^n$ and $t \in \mathbb{T}^k$, $f^{\Delta}(t) \in E^n$ is called the delta derivative of $f(t)$ if for a given $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| < |\sigma(t) - s|$$

for all $s \in U$.

The theorems of the general Lebesgue integration theory also hold for the Δ-integrals $\int_{[a,b]_{\mathbb{T}}} f(t)\Delta t$ on $\mathbb{T}$. For more details, we refer the readers to [3].

Definition 2.2 ([9]) *A time scale is called invariant under translations if*

$$\Pi =: \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

In the following, we always assume that the time scale $\mathbb{T}$ is invariant under translation.

Definition 2.3 ([9]) *A function $f \in C_{rd}(\mathbb{T}, \mathbb{E}^n)$ is called almost automorphic on $\mathbb{T}$ if for every sequence $\{\alpha'_n\} \subset \Pi$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ and a function $\bar{f}$ such that*

$$\lim_{n\to\infty} f(t + \alpha_n) = \bar{f}(t), \text{ and } \lim_{n\to\infty} \bar{f}(t - \alpha_n) = f(t)$$

for every $t \in \mathbb{T}$. We denote the space of all these functions by $AA(\mathbb{T}, \mathbb{E}^n)$.

Now we are in the position to introduce the stepanov-like almost automorphic functions on $\mathbb{T}$. We always assume that $p \ge 1$ afterwards. Let

$$\mathcal{K} := \begin{cases} \inf\{|\tau| : \tau \in \Pi, \tau \neq 0\}, & \text{if } \mathbb{T} \neq \mathbb{R}, \\ 1, & \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

Define the norm $\|\cdot\|_{S^p}$ as

$$\|f\|_{S^p} := \sup_{t\in\mathbb{T}} \left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f(s)|^p \Delta s \right)^{\frac{1}{p}}, \quad \text{for } f \in L^p_{\text{loc}}(\mathbb{T}, \mathbb{E}^n).$$

If $\|f\|_{S^p} < \infty$, we call f is S^p-bounded, and denote by $BS^p(\mathbb{T}, \mathbb{E}^n)$ the space of all these functions.

Definition 2.4 *A function $f \in BS^p(\mathbb{T}, \mathbb{E}^n)$ is called stepanov-like almost automorphic on $\mathbb{T}$, if for every sequence $\{t'_n\} \subset \Pi$, there exist a subsequence $\{t_n\} \subset \{t'_n\}$, and a function $\bar{f}$, such that*

$$\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f(s+\alpha_n) - \bar{f}(s)|^p \Delta s\right)^{\frac{1}{p}} \to 0, \text{ and } \left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}(s-\alpha_n) - f(s)|^p \Delta s\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{T}$. Denote the set of all these functions by $S^pAA(\mathbb{T}, \mathbb{E}^n)$.

It is clear that $AA(\mathbb{T}, \mathbb{E}^n) \subset S^pAA(\mathbb{T}, \mathbb{E}^n)$.

Now we prove some propositions of stepanov-like almost automorphic functions on $\mathbb{T}$. Since $\mathbb{T}$ is invariant under translations, we can easily get the following proposition.

Proposition 2.1 *If $f, g \in S^pAA(\mathbb{T}, \mathbb{E}^n), l \in \Pi$, then*
(i) $f + g \in S^pAA(\mathbb{T}, \mathbb{E}^n)$.
(ii) $fg \in S^pAA(\mathbb{T}, \mathbb{E}^n)$.
(iii) $f(\cdot + l) \in S^pAA(\mathbb{T}, \mathbb{E}^n)$.

Proposition 2.2 *Let $\{f_n : \mathbb{T} \to \mathbb{E}^n\}$ be a sequence of stepanov-like almost automorphic functions on $\mathbb{T}$ such that*

$$\lim_{n\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_n(s) - f(s)|^p \Delta s\right)^{\frac{1}{p}} = 0 \tag{2.1}$$

convergence uniformly on $\mathbb{T}$. Then f is stepanov-like almost automorphic on $\mathbb{T}$.

Proof. Let sequence $\{t'_k\} \subset \Pi$. Since $f_1 \in S^pAA(\mathbb{T}, \mathbb{E}^n)$, there exist a subsequence $\{t^1_k\} \subset \{t'_k\}$ and a function $\bar{f}_1$ such that

$$\lim_{k\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_1(s+t^1_k) - \bar{f}_1(s)|^p \Delta s\right)^{\frac{1}{p}} = 0, \lim_{k\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_1(s-t^1_k) - f_1(s)|^p \Delta s\right)^{\frac{1}{p}} = 0$$

for each $t \in \mathbb{T}$. Since $f_2 \in S^pAA(\mathbb{T}, \mathbb{E}^n)$, there exist $\{t^2_k\} \subset \{t^1_k\}$ and a function $\bar{f}_2$ such that

$$\lim_{k\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_2(s+t^2_k) - \bar{f}_2(s)|^p \Delta s\right)^{\frac{1}{p}} = 0, \lim_{k\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_2(s-t^2_k) - f_2(s)|^p \Delta s\right)^{\frac{1}{p}} = 0$$

for each $t \in \mathbb{T}$. Thus by the diagonal procedure, we can construct a subsequence $\{t_k\} \subset \{t'_k\}$ and a sequence of functions $\{\bar{f}_i\}$ such that

$$\lim_{k\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_i(s+t_k) - \bar{f}_i(s)|^p \Delta s\right)^{\frac{1}{p}} = 0, \lim_{k\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_i(s-t_k) - f_i(s)|^p \Delta s\right)^{\frac{1}{p}} = 0 \tag{2.2}$$

for each $t \in \mathbb{T}$ and $i = 1, 2, \cdots$. Notice that

$$\begin{aligned}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_i(s) - \bar{f}_j(s)|^p \Delta s \le& \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_i(s) - f_i(s+t_k)|^p \Delta s \\ &+ \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_i(s+t_k) - f_j(s+t_k)|^p \Delta s + \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_j(s+t_k) - \bar{f}_j(s)|^p \Delta s.\end{aligned}$$

