Received: 00 Month 0000 | Revised: 00 Month 0000 | Accepted: 00 Month 0000

DOI: xxx/xxxx

ARTICLE TYPE

Numerical solution of coupled Lane-Emden boundary value problems using the Bernstein collocation method

Julee Shahni | Randhir Singh*

[1]BIT Mesra, Ranchi-835215, Jharkhand, India

Correspondence
*Randhir Singh. Email: randhir.math@gmail.com

Abstract

In this paper, we provide an efficient numerical technique based on the Bernstein polynomials for numerical approximation of the coupled Lane-Emden type equation which arises in various fields of applied mathematics, physical and chemical sciences. We consider the equivalent integral form of the coupled Lane-Emden boundary value problems. The Bernstein collocation method is used to convert the integral equation into a system of nonlinear equations. This system is then solved efficiently by suitable iterative method. The error analysis of the current method is discussed. The accuracy of the proposed method is examined by calculating the maximum absolute error L_∞, the L_2 error and the residual error of some numerical examples. The obtained numerical results are compared with the exact solutions and the results obtained by the other known techniques.

KEYWORDS:
Coupled Lane-Emden equations; Bernstein polynomials; Functional approximation; Green's Function; Error analysis.

1 | INTRODUCTION

The Lane-Emden equation was first proposed by J. H. Lane[1] during his analysis on the theory of stellar evolution and studied in details by R. Emden[2]. The Lane-Emden equations are used to model several scientific events in mathematical physics, astrophysics and biochemistry such as, the first kind Lane-Emden equation,

$$u''(t) + \frac{2}{t}u'(t) + u^n(t) = 0,$$

arises in the study of heat explosion[3]. The Emden-Fowler type equation,

$$y''(t) + \frac{2}{t}y'(t) - \frac{\alpha\ y(t)}{y(t) + \beta} = 0,$$

was used in calculation of oxygen concentration inside a spherical cell[4]. The second kind Lane-Emden equation,

$$y''(t) + \frac{k}{t}y'(t) - \alpha\ \exp\left(\frac{y(t)}{1 + \xi\ y(t)}\right) = 0,$$

was used in modeling thermal explosion in a rectangular slab[5] with $k = 0$ and in infinite cylinder and sphere[6] with $k = 1, 2$, respectively.

In this paper, we consider the following class of coupled Lane-Emden boundary value problems

$$\begin{cases} x_1''(t) + \dfrac{k_1}{t} x_1'(t) + \psi_1\big(t, x_1(t), x_2(t)\big) = 0, \quad t \in (0,1), \\ x_2''(t) + \dfrac{k_2}{t} x_2'(t) + \psi_2\big(t, x_1(t), x_2(t)\big) = 0, \\ x_1'(0) = x_2'(0) = 0, \;\; a_1\ x_1(1) + b_1\ x_1'(1) = c_1, \;\; a_2\ x_2(1) + b_2\ x_2'(1) = c_2, \end{cases} \tag{1}$$

where $k_1 > 0$, $k_2 > 0$, $a_1 > 0$, $a_2 > 0$, b_1, b_2, c_1 and c_2 are real constants.

The coupled Lane-Emden equations (1) have significant contribution in various scientific phenomenon such as, in 2011, Flockerzi and Sundmacher[7] studied the existence of the following coupled Lane-Emden equations which arises in dusty fluid model

$$\begin{cases} x_1''(t) + \dfrac{2}{t} x_1'(t) = \alpha_1\ x_1^2(t) + \alpha_2\ x_1(t)\ x_2(t), \;\; t \in (0,1), \\ x_2''(t) + \dfrac{2}{t} x_2'(t) = \alpha_3\ x_1^2(t) + \alpha_4\ x_1(t)\ x_2(t), \\ x_1'(0+) = x_2'(0+) = 0, \;\; x_1(1) = \beta_1, \;\; x_2(1) = \beta_2, \end{cases} \tag{2}$$

where α_1, α_2, α_3, α_4 are non-zero real constants and β_1, β_2 are positive real constants. They provided the integral manifold on which the solutions of (2) necessarily lie by transforming the above boundary value problem to terminal value problem.

In 2014, Muthukumar et. al.[8] constructed a mathematical model on the oxygen and the carbon concentrations in surplus sludge production from water treatment plants which is governed by the following system of Lane-Emden equations

$$\begin{cases} x_1''(\sigma) + \dfrac{2}{\sigma}\ x_1'(\sigma) = -e + a\ \psi_1(x_1(\sigma), x_2(\sigma)) + b\ \psi_2(x_1(\sigma), x_2(\sigma)), \\ x_2''(\sigma) + \dfrac{2}{\sigma}\ x_2'(\sigma) = c\ \psi_1(x_1(\sigma), x_2(\sigma)) + d\ \psi_2(x_1(\sigma), x_2(\sigma)), \\ x_1'(0) = x_2'(0) = 0, \;\; x_1(1) = x_2(1) = 1, \;\; i = 1, 2, \end{cases} \tag{3}$$

where a, b, c, d and e are real constants, $x_1(\sigma)$, $x_2(\sigma)$ are the oxygen and the carbon concentrations, respectively. Here σ is the radius of a spherical floc particle. The nonlinear functions $\psi_1(x_1, x_2)$ and $\psi_2(x_1, x_2)$ are given by

$$\psi_1\ (x_1(\sigma), x_2(\sigma)) = \frac{x_1(\sigma)\ x_2(\sigma)}{(l_1 + x_1(\sigma))(m_1 + x_2(\sigma))},$$
$$\psi_2\ (x_1(\sigma), x_2(\sigma)) = \frac{x_1(\sigma)\ x_2(\sigma)}{(l_2 + x_1(\sigma))(m_2 + x_2(\sigma))},$$

with l_1, l_2, m_1, m_2 are appropriate real constants. They used Adomian decomposition method for approximating analytical expressions for carbon and oxygen substrates for specific values of the parameters.