Let $\varepsilon > 0$. By the uniform convergence of $\{f_n\}$, there exists positive integer N, such that when $i, j > N$, we have

$$\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_i(s+t_k) - f_j(s+t_k)|^p \Delta s < \varepsilon$$

for all $t \in \mathbb{T}$, and $k = 1, 2, \cdots$. Using (2.2) and the completeness of $\mathbb{E}^n$, we can get the pointwise convergence of the sequence $\{\bar{f}_n(t)\}$, say to $\bar{f}(t)$, that is

$$\lim_{n\to\infty} \left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_n(s) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} = 0 \tag{2.3}$$

for each $t \in \mathbb{T}$. By (2.1) and (2.3), for any $\varepsilon > 0$, there exists a positive integer M, such that

$$\left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f(s+t_k) - f_M(s+t_k)|^p \Delta s \right)^{\frac{1}{p}} < \varepsilon, \tag{2.4}$$

$$\left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_M(s) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} < \varepsilon \tag{2.5}$$

for each $t \in \mathbb{T}$, and $k = 1, 2, \cdots$. By (2.2), there exists some positive integer $K = K(t, M)$, such that when $k > K$, we have

$$\left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_M(s+t_k) - \bar{f}_M(s)|^p \Delta s \right)^{\frac{1}{p}} < \varepsilon \tag{2.6}$$

for each $t \in \mathbb{T}$. By (2.4)–(2.6), we get

$$\begin{aligned}
&\left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f(s+t_k) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} \\
\le &\left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f(s+t_k) - f_M(s+t_k)|^p \Delta s \right)^{\frac{1}{p}} \\
&+ \left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f_M(s+t_k) - \bar{f}_M(s)|^p \Delta s \right)^{\frac{1}{p}} + \left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}_M(s) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} \\
<&3\varepsilon
\end{aligned}$$

for $k > K$, and $t \in \mathbb{T}$. Similarly, we can easily get

$$\lim_{k\to\infty} \left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}(s-t_k) - f(s)|^p \Delta s \right)^{\frac{1}{p}} = 0 \text{ for each } t \in \mathbb{T}.$$

Thus $f \in S^pAA(\mathbb{T}, \mathbb{E}^n)$, and the proof is complete.

By the above argument, it is to see that the space $S^pAA(\mathbb{T}, \mathbb{E}^n)$ is an Banach space under the norm $\|\cdot\|_{S^p}$.

2.1 Composition theorem of stepanov-like almost automorphic functions on $\mathbb{T}$

Theorem 2.1 *Assume $f : \mathbb{R} \to \mathbb{R}$ sastifies that there exists a constant $L > 0$ such that*

$$|f(x) - f(y)| \le L|x - y|. \tag{2.7}$$

If $\varphi \in S^pAA(\mathbb{T},\mathbb{R})$, $\gamma \in AA(\mathbb{T},\mathbb{R})$, and $t-\gamma(t) \in \mathbb{T}$, then $f(\varphi(t-\gamma(t)))$ is stepanov-like almost automorphic.

Proof. Since $\varphi \in S^pAA(\mathbb{T},\mathbb{R})$ and $\gamma \in AA(\mathbb{T},\mathbb{R})$, for every sequence $\{t'_n\} \subset \Pi$, there exist a subsequence $\{t_n\} \subset \{t'_n\}$ and functions $\bar{\varphi}, \bar{\gamma}$, such that

$$\lim_{n\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}|\varphi(s+t_n)-\bar{\varphi}(s)|^p\Delta s\right)^{\frac{1}{p}}=0,\ \lim_{n\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}|\bar{\varphi}(s-t_n)-\varphi(s)|^p\Delta s\right)^{\frac{1}{p}}=0 \quad (2.8)$$

and

$$\lim_{n\to\infty}|\gamma(s+t_n)-\bar{\gamma}(s)|=0, \quad \lim_{n\to\infty}|\bar{\gamma}(s-t_n)-\gamma(s)|=0 \quad (2.9)$$

for $t \in \mathbb{T}$. By (2.7)–(2.9), we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}|f(\varphi(s+t_n-\gamma(s+t_n)))-f(\bar{\varphi}(s-\bar{\gamma}(s)))|^p\Delta s\right)^{\frac{1}{p}}\\
\le&\lim_{n\to\infty}L\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}|\varphi(s+t_n-\gamma(s+t_n))-\bar{\varphi}(s-\bar{\gamma}(s))|^p\Delta s\right)^{\frac{1}{p}}\\
\le&K_1+K_2,
\end{aligned}$$

where

$$\begin{aligned}
K_1&:=\lim_{n\to\infty}L\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}|\varphi(s+t_n-\gamma(s+t_n))-\varphi(s+t_n-\bar{\gamma}(s))|^p\Delta s\right)^{\frac{1}{p}},\\
K_2&:=\lim_{n\to\infty}L\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}|\varphi(s+t_n-\bar{\gamma}(s))-\bar{\varphi}(s-\bar{\gamma}(s))|^p\Delta s\right)^{\frac{1}{p}},
\end{aligned}$$

for $t \in \mathbb{T}$. By (2.8) and (2.9), we can get $K_1 = 0, K_2 = 0$. By the similar argument, we have

$$\lim_{n\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}|f(\bar{\phi}(s-t_n-\gamma(s-t_n)))-f(\phi(s-\gamma(s)))|^p\Delta s\right)^{\frac{1}{p}}=0$$

for $t \in \mathbb{T}$. Thus $f(\varphi(t-\gamma(t)))$ is stepanov-like almost automorphic, and the proof is complete.

2.2 The connection between stepanov-like almost automorphic functions on $\mathbb{T}$ and $\mathbb{R}$

In order to present the relationship of stepanov-like almost automorphic functions on $\mathbb{T}$ and on $\mathbb{R}$, we need to introduce the following notation. Similarly, as in [13], for a given $t \in \mathbb{T}$ we define t_* by

$$t_* = \sup\{s \in \mathbb{T}; s \le t\}.$$

Lemma 2.1 $\Delta t = \Delta t_*$ *for* $t \in \mathbb{R}$.

Proof. Since $\mathbb{T}$ is invariant under translations, $\Pi \neq \{0\}$. Let $\tau \in \Pi$ and $\tau > 0$. By the definition of t_*, we have $t_* \le t \le t_* + \tau$. By differentiation on the two sides, we get $\Delta t = \Delta t_*$, and the proof is complete.

Theorem 2.2 *A continuous function $g : \mathbb{T} \to \mathbb{E}^n$ to be stepanov-like almost automorphic on $\mathbb{T}$ if and only if there is a stepanov-like almost automorphic function $f : \mathbb{R} \to \mathbb{E}^n$ such that $f(t) = g(t)$ for every $t \in \mathbb{T}$.*

Proof. It is clear to see that the condition is sufficient. On the other hand, suppose that a continuous function $g : \mathbb{T} \to \mathbb{E}^n$ is stepanov-like almost automorpic on $\mathbb{T}$. Then, define a function $f : \mathbb{R} \to \mathbb{E}^n$ as follows:

$$f(t) = \begin{cases} \left(1 - \frac{t-t_*}{\mu(t_*)}\right) g(t_*) + \frac{t-t_*}{\mu(t_*)} g(\sigma(t_*)), & t \in \mathbb{R} \setminus \mathbb{T}, \\ g(t), & t \in \mathbb{T}. \end{cases}$$

It is easy to see that the function f is well defined, and $f \in BS^p(\mathbb{R}, \mathbb{E}^n)$.

Now we prove f is stepanov-like almost automorphic on $\mathbb{R}$. Since g is stepanov-like almost automorphic, for every sequence $\{\tau_n'\} \subset \Pi$, there exist a subsequence $\{\tau_n\} \subset \{\tau_n'\}$ and a function $\bar{g}$, such that

$$\lim_{n\to\infty} \left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |g(s+\tau_n) - \bar{g}(s)|^p \Delta s\right)^{\frac{1}{p}} = 0, \ \lim_{n\to\infty} \left(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{g}(s-\tau_n) - g(s)|^p \Delta s\right)^{\frac{1}{p}} = 0$$

for each $t \in \mathbb{T}$. Define the function $\bar{f} : \mathbb{R} \to \mathbb{E}^n$ by

$$\bar{f}(t) = \begin{cases} \left(1 - \frac{t-t_*}{\mu(t_*)}\right) \bar{g}(t_*) + \frac{t-t_*}{\mu(t_*)} \bar{g}(\sigma(t_*)), & t \in \mathbb{R} \setminus \mathbb{T}, \\ \bar{g}(t), & t \in \mathbb{T}. \end{cases}$$

We have

$$\begin{aligned} &\left(\int_t^{t+1} |f(u+\tau_n) - \bar{f}(u)|^p du\right)^{\frac{1}{p}} \\ =&\left(\int_{[t,t+1]_{\mathbb{T}}} |f(u+\tau_n) - \bar{f}(u)|^p \Delta u + \int_{[t,t+1]\setminus\mathbb{T}} |f(u+\tau_n) - \bar{f}(u)|^p \Delta u\right)^{\frac{1}{p}} \\ \leq&\left(\int_{[t,t+1]_{\mathbb{T}}} |f(u+\tau_n) - \bar{f}(u)|^p \Delta u\right)^{\frac{1}{p}} + \left(\int_{[t,t+1]\setminus\mathbb{T}} |f(u+\tau_n) - \bar{f}(u)|^p \Delta u\right)^{\frac{1}{p}} \\ :=&I_1 + I_2. \end{aligned}$$

Since g is stepanov-like almost automorphic on $\mathbb{T}$, we have

$$I_1 = \left(\int_{[t,t+1]_{\mathbb{T}}} |g(u+\tau_n) - \bar{g}(u)|^p \Delta u\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ for each $t \in \mathbb{R}$. Since $(u+\tau_n)_* = u_* + \tau_n$, $\sigma(u_* + \tau_n) = \sigma(u_*) + \tau_n, \mu(u_* + \tau_n) = \mu(u_*)$ for $u \in \mathbb{R}, \tau_n \in \Pi$ ([7], Lemma 3.1, Lemma2.7),

$$
\begin{aligned}
I_2 =& \left(\int_{[t,t+1]\setminus\mathbb{T}} \left| \left(1 - \frac{u+\tau_n - (u+\tau_n)_*}{\mu((u+\tau_n)_*)}\right) g((u+\tau_n)_*) + \frac{u+\tau_n-(u+\tau_n)_*}{\mu((u+\tau_n)_*)} g(\sigma((u+\tau_n)_*)) \right.\right. \\
&\left.\left. - \left(1 - \frac{u-u_*}{\mu(u_*)}\right) \bar{g}(u_*) - \frac{u-u_*}{\mu(u_*)} \bar{g}(\sigma(u_*)) \right|^p \Delta u \right)^{\frac{1}{p}} \\
=& \left(\int_{[t,t+1]\setminus\mathbb{T}} \left| \left(1 - \frac{u-u_*}{\mu(u_*)}\right) g(u_*+\tau_n) + \frac{u-u_*}{\mu(u_*)} g(\sigma(u_*)+\tau_n) \right.\right. \\
&\left.\left. - \left(1 - \frac{u-u_*}{\mu(u_*)}\right) \bar{g}(u_*) - \frac{u-u_*}{\mu(u_*)} \bar{g}(\sigma(u_*)) \right|^p \Delta u \right)^{\frac{1}{p}} \\
=& \left(\int_{[t,t+1]\setminus\mathbb{T}} \left| g(u_*+\tau_n) - \bar{g}(u_*) - \frac{u-u_*}{\mu(u_*)}(g(u_*+\tau_n) - \bar{g}(u_*)) + \frac{u-u_*}{\mu(u_*)}(g(\sigma(u_*)+\tau_n) - \bar{g}(\sigma(u_*))) \right|^p \Delta u \right)^{\frac{1}{p}} \\
\le& J_1 + J_2 + J_3,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &:= \left(\int_{[t,t+1]\setminus\mathbb{T}} |g(u_*+\tau_n) - \bar{g}(u_*)|^p \Delta u \right)^{\frac{1}{p}}, \\
J_2 &:= \left(\int_{[t,t+1]\setminus\mathbb{T}} \left| \frac{u-u_*}{\mu(u_*)}(g(u_*+\tau_n) - \bar{g}(u_*)) \right|^p \Delta u \right)^{\frac{1}{p}}, \\
J_3 &:= \left(\int_{[t,t+1]\setminus\mathbb{T}} \left| \frac{u-u_*}{\mu(u_*)}(g(\sigma(u_*)+\tau_n) - \bar{g}(\sigma(u_*))) \right|^p \Delta u \right)^{\frac{1}{p}}.
\end{aligned}
$$