There is a huge literature on the numerical solution of Lane-Emden equations with various initial and boundary conditions such as the finite difference method[9,10,11], spline finite difference method[12], parametric-spline method[13], cubic spline method[14], Tau method[15], B-spline collocation method[16], Adomian decomposition method with Green's function[17,18,19,20], Laguerre wavelets collocation method[21] and the Haar-wavelet collocation method[22,23].

The system of Lane-Emden type equations with initial conditions were solved by the Hermite spline method[24], Adomian decomposition method[25], Chebyshev operational method[26], differential transform method[27], and Taylor series method[28].

In[29,30], authors used modified Adomian decomposition method for finding numerical solution of (2) and (3), respectively. In[31], authors provided numerical solution of (3) using variational iteration method. In[32], authors applied the homotopy analysis method with Green's function for the approximate series solutions of (1). Recently, in[33,34], authors used an numerical method based on series approximation for solving (2).

The Bernstein polynomials are one of the famous bases of polynomials space. Such polynomials have several meaningful properties, such as the continuity, the positivity and complete basis formation. Numerical methods based on the Bernstein polynomials have been used to solve various differential and integral equations[35,36,37,38]. The numerical methods based on the Bernstein polynomials have been applied to solve Lane-Emden equation with initial value in[39,40] and to solve some linear integral system of equations in[41].

To the best of our knowledge, the the Bernstein collocation method has not been applied for numerical solution of the coupled Lane-Emden equations (1). In this paper, we propose an efficient collocation method based on the Bernstein polynomials for numerical solution of the coupled Lane-Emden boundary value problems (1). In order to avoid singularity, we first transform the coupled Lane-Emden BVPs (1) into the equivalent integral equations. The Bernstein collocation method is used to convert the integral equations into a system of nonlinear equations. Then the iterative numerical technique is used to find solutions of the system of nonlinear equations. In addition, the error analysis of the proposed method is provided under quite general conditions. The accuracy of the proposed method is examined by calculating the maximum absolute error L_∞, the L_2 error and the residual error of five numerical examples. The obtained numerical results are compared with the exact solutions and the results obtained by the other known techniques.

2 | INTEGRAL FORM OF THE COUPLED LANE-EMDEN BVPS

In this section, we establish the equivalent integral form of the coupled Lane-Emden BVPs

$$\begin{cases} x_1''(t) + \dfrac{k_1}{t} x_1'(t) + \psi_1\big(t, x_1(t), x_2(t)\big) = 0, \quad t \in (0,1), \\ x_2''(t) + \dfrac{k_2}{t} x_2'(t) + \psi_2\big(t, x_1(t), x_2(t)\big) = 0, \\ x_1'(0) = x_2'(0) = 0, \;\; a_1\, x_1(1) + b_1\, x_1'(1) = c_1, \;\; a_2\, x_2(1) + b_2\, x_2'(1) = c_2, \end{cases} \tag{4}$$

which is obtained as

$$\begin{cases} x_1(t) = \dfrac{c_1}{a_1} + \displaystyle\int_0^1 G_1(t,s)\, s^{k_1}\, \psi_1\big(s, x_1(s), x_2(s)\big)\, ds, \quad t \in (0,1), \\ x_2(t) = \dfrac{c_2}{a_2} + \displaystyle\int_0^1 G_2(t,s)\, s^{k_2}\, \psi_2\big(s, x_1(s), x_2(s)\big)\, ds, \end{cases} \tag{5}$$

where $G_1(t,s)$ and $G_2(t,s)$ are the Green's functions given by

$$G_1(t,s) = \begin{cases} \ln(s) - \dfrac{b_1}{a_1}, \; t \le s, \\ \ln(t) - \dfrac{b_1}{a_1}, \;\; s \le t, \end{cases} \quad \text{and} \quad G_2(t,s) = \begin{cases} \ln(s) - \dfrac{b_2}{a_2}, \; t \le s, \\ \ln(t) - \dfrac{b_2}{a_2}, \;\; s \le t, \end{cases} \tag{6}$$

for $k_1 = k_2 = 1$ and

$$G_1(t,s) = \begin{cases} \dfrac{s^{1-k_1} - 1}{1-k_1} - \dfrac{b_1}{a_1}, \; t \le s, \\ \dfrac{t^{1-k_1} - 1}{1-k_1} - \dfrac{b_1}{a_1}, \;\; s \le t, \end{cases} \quad \text{and} \; G_2(t,s) = \begin{cases} \dfrac{s^{1-k_2} - 1}{1-k_2} - \dfrac{b_2}{a_2}, \; t \le s, \\ \dfrac{t^{1-k_2} - 1}{1-k_2} - \dfrac{b_2}{a_2}, \;\; s \le t, \end{cases} \tag{7}$$

for $k_1 \neq 1$, $k_2 \neq 1$, respectively.