By Lemma 2.1, we have

$$
J_1 = \left(\int_{[t,t+1]_{\mathbb{T}}} |g(u_*+\tau_n) - \bar{g}(u_*)|^p \Delta u_* \right)^{\frac{1}{p}} \to 0, \text{ as } n \to \infty \text{ for each } t \in \mathbb{R}.
$$

Since $u \in \mathbb{R} \setminus \mathbb{T}$, we have $u_* < u < \sigma(u_*)$, and then $0 < u - u_* < \sigma(u_*) - u_* = \mu(u_*)$, thus

$$
0 < \frac{u-u_*}{\mu(u_*)} < 1. \tag{2.1}
$$

By Lemma 2.1 and (2.1), we have

$$
J_2 < \left(\int_{[t,t+1]\setminus\mathbb{T}} |g(u_*+\tau_n) - \bar{g}(u_*)|^p \Delta u \right)^{\frac{1}{p}} = \left(\int_{[t,t+1]_{\mathbb{T}}} |g(u_*+\tau_n) - \bar{g}(u_*)|^p \Delta u_* \right)^{\frac{1}{p}} \to 0
$$

as $n \to \infty$ for each $t \in \mathbb{R}$. Similarly, we can prove that $J_3 \to 0$ as $n \to \infty$ for each $t \in \mathbb{R}$. Therefore,

$$
\lim_{n\to\infty} \left(\int_t^{t+1} |f(u+\tau_n) - \bar{f}(u)|^p du \right)^{\frac{1}{p}} = 0
$$

for each $t \in \mathbb{R}$. By the similar argument, we can get

$$
\lim_{n\to\infty} \left(\int_t^{t+1} |\bar{f}(u-\tau_n) - f(u)|^p du \right)^{\frac{1}{p}} = 0 \text{ for each } t \in \mathbb{R}.
$$

Thus f is stepanov-like almost automorphic on $\mathbb{R}$, and the proof is complete.

Remark 2.1 *The construction of the function f in* Theorem 2.2 *is not unique. For example,*

$$f(t)=\begin{cases} \left(1-\frac{t^*-t}{t^*-\rho(t^*)}\right)g(t^*)+\frac{t^*-t}{t^*-\rho(t^*)}g(\rho(t^*)), & t\in\mathbb{R}\setminus\mathbb{T},\\ g(t), & t\in\mathbb{T},\end{cases}$$

is stepanov-like almost automorphic function on $\mathbb{T}$, and it is $g(t)$ restricted on $\mathbb{R}$, where $t^*=\inf\{s\in\mathbb{T}, s\geq t\}$.

3 Almost automorphic solutions of linear dynamic equations on time scales

We say that a function $p:\mathbb{T}\to\mathbb{R}$ is regressive provided $1+\mu(t)p(t)\neq 0$ for $t\in\mathbb{T}^k$. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}=\mathcal{R}(\mathbb{T})=\mathcal{R}(\mathbb{T},\mathbb{R})$, and $\mathcal{R}^+=\mathcal{R}(\mathbb{T},\mathbb{R})=\{p\in\mathcal{R}, 1+\mu(t)p(t)>0, t\in\mathbb{T}\}$. Suppose that $p,q\in\mathcal{R}$, then we define $p\oplus q$ and $\ominus p$ as follows:

$$(p\oplus q)(t):=p(t)+q(t)+\mu(t)p(t)q(t),\quad (\ominus p)(t):=\frac{-p(t)}{1+\mu(t)p(t)},$$

for all $t\in\mathbb{T}^k$. It is clear that $(\mathcal{R},\oplus)$ is an Abelian group. In the sequel, we define the generalized exponential function $e_p(t,s)$.

Definition 3.1 ([2]) *If $p\in\mathcal{R}$, then we define the generalized exponential function by*

$$e_p(t,s)=exp\left(\int_{[s,t]_{\mathbb{T}}}\xi_{\mu(\tau)}p(\tau)\Delta\tau\right),\ s,t\in\mathbb{T}.$$

where the cylinder transformation $\xi_h:C_h\to Z_h$ is given by

$$\xi_h(z)=\begin{cases}\frac{1}{h}log(1+zh), & h\neq 0,\\ z, & h=0.\end{cases}$$

where log is the principal logarithm function.

Proposition 3.1 ([2]) *Let $t,s\in\mathbb{T}$.*
(i) $e_p(t,t)=1, e_A(t,t)=I$.
(ii) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$.
(iii) $e_p(t,s)e_p(s,r)=e_p(t,r), e_A(t,s)e_A(s,r)=e_A(t,r)$.

Proposition 3.2 ([12]) *Let $a>0$ be a constant and $t,s\in\mathbb{T}$.*
(i) $e_{\ominus a}(t,s)\leq 1$ *if $t\geq s$.*
(ii) $e_{\ominus a}(t+\tau,s+\tau)=e_{\ominus a}(t,s)$ *for $\tau\in\Pi$.*
(iii) *There exists $N>0$ such that $(t-s)e_{\ominus a}(t,s)\leq N$ for $t\geq s$.*
(iv) *For $t_0\in\mathbb{T}$, $e_{\ominus a}(t_0,\cdot)$ is increasing on $(-\infty,t_0]_{\mathbb{T}}$.*
(v) *The series $\sum_{j=1}^{\infty}e_{\ominus a}(t,\sigma(t)-(j-1)\mathcal{K})$ converges uniformly for $t\in\mathbb{T}$. Moreover, for all $t\in\mathbb{T}$,*

$$\sum_{j=1}^{\infty}e_{\ominus a}(t,\sigma(t)-(j-1)\mathcal{K})\leq\lambda_a:=\begin{cases}\frac{1}{1-e^{-a}}, & \mathbb{T}=\mathbb{R},\\ 2+a\bar{\mu}+\frac{1}{a\bar{\mu}}, & \mathbb{T}\neq\mathbb{R},\end{cases}$$

where $\bar{\mu}=\sup_{t\in\mathbb{T}}\mu(t)$.

Now we are in the position to consider the following equation:

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T}, \tag{3.1}$$

where $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $f \in S^pAA(\mathbb{T}, \mathbb{R}^n)$ is continuous on $\mathbb{T}$.

Theorem 3.1 *Assume that A is almost automorphic on $\mathbb{T}$, and the following condition is satisfied*

$$\|e_A(t,s)\| \le Ce_{\ominus a}(t,s), t \ge s. \tag{3.2}$$

Then the equation (3.1) *admits a unique almost automorphic solution $x(t)$ given by*

$$x(t) = \int_{(-\infty,t]_{\mathbb{T}}} e_A(t,\sigma(s))f(s)\Delta s, \quad t \in \mathbb{T}. \tag{3.3}$$

Proof. By the similar argument of Lemma 4.7 in [12], we can show that (3.3) is the unique continuous solution of (3.1), and here we omit the details. Now we prove it is almost automorphic on $\mathbb{T}$. For each sequence $\{t'_n\} \subset \Pi$, there exist a subsequence $\{t_n\} \subset \{t'_n\}$, and functions $\bar{A}, \bar{f}$, such that

$$\lim_{n\to\infty} |A(t+t_n) - \bar{A}(t)| = 0, \lim_{n\to\infty} |\bar{A}(t-t_n) - A(t)| = 0 \tag{3.4}$$

and

$$\lim_{n\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |f(s+t_n) - \bar{f}(s)|^p\Delta s\right)^{\frac{1}{p}} = 0, \lim_{n\to\infty}\left(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}} |\bar{f}(s-t_n) - f(s)|^p\Delta s\right)^{\frac{1}{p}} = 0, \tag{3.5}$$

for each $t \in \mathbb{T}$. Let

$$\begin{aligned}\bar{x}(t) &:= \int_{(-\infty,t]_{\mathbb{T}}} e_{\bar{A}}(t,\sigma(s))\bar{f}(s)\Delta s,\\ U(t,\sigma(s)) &:= e_A(t+t_n,\sigma(s+t_n)) - e_{\bar{A}}(t,\sigma(s)),\ t \in \mathbb{T}, 0 < t-s < \mathcal{K}.\end{aligned}$$

It is easy to get that

$$\begin{aligned}\frac{\partial\Delta U(t,\sigma(s))}{\partial\Delta t} &= A(t+t_n)e_A(t+t_n,\sigma(s)+t_n) - \bar{A}(t)e_{\bar{A}}(t,\sigma(s))\\ &= (A(t+t_n) - \bar{A}(t))e_A(t+t_n,\sigma(s)+t_n) + \bar{A}(t)(e_A(t+t_n,\sigma(s)+t_n) - e_{\bar{A}}(t,\sigma(s)))\\ &= (A(t+t_n) - \bar{A}(t))e_A(t+t_n,\sigma(s)+t_n) + \bar{A}(t)U(t,\sigma(s)).\end{aligned}$$

By the variation of constants formula, we have

$$U(t,\sigma(s)) = \int_{[\sigma(s),t]_{\mathbb{T}}} e_{\bar{A}}(t,\sigma(\tau))(A(\tau+t_n) - \bar{A}(\tau))e_A(\tau+t_n,\sigma(s)+t_n)\Delta\tau,$$

since $U(\sigma(s),\sigma(s)) = 0$. By (3.2), we have $\|e_{\bar{A}}(t,s)\| \le Ce_{\ominus a}(t,s), t \ge s$. Then

$$\begin{aligned}|U(t,\sigma(s))| &\le \int_{[\sigma(s),t]_{\mathbb{T}}} |e_{\bar{A}}(t,\sigma(\tau))| \cdot |A(\tau+t_n) - \bar{A}(\tau)| \cdot |e_A(\tau+t_n,\sigma(s)+t_n)|\Delta\tau\\ &\le \int_{[\sigma(s),t]_{\mathbb{T}}} C^2|A(\tau+t_n) - \bar{A}(\tau)| \cdot |e_{\ominus a}(t,\sigma(\tau))e_{\ominus a}(\tau+t_n,\sigma(s)+t_n)|\Delta\tau\\ &= C^2e_{\ominus a}(t,\sigma(s))\int_{[\sigma(s),t]_{\mathbb{T}}} |A(\tau+t_n) - \bar{A}(\tau)| \cdot |e_{\ominus a}(\tau,\sigma(\tau))|\Delta\tau\\ &= C^2e_{\ominus a}(t,\sigma(s))\int_{[\sigma(s),t]_{\mathbb{T}}} |A(\tau+t_n) - \bar{A}(\tau)| \cdot |1 + a\mu(\tau)|\Delta\tau.\end{aligned}$$

By (3.4) and [Theorem 2.1, [10]], for any $\varepsilon > 0$, there exists a positive integer N_1, such that when $n > N_1$, we have

$$\int_{[\sigma(s),t]_{\mathbb{T}}} |A(\tau + t_n) - \bar{A}(\tau)| \cdot |1 + a\mu(\tau)|\Delta\tau < \varepsilon,$$

since $A(\tau), \bar{A}(\tau)$ and $1 + a\mu(\tau)$ are bounded on $[\sigma(s), t]_{\mathbb{T}}$. Thus

$$|e_A(t + t_n, \sigma(s + t_n)) - e_{\bar{A}}(t, \sigma(s))| \leq C^2 \varepsilon e_{\ominus a}(t, \sigma(s)) \tag{3.6}$$

for $n > N_1$, and each $t \in \mathbb{T}, 0 < t - s < \mathcal{K}$. Notice that

$$\begin{aligned}
& x(t + t_n) - \bar{x}(t) \\
= & \int_{(-\infty, t+t_n]_{\mathbb{T}}} e_A(t + t_n, \sigma(s)) f(s)\Delta s - \int_{(-\infty, t]_{\mathbb{T}}} e_{\bar{A}}(t, \sigma(s))\bar{f}(s)\Delta s \\
= & \sum_{j=1}^{\infty} \int_{[t+t_n - j\mathcal{K}, t+t_n-(j-1)\mathcal{K}]_{\mathbb{T}}} e_A(t + t_n, \sigma(s)) f(s)\Delta s - \sum_{j=1}^{\infty} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} e_{\bar{A}}(t, \sigma(s))\bar{f}(s)\Delta s \\
= & \sum_{j=1}^{\infty} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} [e_A(t + t_n, \sigma(s + t_n)) f(s + t_n) - e_{\bar{A}}(t, \sigma(s))\bar{f}(s)]\Delta s \\
= & \sum_{j=1}^{\infty} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} e_A(t + t_n, \sigma(s + t_n))(f(s + t_n) - \bar{f}(s))\Delta s \\
& + \sum_{j=1}^{\infty} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} (e_A(t + t_n, \sigma(s + t_n)) - e_{\bar{A}}(t, \sigma(s)))\bar{f}(s)\Delta s \\
:= & K_1 + K_2.
\end{aligned}$$