3 | THE BERNSTEIN COLLOCATION METHOD (BCM)

In this section, we will provide some preliminaries and notations of the Bernstein polynomials technique. We will also derive the BCM for the numerical approximation of the integral equation (5).

Definition 1. The Bernstein basis polynomials of degree n [35] are defined as

$$B_i^n(t) = \begin{cases} \dbinom{n}{i} t^i (1-t)^{n-i}, \; 0 \le i \le n, \\ 0, \qquad\qquad\qquad\quad i < 0, \; i > n, \end{cases} \tag{8}$$

where $\dbinom{n}{i} = \dfrac{n!}{i!(n-i)!}$, $n \in \mathbb{N}$, $i = 0, 1, 2, \cdots n$ and $t \in [0,1]$.

These polynomials form a complete basis and have several useful properties. Some of the important properties are

(i) $B_i^n(t) \geq 0, \;\; \forall\, t \in [0,1]$ and $i = 0, 1, 2, \ldots, n$.

(ii) $B_0^n(0) = B_n^n(1) = 1$ and $B_i^n(0) = B_i^n(1) = 0, \;$ for $1 \leq i \leq n-1$.

(iii) $\sum_{i=0}^{n} B_i^n(t) = 1$.

Any function $g(t) \in L^2[0,1]$ can be approximated by the Bernstein basis polynomials as

$$g(t) = \sum_{i=0}^{\infty} a_i B_i^n(t). \tag{9}$$

For numerical purpose, we consider the first $(n+1)$ terms of the above expansion as

$$g(t) \approx \sum_{i=0}^{n} a_i B_i^n(t) = \mathbf{A}^T \mathbf{B}(t), \tag{10}$$

where $\mathbf{A}$ and $\mathbf{B}(t)$ are $(n+1) \times 1$ column vectors defined as

$$\mathbf{A} = [a_0, a_1, \cdots, a_n]^T, \;\; \mathbf{B}(t) = [B_0^n(t), B_1^n(t), \cdots, B_n^n(t)]^T. \tag{11}$$

3.1 | Derivation of the Bernstein Collocation Method

Let us again consider the integral equations of the coupled Lane-Emden BVPs (1) as

$$\begin{cases} x_1(t) = \dfrac{c_1}{a_1} + \displaystyle\int_0^1 G_1(t,s)\; s^{k_1}\; \psi_1\big(s, x_1(s), x_2(s)\big)\; ds, \qquad t \in (0,1), \\[2ex] x_2(t) = \dfrac{c_2}{a_2} + \displaystyle\int_0^1 G_2(t,s)\; s^{k_2}\; \psi_2\big(s, x_1(s), x_2(s)\big)\; ds. \end{cases} \tag{12}$$

For convenience, we consider $\psi_1(t, x_1, x_2)$ and $\psi_2(t, x_1, x_2)$ as

$$z_1(t) = \psi_1\big(t, x_1(t), x_2(t)\big), \;\; z_2(t) = \psi_2\big(t, x_1(t), x_2(t)\big). \tag{13}$$

On approximating $x_1(t)$, $x_2(t)$, $z_1(t)$ and $z_2(t)$ by the Bernstein polynomial approximation, we get

$$x_1(t) \approx \mathbf{A}_1^T\; \mathbf{B}(t), \;\;\; x_2(t) \approx \mathbf{A}_2^T\; \mathbf{B}(t), \tag{14}$$

$$z_1(t) \approx \mathbf{C}_1^T\; \mathbf{B}(t), \;\;\; z_2(t) \approx \mathbf{C}_2^T\; \mathbf{B}(t), \tag{15}$$

where $\mathbf{A}_i^T = [a_{i0}, a_{i1}, a_{i2}, \cdots, a_{in}]$ and $\mathbf{C}_i^T = [c_{i0}, c_{i1}, c_{i2}, \cdots, c_{in}]$, for $i = 1, 2$.

Using (13), (14) and (15), the integral equation (12) becomes

$$\begin{cases} \mathbf{A}_1^T\; \mathbf{B}(t) = \dfrac{c_1}{a_1} + \displaystyle\int_0^1 G_1(t,s)\; s^{k_1}\; \mathbf{C}_1^T\; \mathbf{B}(s) ds, \\[2ex] \mathbf{A}_2^T\; \mathbf{B}(t) = \dfrac{c_2}{a_2} + \displaystyle\int_0^1 G_2(t,s)\; s^{k_2}\; \mathbf{C}_2^T\; \mathbf{B}(s) ds, \end{cases} \tag{16}$$

which can further be written as

$$\begin{cases} \mathbf{A}_1^T\; \mathbf{B}(t) = \dfrac{c_1}{a_1} + \mathbf{C}_1^T\; K_1(t), \\[1ex] \mathbf{A}_2^T\; \mathbf{B}(t) = \dfrac{c_2}{a_2} + \mathbf{C}_2^T\; K_2(t), \end{cases} \tag{17}$$

where

$$K_1(t) = \int_0^1 G_1(t,s)\; s^{k_1}\; \mathbf{B}(s)\; ds, \quad K_2(t) = \int_0^1 G_2(t,s)\; s^{k_2}\; \mathbf{B}(s)\; ds. \tag{18}$$