By Hölder inequality, (3.2), and Propersition 3.2, we have

$$\begin{aligned}
K_1 \leq & \sum_{j=1}^{\infty} \left(\int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |e_A(t + t_n, \sigma(s + t_n))|^q \Delta s \right)^{\frac{1}{q}} \cdot \left(\int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |f(s + t_n) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} \\
\leq & \sum_{j=1}^{\infty} C \left(\int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |e_{\ominus a}(t + t_n, \sigma(s + t_n))|^q \Delta s \right)^{\frac{1}{q}} \\
& \cdot \mathcal{K}^{\frac{1}{p}} \left(\frac{1}{\mathcal{K}} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |f(s + t_n) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} \\
= & \sum_{j=1}^{\infty} C \left(\int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |e_{\ominus a}(t, \sigma(s))|^q \Delta s \right)^{\frac{1}{q}} \mathcal{K}^{\frac{1}{p}} \left(\frac{1}{\mathcal{K}} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |f(s + t_n) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} \\
\leq & \sum_{j=1}^{\infty} C |e_{\ominus a}(t, \sigma(t - (j-1)\mathcal{K})| \cdot \left(\frac{1}{\mathcal{K}} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |f(s + t_n) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}} \\
\leq & C\lambda_a \left(\frac{1}{\mathcal{K}} \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K}]_{\mathbb{T}}} |f(s + t_n) - \bar{f}(s)|^p \Delta s \right)^{\frac{1}{p}},
\end{aligned}$$

where $1/p+1/q=1$. By (3.4), for the ε given above, there exists a positive integer $N>N_1$, such that when $n>N$, we have $K_1\le C\lambda_a\varepsilon$. By (3.6) and Propersition 3.2, we get

$$\begin{aligned}
K_2\le&\sum_{j=1}^{\infty}\left(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K}]_{\mathbb{T}}}|\bar{f}(s)|^p\Delta s\right)^{\frac{1}{p}}\cdot\left(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K}]_{\mathbb{T}}}|e_A(t+t_n,\sigma(s+t_n))-e_{\bar{A}}(t,\sigma(s))|^q\Delta s\right)^{\frac{1}{q}}\\
\le&\sum_{j=1}^{\infty}\left(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K}]_{\mathbb{T}}}|e_A(t+t_n,\sigma(s+t_n))-e_{\bar{A}}(t,\sigma(s))|^q\Delta s\right)^{\frac{1}{q}}\mathcal{K}^{\frac{1}{p}}\|\bar{f}(s)\|_{S^p}\\
\le&\sum_{j=1}^{\infty}\left(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K}]_{\mathbb{T}}}|C^2\varepsilon e_{\ominus a}(t,\sigma(s))|^q\Delta s\right)^{\frac{1}{q}}\mathcal{K}^{\frac{1}{p}}\|\bar{f}(s)\|_{S^p}\\
\le&C^2\|\bar{f}(s)\|_{S^p}\varepsilon\mathcal{K}^{\frac{1}{p}}\sum_{j=1}^{\infty}e_{\ominus a}(t,\sigma(t)-(j-1)\mathcal{K})\mathcal{K}^{\frac{1}{q}}\\
\le&C^2\lambda_a\|\bar{f}(s)\|_{S^p}\varepsilon
\end{aligned}$$

for $n>N$, and $t\in\mathbb{T}$, where $1/p+1/q=1$. Thus we get $\lim_{n\to\infty}|x(t+t_n)-\bar{x}(t)|=0$ for each $t\in\mathbb{T}$. By the similar argument, we can prove that $\lim_{n\to\infty}|\bar{x}(t-t_n)-x(t)|=0$ for each $t\in\mathbb{T}$. Thus x is almost automorphic on $\mathbb{T}$, and the proof is complete.

Remark 3.1 *When $\mathbb{T}=\mathbb{R}$, it is easily to see that the condition* (3.9) *admits exponential dichotomy condition on $\mathbb{R}$, and the* Theorem 3.1 *is the generalization of* Theorem 3.8 *in* [14].

4 Almost automorphic solutions of cellular neural networks on time scales

In this section, we will study the existence and exponentially stable of almost automorphic solutions for system

$$x_i^{\Delta}(t)=-c_i(t)x_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}(t)f_j(x_j(t-\gamma_{ij}(t)))+I_i(t),\quad t\in\mathbb{T},\tag{4.1}$$

where $f_j\in C(\mathbb{R},\mathbb{R})$, and c_i,γ_{ij} are almost automorphic, and a_{ij},b_{ij},I_i are stepanov-like almost automorphic on $\mathbb{T}$, $i,j=1,2,\cdots,n$. The system (4.1) is supplemented with the initial values given by

$$x_i(s)=\varphi_i(s),\quad s\in[-\gamma,0]_{\mathbb{T}},\tag{4.2}$$

where $\varphi_i\in C_{rd}([-\gamma,0]_{\mathbb{T}},\mathbb{R})$, and $\gamma=\max_{1\le i,j\le n}\{\sup_{t\in\mathbb{T}}|\gamma_{ij}(t)|\}$.

Definition 4.1 *System* (4.1) *with initial condition* (4.2) *is said to be globally exponentially stable, if there are constants $\lambda>0$ and $M\ge 1$ such that for any two solutions $x(t,\varphi)$ and $x(t,\psi)$ with the initial functions φ,ψ, respectively, one has*

$$\|x(t,\varphi)-x(t,\psi)\|_{\infty}\le Me_{\ominus\lambda}(t,t_0)\|\varphi-\psi\|_{\infty},\quad t\ge 0,$$

with $\|f(\cdot)\|_{\infty}=\sup_{t\in\mathbb{T}}|f(t)|$.

By the similar argument of [Lemma 2.15, [6]], we can get the following result:

Lemma 4.1 *Let $c_i(t)$ be an almost automorphic function on $\mathbb{T}$, where $c_i(t) > 0, -c_i(t) \in \mathcal{R}^+, i = 1, 2, \cdots, n$, $t \in \mathbb{T}$, and $\min_{1\le i\le n}\{\inf_{t\in\mathbb{T}} c_i(t)\} = \tilde{m} > 0$, then the linear system*

$$x^{\Delta}(t) = diag(-c_1(t), -c_2(t), \cdots, -c_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

In the following, we denote $\underline{c_i} = \inf_{t\in\mathbb{T}} c_i(t)$. In order to proof the following theorem, we need the following conditions:

(H_1) There exist positive constants M_j and α_j such that

$$|f_j(x)| \le M_j, \quad |f_j(x) - f_j(y)| \le \alpha_j |x - y|, \quad x, y \in \mathbb{R}, \, j = 1, 2, \cdots, n.$$

(H_2) $\inf_{t\in\mathbb{T}} c_i(t) > 0, -c_i(t) \in \mathcal{R}^+, i = 1, 2, \cdots, n.$

(H_3) $\lambda_{\underline{c_i}} \left(\sum\limits_{j=1}^{n} \alpha_j (\|a_{ij}\|_{S^p} + \|b_{ij}\|_{S^p}) \right) < 1$, where $\lambda_{\underline{c_i}}$ is given in Proposition 3.2.

Theorem 4.1 *Assume* (H_1)–(H_3) *hold, and $t - \gamma_{ij}(t) \in \mathbb{T}$ for $i, j = 1, 2, \cdots, n$, $t \in \mathbb{T}$. Then the system* (4.1) *has a unique almost automorphic solution $x^*(t)$. Moreover, the unique almost automorphic solution of* (4.1) *with the initial condition* (4.2) *is exponentially stable.*

Proof. Let $E = \{\varphi \in S^pAA(\mathbb{T}, \mathbb{R}^n) : \|\varphi\|_{S^p} \le r\}$, where $r = \max_i\{\lambda_{\underline{c_i}} \sum_{j=1}^{n}(M_j(\|a_{ij}\|_{S^p} + \|b_{ij}\|_{S^p}) + \|I_i\|_{S^p})$. For any given $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n)^T \in E$, consider the following equation:

$$x_i^{\Delta}(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(\varphi_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(\varphi_j(t - \gamma_{ij}(t))) + I_i(t), \quad t \in \mathbb{T}. \qquad (4.3)$$

By Lemma 4.1, the associated homogeneous equation

$$x_i^{\Delta}(t) = -c_i(t)x_i(t), \quad i = 1, 2, \cdots, n.$$

admits an exponential dichotomy. By Proposition 2.1 and Theorem 2.1, we get the function

$$F(t) = (F_1(t), F_2(t), \cdots, F_n(t))^T$$

is stepanov-like almost automorphic, where

$$F_i(t) = \sum_{j=1}^{n} a_{ij}(t) f_j(\varphi_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(\varphi_j(t - \gamma_{ij}(t))) + I_i(t), i = 1, 2, \cdots, n.$$

By Theorem 3.1, we know the system (4.3) has an unique almost automorphic solution

$$x_{\varphi}(t) = \int_{(-\infty, t]_{\mathbb{T}}} e_{-c}(t, \sigma(s)) F(s) \Delta s = (x_{\varphi_1}(t), x_{\varphi_2}(t), \cdots, x_{\varphi_n}(t))^T, \quad t \in \mathbb{T},$$

where

$$x_{\varphi_i}(t)=\int_{(-\infty,t]_{\mathbb{T}}} e_{-c_i}(t,\sigma(s))\left(\sum_{j=1}^{n} a_{ij}(s)f_j(\varphi_j(s))+\sum_{j=1}^{n} b_{ij}(s)f_j(\varphi_j(s-\gamma_{ij}(s)))+I_i(s)\right)\Delta s.$$

Define the operator by

$$T(\varphi)(t)=(x_{\varphi_1}(t),x_{\varphi_2}(t),\cdots,x_{\varphi_n}(t))^T,\varphi\in E.$$

From Proposition 3.2, and (H_1)–(H_3), we get

$$\begin{aligned}
|x_{\varphi_i}(t)|=&\left|\int_{(-\infty,t]_{\mathbb{T}}} e_{-c_i}(t,\sigma(s))\left(\sum_{j=1}^{n} a_{ij}(s)f_j(\varphi_j(s))+\sum_{j=1}^{n} b_{ij}(s)f_j(\varphi_j(s-\gamma_{ij}(s)))+I_i(s)\right)\Delta s\right|\\
\leq&\left|\int_{(-\infty,t]_{\mathbb{T}}} e_{-\underline{c_i}}(t,\sigma(s))\left(\sum_{j=1}^{n}(a_{ij}(s)+b_{ij}(s))M_j+I_i(s)\right)\Delta s\right|\\
=&\left|\sum_{l=1}^{\infty}\int_{[t-l\mathcal{K},t-(l-1)\mathcal{K}]_{\mathbb{T}}} e_{-\underline{c_i}}(t,\sigma(s))\left(\sum_{j=1}^{n}(a_{ij}(s)+b_{ij}(s))M_j+I_i(s)\right)\Delta s\right|\\
\leq&\sum_{l=1}^{\infty}\left(\int_{[t-l\mathcal{K},t-(l-1)\mathcal{K}]_{\mathbb{T}}}|e_{-\underline{c_i}}(t,\sigma(s))|^q\Delta s\right)^{\frac{1}{q}}\\
&\cdot\mathcal{K}^{\frac{1}{p}}\left(\frac{1}{\mathcal{K}}\int_{[t-l\mathcal{K},t-(l-1)\mathcal{K}]_{\mathbb{T}}}|\sum_{j=1}^{n}(a_{ij}(s)+b_{ij}(s))M_j+I_i(s)|^p\Delta s\right)^{\frac{1}{p}}\\
\leq&\sum_{l=1}^{\infty} e_{-\underline{c_i}}(t,\sigma(t-(l-1)\mathcal{K}))\sum_{j=1}^{n}(M_j(\|a_{ij}\|_{S^p}+\|b_{ij}\|_{S^p})+\|I_i\|_{S^p})\\
\leq&\lambda_{\underline{c_i}}\sum_{j=1}^{n}(M_j(\|a_{ij}\|_{S^p}+\|b_{ij}\|_{S^p})+\|I_i\|_{S^p})=r,\ i=1,2,\cdots,n.
\end{aligned}$$

Thus $\|T(\varphi)\|_\infty\leq r$, and then $T(E)\subset E$.