Substituting the expressions from equation (15) into equation (13) and using (17), we have

$$\begin{cases} \mathbf{C}_1^T\ \mathbf{B}(t) = \psi_1\left(t, \dfrac{c_1}{a_1} + \mathbf{C}_1^T\ K_1(t), \dfrac{c_2}{a_2} + \mathbf{C}_2^T\ K_2(t)\right), \\ \mathbf{C}_2^T\ \mathbf{B}(t) = \psi_2\left(t, \dfrac{c_1}{a_1} + \mathbf{C}_1^T\ K_1(t), \dfrac{c_2}{a_2} + \mathbf{C}_2^T\ K_2(t)\right). \end{cases} \tag{19}$$

Inserting the collocation points $t_j = t_0 + {}^{j}/_{n}$, $j = 0, 1, 2 \dots, n$ into (19), we get the nonlinear system of equations as

$$\begin{cases} \mathbf{C}_1^T\ \mathbf{B}(t_j) - \psi_1\left(t_j, \dfrac{c_1}{a_1} + \mathbf{C}_1^T\ K_1(t_j), \dfrac{c_2}{a_2} + \mathbf{C}_2^T\ K_2(t_j)\right) = 0, \\ \mathbf{C}_2^T\ \mathbf{B}(t_j) - \psi_2\left(t_j, \dfrac{c_1}{a_1} + \mathbf{C}_1^T\ K_1(t_j), \dfrac{c_2}{a_2} + \mathbf{C}_2^T\ K_2(t_j)\right) = 0, \end{cases} \tag{20}$$

with the unknowns $\mathbf{C}_1^T$ and $\mathbf{C}_2^T$. We use Newton's iteration method to find these unknowns. Firstly we write the system of equation (20) in matrix form as

$$\mathbf{F}(\mathbf{C}) = \mathbf{0}, \tag{21}$$

where

$$\mathbf{C} = [\mathbf{C}_1, \mathbf{C}_2]^T, \tag{22}$$

$$\mathbf{F}(\mathbf{C}) = [F_{10}(\mathbf{C}), F_{11}(\mathbf{C}), \cdots, F_{1n}(\mathbf{C}), F_{20}(\mathbf{C}), F_{21}(\mathbf{C}), \cdots, F_{2n}(\mathbf{C})]^T, \tag{23}$$

$$F_{ij}(\mathbf{C}) = \mathbf{C}_i^T\ \mathbf{B}(t_j) - f_i\left(t_j, g(t_j) + \mathbf{C}^T\ K(t_j)\right), \quad i = 1, 2, \tag{24}$$

$$\mathbf{0} = [0, 0, \cdots, 0]^T \in \mathbb{R}^{2n+2}. \tag{25}$$

To find the numerical value of the unknowns, the Newton's iteration method is applied to the nonlinear system (21) as

$$\mathbf{C}^{[m+1]} - \mathbf{C}^{[m]} = -J^{-1}(\mathbf{C}^{[m]})\mathbf{F}(\mathbf{C}^{[m]}), \quad m = 0, 1, 2, \cdots \tag{26}$$

where

$$J(\mathbf{C}) = \begin{bmatrix} \frac{\partial F_{10}}{\partial c_{10}} & \dots & \frac{\partial F_{1n}}{\partial c_{10}} & \frac{\partial F_{20}}{\partial c_{10}} & \dots & \frac{\partial F_{2n}}{\partial c_{10}} \\ \frac{\partial F_{10}}{\partial c_{11}} & \dots & \frac{\partial F_{1n}}{\partial c_{11}} & \frac{\partial F_{20}}{\partial c_{11}} & \dots & \frac{\partial F_{2n}}{\partial c_{11}} \\ \vdots & & & & & \\ \frac{\partial F_{10}}{\partial c_{1n}} & \dots & \frac{\partial F_{1n}}{\partial c_{1n}} & \frac{\partial F_{20}}{\partial c_{1n}} & \dots & \frac{\partial F_{2n}}{\partial c_{1n}} \\ \frac{\partial F_{10}}{\partial c_{20}} & \dots & \frac{\partial F_{1n}}{\partial c_{20}} & \frac{\partial F_{20}}{\partial c_{20}} & \dots & \frac{\partial F_{2n}}{\partial c_{20}} \\ \vdots & & & & & \\ \frac{\partial F_{10}}{\partial c_{2n}} & \dots & \frac{\partial F_{1n}}{\partial c_{2n}} & \frac{\partial F_{20}}{\partial c_{2n}} & \dots & \frac{\partial F_{2n}}{\partial c_{2n}} \end{bmatrix}$$

and $\mathbf{C}^{[m]}$ is the m-th iterative solution of (20). The numerical values of the unknown coefficients will be substituted in equation (17) to get the numerical solution of (12).

4 | ERROR ANALYSIS

In this section, the error bound of the BCM is provided. To do this, we consider the following integral equation

$$\mathbf{x} = \frac{\mathbf{c}}{\mathbf{a}} + \int_0^1 \mathbf{G}(t, s)\ s^{\mathbf{k}}\ \mathbf{f}(s, \mathbf{x}(s))\ ds = 0, \quad t \in (0, 1), \tag{27}$$

where $\mathbf{x} = [x_1, x_2]^T$, $\frac{\mathbf{c}}{\mathbf{a}} = [\frac{\mathbf{c_1}}{\mathbf{a_1}}, \frac{\mathbf{c_2}}{\mathbf{a_2}}]^T$, $\mathbf{G} = [G_1, G_2]^T$, $\mathbf{k} = [k_1, k_2]^T$, $\mathbf{f} = [\psi_1, \psi_2]^T$.