Let $\varphi,\psi\in E$, by (H_1), we have

$$\begin{aligned}
&|x_{\varphi_i}-x_{\psi_i}|\\
=&\left|\int_{(-\infty,t]_{\mathbb{T}}} e_{-c_i}(t,\sigma(s))\left(\sum_{j=1}^{n} a_{ij}(s)(f_j(\varphi_j(s))-f_j(\psi_j(s)))\right.\right.\\
&\left.\left.+\sum_{j=1}^{n} b_{ij}(s)(f_j(\varphi_j(s-\gamma_{ij}(s)))-f_j(\psi_j(s-\gamma_{ij}(s))))\right)\Delta s\right|\\
\leq&\left|\int_{(-\infty,t]_{\mathbb{T}}} e_{-c_i}(t,\sigma(s))\left(\sum_{j=1}^{n}\alpha_j[a_{ij}(s)(\varphi_j(s)-\psi_j(s))+b_{ij}(s)(\varphi_j(s-\gamma_{ij}(s))-\psi_j(s-\gamma_{ij}(s)))]\right)\Delta s\right|\\
\leq&\lambda_{\underline{c_i}}\left(\sum_{j=1}^{n}\alpha_j(\|a_{ij}\|_{S^p}+\|b_{ij}\|_{S^p})\right)\|\varphi-\psi\|_\infty.
\end{aligned}$$

By (H_3), we know T is a contraction mapping from E to E, and then T has a fixed point $x^*(t)$ in E, which is the almost automorphic solution for (4.1).

By the similar argument of [Theorem 6.1 [10]], we can prove that the almost automorphic solution $x^*(t)$ is exponentially stable, and here we omit the details. The proof is complete.

At last, we give the following example to illustrate our results Theorem 4.1.

Example 4.1 *Consider the following neural network:*

$$x_i^{\Delta}(t) = -c_i(t)x_i(t) + \sum_{j=1}^{2} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{2} b_{ij}(t)f_j(x_j(t-\gamma_{ij}(t))) + I_i(t), \quad t \in \mathbb{R}, i = 1, 2,$$

with the initial codition

$$x_1(s) = \sin s, \quad x_2(s) = \cos s, \quad s \in [-1, 0],$$

where

$$\begin{aligned}
&c_1(t) = \sin\frac{1}{2+\cos t+\cos \pi t} + 2, \quad c_2(t) = \sin\frac{1}{2+\cos t+\cos\sqrt{t}} + 2,\\
&f_1(t) = \frac{1}{12}\cos^2 t, \quad f_2(t) = \frac{1}{12}\sin^2 t,\\
&(a_{ij}(t))_{2\times 2} = \begin{pmatrix} \frac{1}{2}|\sin t|, & \frac{1}{2}|\cos t| \\ \frac{1}{3}|\sin t|, & \frac{1}{3}|\cos\sqrt{2}t| \end{pmatrix}, \quad (b_{ij}(t))_{2\times 2} = \begin{pmatrix} \frac{9}{7}|\cos t|, & \frac{9}{7}|\sin t| \\ \frac{10}{7}|\cos t|, & \frac{10}{7}|\sin\sqrt{2}t| \end{pmatrix},\\
&(\gamma_{ij}(t))_{2\times 2} = \begin{pmatrix} \cos\frac{1}{1+|\sin t|}, & \sin\frac{1}{1+|\cos t|} \\ \cos\frac{1}{1+|\sin\sqrt{2}t|}, & \sin\frac{1}{1+|\cos\sqrt{2}t|} \end{pmatrix},\\
&I_1(t) = g_1(t)h_1(t), \quad I_2(t) = g_2(t)h_2(t),
\end{aligned}$$

with

$$\begin{aligned}
&g_1(t) = \begin{cases} \frac{1}{1+t^2}, & t \in \mathbb{Z} \\ \cos t, & t \in \mathbb{T}\setminus\mathbb{Z} \end{cases}, \quad g_2(t) = \begin{cases} \frac{2}{1+t^2}, & t \in \mathbb{Z} \\ \cos 2t, & t \in \mathbb{T}\setminus\mathbb{Z} \end{cases},\\
&h_1(t) = \begin{cases} 1, & t \in \mathbb{Z} \\ \sin\frac{1}{2+\sin\sqrt{2}t}. & t \in \mathbb{T}\setminus\mathbb{Z} \end{cases}, \quad h_2(t) = \begin{cases} 1, & t \in \mathbb{Z} \\ \cos\frac{1}{2+\cos\sqrt{2}t}. & t \in \mathbb{T}\setminus\mathbb{Z} \end{cases}
\end{aligned}$$

It is easy to see that f_1, f_2 satisfy (H_1), and $\alpha_1 = \alpha_2 = 1/6$. It is not difficult to calculate that $\underline{c_1} = 1, \underline{c_2} = 1$, and the condition (H_2) is true. Since $\lambda_{\underline{c_1}} = \frac{1}{1-e^{-1}}, \lambda_{\underline{c_2}} = \frac{1}{1-e^{-1}}$, $\|a_{1j}\|_{S^p} = \frac{1}{2}, \|a_{2j}\|_{S^p} = \frac{1}{3}$, $\|b_{1j}\|_{S^p} = \frac{9}{7}, \|b_{2j}\|_{S^p} = \frac{10}{7}$, for $j = 1, 2$, we have

$$\lambda_{\underline{c_1}} \sum_{j=1}^{2} \alpha_1(\|a_{1j}\|_{S^p} + \|b_{1j}\|_{S^p}) < 1, \quad \lambda_{\underline{c_2}} \sum_{j=1}^{2} \alpha_1(\|a_{2j}\|_{S^p} + \|b_{2j}\|_{S^p}) < 1,$$

and (H_3) holds. . Now, by Theorems 4.1, system (4.1) has a unique almost automorphic solution in the region $E = \{\varphi \in S^pAA(\mathbb{T}, \mathbb{R}^2) : \|\varphi\|_\infty \le r\}$, with $r = \frac{25}{7(1-e^{-1})}$, which is globally exponential stable.

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