Let $\mathbf{Y} = (C[0, 1], \|\mathbf{x}\|)$ be the Banach space with the norm defined as

$$\|\mathbf{x}\| = \max\left\{\|x_1\|, \|x_2\|\right\}, \tag{28}$$

where $\|x_i\| = \max_{t\in[0,1]} |x_i(t)|$ for $i = 1, 2$.

Theorem 1. For all functions $g(t)$ in $C[0,1]$, the sequence $\{B_n(g)\}$ converges uniformly to g, where $B_n(g) = \sum_{i=0}^{n} a_i B_i^n(t)$ is the Bernstein approximation function.

Proof. For more details see[42]. From this theorem we conclude that for any $\epsilon > 0$ there exists an n such that

$$\|B_n(g) - g\| < \epsilon.$$

□

Theorem 2. If g is bounded and and g'' exists in $[0,1]$ then the error bound for the Bernstein's approximation function is given as

$$\|B_n(g) - g\| \le \frac{1}{2n} t(1-t)\|g''\|, \quad t \in (0,1). \tag{29}$$

Proof. For more details see[43]. □

Theorem 3. Let $\mathbf{Y}$ be the Banach space with the norm defined by (28) and let $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{x}_n = [x_{1n}, x_{2n}]^T$ be the exact and the approximate solutions of (27). Assume that the nonlinear function $\mathbf{f}(t, \mathbf{x})$ satisfies the Lipschitz condition

$$|\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{x}^*)| \le \sum_{j=1}^{2} l_j \left| x_j - x_j^* \right|, \quad j = 1, 2, \tag{30}$$

where l_1 and l_2 are the Lipschitz constants. Then the error bound for the Bernstein collocation method is estimated as

$$\|\mathbf{x} - \mathbf{x}_n\| \le Ml \frac{\max\{\|x_1''\|, \|x_2''\|\}}{4n}. \tag{31}$$

Proof. Consider

$$\begin{aligned}
\|\mathbf{x} - \mathbf{x}_n\| &= \max_{t\in[0,1]} \left| \frac{\mathbf{c}}{\mathbf{a}} + \int_0^1 \mathbf{G}(t,s) s^{\mathbf{k}}\, \mathbf{f}(s, \mathbf{x}(s)) ds - \frac{\mathbf{c}}{\mathbf{a}} - \int_0^1 \mathbf{G}(t,s) s^{\mathbf{k}}\, \mathbf{f}(s, \mathbf{x}_n(s)) ds \right| \\
&= \max_{t\in[0,1]} \left| \int_0^1 \mathbf{G}(t,s) s^{\mathbf{k}} \left(\mathbf{f}(s,\mathbf{x}(s)) - \mathbf{f}(s,\mathbf{x}_n(s)) \right) ds \right| \\
&\le \max_{t\in[0,1]} \left| \int_0^1 \mathbf{G}(t,s) s^{\mathbf{k}} ds \right| \left| \mathbf{f}(s,\mathbf{x}(s)) - \mathbf{f}(s,\mathbf{x}_n(s)) \right|.
\end{aligned}$$

Since $\mathbf{f}(t, \mathbf{x})$ satisfies the Lipschitz condition, therefore the above inequality becomes

$$\begin{aligned}
\|\mathbf{x} - \mathbf{x}_n\| &\le \max_{t\in[0,1]} \left| \int_0^1 \mathbf{G}(t,s) s^{\mathbf{k}} ds \right| \times \sum_{j=1}^{2} l_j \max_{s\in[0,1]} \left| x_j(s) - x_{jn}(s) \right| \\
&\le \max_{t\in[0,1]} \left| \int_0^1 \mathbf{G}(t,s) s^{\mathbf{k}} ds \right| \times 2l \max_{s\in[0,1]} \left\{ \left| x_1(s) - x_{1n}(s) \right|, \left| x_2(s) - x_{2n}(s) \right| \right\},
\end{aligned}$$

where $l = \max\{l_1, l_2\}$. Since, $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{x}_n = [x_{1n}, x_{2n}]^T$, so we have

$$\|\mathbf{x} - \mathbf{x}_n\| \le 2Ml \max_{s\in[0,1]} |\mathbf{x}(s) - \mathbf{x}_n(s)|, \tag{32}$$

with

$$M = \max_{t\in[0,1]} \left| \int_0^1 \mathbf{G}(t,s)\, s^{\mathbf{k}} ds \right|.$$

Applying the Bernstein collocation method, the approximate solution of (27) is $B_n(\mathbf{x})$. Hence replacing $\mathbf{x}_n(s)$ by $B_n(\mathbf{x}(s))$, equation (32) reduces to

$$\|\mathbf{x}-\mathbf{x}_n\| \le 2Ml \max_{s\in[0,1]} |\mathbf{x}(s) - B_n(\mathbf{x}(s))|. \tag{33}$$

Using the result from equation (29) into equation (33), we obtain

$$\|\mathbf{x}-\mathbf{x}_n\| \le 2Ml\|\mathbf{x}-B_n(\mathbf{x})\| \le 2Ml\frac{\|\mathbf{x}''\|}{2n} \max_{s\in[0,1]}[s(1-s)]. \tag{34}$$

Hence, we have

$$\|\mathbf{x}-\mathbf{x}_n\| \le Ml\frac{\|\mathbf{x}''\|}{4n} = Ml\frac{\max\{\|x_1''\|, \|x_2''\|\}}{4n}. \tag{35}$$

□

5 | NUMERICAL RESULTS

In this section, we compare our numerical results with the exact solutions and the results obtained by some other known methods. For comparison purpose, we define the maximum absolute error and the L_2 absolute error as

$$L^i_\infty := \max_{t\in[0,1]} |x_i(t) - x_{in}(t)|, \tag{36}$$

$$L^i_2 := \left(\sum_{j=1}^{m} |x_i(t_j) - x_{in}(t_j)|^2\right)^{1/2}, \quad i = 1,2, \tag{37}$$

where $x_i(t)$ and $x_{in}(t)$ are the exact and the approximate solutions, respectively. When the exact solutions are not available in the literature, we define the absolute residual errors as

$$r_{in}(t) := |x''_{in}(t) + \frac{k_i}{t}x'_{in}(t) + \psi_i(t, x_{1n}(t), x_{2n}(t))|, \tag{38}$$

$$R_{im}(t) := |\phi''_{im}(t) + \frac{k_i}{t}\phi'_{im}(t) + \psi_i(t, \phi_{1m}(t), \phi_{2m}(t))|, \quad i = 1,2, \tag{39}$$

and the maximal residual errors as

$$Mr_{in} := \max_{t\in[0,1]} |r_{in}(t)|, \tag{40}$$

$$MR_{im} := \max_{t\in[0,1]} |R_{im}(t))|, \quad i = 1,2, \tag{41}$$

where $\phi_{im}(t)$ are the approximate solutions obtained by the series approximation method (SAM)[33].

Problem 1. Consider the following coupled Lane-Emden BVPs

$$\begin{cases} x_1''(t) + \dfrac{5}{t}x_1'(t) = -8e^{x_1(t)} - 16e^{-x_2(t)/2}, \quad t \in (0,1), \\ x_2''(t) + \dfrac{3}{t}x_2'(t) = 8e^{-x_2(t)} + 8e^{x_1(t)/2}, \\ x_1'(0) = x_2'(0) = 0, \quad x_1(1) = -2\ln(2), \; x_2(1) = 2\ln(2). \end{cases} \tag{42}$$

Clearly, (1) is equivalent to

$$\begin{cases} x_1(t) = -2\ln(2) + \displaystyle\int_0^1 G_1(t,s)\, s^5 \left(8e^{x_1(s)} + 16e^{-x_2(s)/2}\right) ds, \quad t \in (0,1), \\ x_2(t) = 2\ln(2) + \displaystyle\int_0^1 G_2(t,s)\, s^3 \left(-8e^{-x_2(s)} - 8e^{x_1(s)/2}\right) ds. \end{cases} \tag{43}$$

The exact solutions are

$$x_1(t) = -2\ln(1+t^2) \text{ and } x_2(t) = 2\ln(1+t^2)$$

and the Green's kernel functions are

$$G_1(t,s)=\begin{cases}\frac{1}{4}\left(1-\frac{1}{s^4}\right), & t\le s,\\ \frac{1}{4}\left(1-\frac{1}{t^4}\right), & s\le t,\end{cases}\quad\text{and } G_2(t,s)=\begin{cases}\frac{1}{2}\left(1-\frac{1}{s^2}\right), & t\le s,\\ \frac{1}{2}\left(1-\frac{1}{t^2}\right), & s\le t.\end{cases}\tag{44}$$

In Tables 1 -6 , the numerical results obtained by the BCM and the exact solutions of problems (1)–(3) are shown. It has been observed that the results obtained by the present method are very close to the exact solutions. In the same tables, the numerical results of errors, L^i_∞ and L^i_2 (for $i=1,2$) are provided. It can be seen that as the degree of the Bernstein polynomial increases, the numerical errors decreases significantly.

TABLE 1 Numerical results of solutions of problem 1 with $n=5$

	BCM		Exact solutions	
t	$x_{15}(t)$	$x_{25}(t)$	$x_1(t)$	$x_2(t)$
0.1	-0.01973810	0.01972106	-0.01990066	0.01990066
0.2	-0.07831193	0.07830596	-0.07844143	0.07844142
0.3	-0.17223018	0.17223424	-0.17235539	0.17235539
0.4	-0.29670249	0.29670874	-0.29684001	0.29684001
0.5	-0.44615847	0.44616104	-0.44628710	0.44628710
0.6	-0.61487205	0.61487237	-0.61496940	0.61496940
0.7	-0.79747198	0.79747640	-0.79755224	0.79755224
0.8	-0.98929866	0.98930995	-0.98939248	0.98939248
0.9	-1.18656685	1.18657734	-1.18665369	1.18665369

TABLE 2 Numerical results of the errors of problem 1 for $n=5,6,\dots,10$

n	L^1_∞	L^2_∞	L^1_2	L^2_2
5	1.62E-04	1.79E-04	3.80E-04	2.71E-04
6	3.17E-05	1.41E-05	7.61E-05	5.68E-05
7	1.69E-06	2.03E-06	4.34E-06	2.92E-06
8	2.25E-06	2.39E-06	5.31E-06	3.88E-06
9	3.66E-07	3.98E-07	8.85E-07	6.71E-07
10	6.78E-08	6.70E-08	1.53E-07	1.08E-07

Problem 2. Consider the following coupled Lane-Emden BVPs

$$\begin{cases} x_1''(t)+\dfrac{2}{t}x_1'(t)=-6(e^{x_2(t)/3}+4)e^{2x_1(t)/3}, & t\in(0,1),\\ x_2''(t)+\dfrac{2}{t}x_2'(t)=6(e^{-x_1(t)/3}+4)e^{-2x_2(t)/3}, \\ x_1'(0)=x_2'(0)=0, \quad x_1(1)=-3\ln(3),\ x_2(1)=3\ln(3), \end{cases}\tag{45}$$

which are equivalent to the integral equations

$$\begin{cases} x_1(t)=-3\ln(3)+\displaystyle\int_0^1 G_1(t,s)\, s^2\left(6(e^{x_2(s)/3}+4)e^{2x_1(s)/3}\right)ds, \quad t\in(0,1),\\ x_2(t)=3\ln(3)+\displaystyle\int_0^1 G_2(t,s)\, s^2\left(-6(e^{-x_1(s)/3}+4)e^{-2x_2(s)/3}\right)ds. \end{cases}\tag{46}$$

The exact solutions are

$$x_1(t) = -3\ln(2+t^2) \text{ and } x_2(t) = 3\ln(2+t^2)$$

and the Green's kernel functions are

$$G_1(t,s) = \begin{cases} 1-\frac{1}{s}, & t \le s, \\ 1-\frac{1}{t}, & s \le t, \end{cases} \text{ and } G_2(t,s) = \begin{cases} 1-\frac{1}{s}, & t \le s, \\ 1-\frac{1}{t}, & s \le t. \end{cases} \tag{47}$$

TABLE 3 Numerical results of solutions of problem 2 with $n = 5$

	BCM		Exact solutions	
t	$x_{15}(t)$	$x_{25}(t)$	$x_1(t)$	$x_2(t)$
0.1	-2.09438417	2.09438417	-2.09440417	2.09440417
0.2	-2.13883420	2.13883420	-2.13884942	2.13884942
0.3	-2.21147901	2.21147901	-2.21149220	2.21149220
0.4	-2.31031035	2.31031035	-2.31032467	2.31032467
0.5	-2.43277623	2.43277623	-2.43279065	2.43279065
0.6	-2.57597310	2.57597310	-2.57598486	2.57598486
0.7	-2.73683841	2.73683841	-2.73684813	2.73684813
0.8	-2.91232538	2.91232538	-2.91233675	2.91233675
0.9	-3.09954214	3.09954214	-3.09955345	3.09955345

TABLE 4 Numerical results of the errors of problem 2 for $n = 5, 6, \dots, 10$

n	L^1_∞	L^2_∞	L^1_2	L^2_2
5	2.00E-05	2.00E-05	4.48E-05	3.17E-05
6	7.08E-07	7.08E-07	1.01E-06	7.16E-07
7	6.37E-07	6.37E-07	1.43E-06	1.01E-06
8	6.36E-08	6.36E-08	1.66E-07	1.17E-07
9	1.34E-08	1.34E-08	3.00E-08	2.12E-08
10	3.56E-09	3.56E-09	9.33E-09	6.60E-08

Problem 3. Consider the following coupled Lane-Emden BVPs

$$\begin{cases} x_1''(t) + \dfrac{3}{t}x_1'(t) = -(3 + x_2^2(t))x_1^5(t), \quad t \in (0,1), \\ x_2''(t) + \dfrac{4}{t}x_2'(t) = (4x_1^{-2}(t) + 1)x_2^{-3}(t), \\ x_1'(0) = x_2'(0) = 0, \quad x_1(1) = \frac{1}{\sqrt{2}}, \quad x_2(1) = \sqrt{2}, \end{cases} \tag{48}$$

which is equivalent to

$$\begin{cases} x_1(t) = \dfrac{1}{\sqrt{2}} + \displaystyle\int_0^1 G_1(t,s)\, s^3 \left(\left(3 + x_2^2(s)\right) x_1^5(s) \right) ds, \quad t \in (0,1), \\ x_2(t) = \sqrt{2} + \displaystyle\int_0^1 G_2(t,s)\, s^4 \left(-\left(4x_1^{-2}(s) + 1\right) x_2^{-3}(s) \right) ds. \end{cases} \tag{49}$$

The exact solutions are

$$x_1(t) = \frac{1}{\sqrt{1+t^2}} \text{ and } x_2(t) = \sqrt{1+t^2}$$

and the Green's kernel functions are

$$G_1(t,s) = \begin{cases} \frac{1}{2}\left(1-\frac{1}{s^2}\right), & t \le s, \\ \frac{1}{2}\left(1-\frac{1}{t^2}\right), & s \le t, \end{cases} \text{ and } G_2(t,s) = \begin{cases} \frac{1}{3}\left(1-\frac{1}{s^3}\right), & t \le s, \\ \frac{1}{3}\left(1-\frac{1}{s^3}\right), & s \le t. \end{cases} \tag{50}$$

TABLE 5 Numerical results of solutions of problem 3 with $n = 5$

	BCM		Exact solutions	
t	$x_{15}(t)$	$x_{25}(t)$	$x_1(t)$	$x_2(t)$
0.1	0.996045462	1.00580008	0.995037190	1.00498756
0.2	0.981491891	1.02056051	0.980580676	1.01980390
0.3	0.958620543	1.04469848	0.957826285	1.04403065
0.4	0.929146418	1.07759026	0.928476691	1.07703296
0.5	0.894959704	1.11847341	0.894427191	1.11803399
0.6	0.857883415	1.16651533	0.857492926	1.16619038
0.7	0.819498431	1.22087460	0.819231921	1.22065556
0.8	0.781042206	1.28075006	0.780868809	1.28062485
0.9	0.743387411	1.34541250	0.743294146	1.34536240

TABLE 6 Numerical results of the errors of problem 3 for $n = 5, 6, \ldots, 10$

n	L_∞^1	L_∞^2	L_2^1	L_2^2
5	1.01E-03	8.13E-04	2.01E-03	1.16E-03
6	3.08E-04	2.55E-04	6.19E-04	3.64E-04
7	6.58E-06	5.97E-06	1.34E-05	8.58E-06
8	1.93E-05	1.59E-05	3.86E-05	2.27E-05
9	4.77E-06	3.96E-06	9.58E-06	5.66E-06
10	2.79E-07	2.24E-07	5.58E-07	3.19E-07

Problem 4. Consider the following coupled Lane-Emden BVPs[7]

$$\begin{cases} x_1''(t) + \frac{2}{t}x_1'(t) = \alpha_1\, x_1^2(t) + \alpha_2\, x_1(t)\, x_2(t), \quad t \in (0,1), \\ x_2''(t) + \frac{2}{t}x_2'(t) = \alpha_3\, x_1^2(t) + \alpha_4\, x_1(t)\, x_2(t), \\ x_1'(0) = x_2'(0) = 0, \quad x_1(1) = 1, \; x_2(1) = 2, \end{cases} \tag{51}$$

where α_1, α_2, α_3, α_4 are chemical constants.

The equivalent integral equations are

$$\begin{cases} x_1(t) = 1 + \int_0^1 G_1(t,s)\, s^2 \Big(-\alpha_1\, x_1^2(s) - \alpha_2\, x_1(s)\, x_2(s) \Big)\, ds, \quad t \in (0,1), \\ x_2(t) = 2 + \int_0^1 G_2(t,s)\, s^2 \Big(-\alpha_3\, x_1^2(s) - \alpha_4\, x_1(s)\, x_2(s) \Big)\, ds, \end{cases} \tag{52}$$

where the Green's kernel functions are

$$G_1(t,s) = \begin{cases} 1 - \frac{1}{s}, & t \le s, \\ 1 - \frac{1}{t}, & s \le t, \end{cases} \quad \text{and } G_2(t,s) = \begin{cases} 1 - \frac{1}{s}, & t \le s, \\ 1 - \frac{1}{t}, & s \le t. \end{cases} \tag{53}$$

The numerical results of the solutions and the residual errors obtained by the BCM are given in Tables 7 , 8 and 9 of problem 4. The comparison of the maximal residual error of the BCM and the SAM[33] are shown in Table 10 where n is the degree of Bernstein polynomial and m is number of terms in the solution series of the SAM. It can be observed that the present method converges faster than the other method. Also we observe that as the value of n increases the maximal residual error decreases rapidly.

TABLE 7 Numerical results of solutions for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ of problem 4

t	x_{14}	x_{24}	r_{14}	r_{24}
0.1	1.18641543	2.18641543	2.07E-08	2.07E-08
0.2	1.18048828	2.18048828	3.23E-08	3.23E-08
0.3	1.17064920	2.17064920	5.39E-08	5.39E-08
0.4	1.15695710	2.15695710	1.32E-07	1.32E-07
0.5	1.13949391	2.13949391	8.67E-07	4.34E-07
0.6	1.11836390	2.11836390	3.16E-07	3.16E-07
0.7	1.09369305	2.09369305	3.33E-07	3.33E-07
0.8	1.06562806	2.06562806	6.26E-07	6.26E-07
0.9	1.03433538	2.03433538	2.17E-06	2.17E-06

TABLE 8 Maximal residual error for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ of problem 4

n	3	4	5	6	7	8	9	10
Mr_{1n}	8.32E-05	2.17E-06	1.58E-07	2.59E-09	1.20E-10	1.22E-12	4.75E-14	3.34E-13
Mr_{2n}	8.32E-05	2.17E-06	1.58E-07	2.59E-09	1.20E-10	1.22E-12	4.87E-14	8.67E-14

Problem 5. Consider the following coupled Lane-Emden BVPs[8]

$$\begin{cases} x_1''(t) + \dfrac{k_1}{t} x_1'(t) = -\mu_2 + \dfrac{\mu_1\, x_1(t)\, x_2(t)}{\big(l_1 + x_1(t)\big)\big(m_1 + x_2(t)\big)} + \dfrac{\mu_3\, x_1(t)\, x_2(t)}{\big(l_2 + x_1(t)\big)\big(m_2 + x_2(t)\big)}, \\ x_2''(t) + \dfrac{k_2}{t} x_2'(t) = \dfrac{\mu_4\, x_1(t)\, x_2(t)}{\big(l_1 + x_1(t)\big)\big(m_1 + x_2(t)\big)} + \dfrac{\mu_5\, x_1(t)\, x_2(t)}{\big(l_2 + x_1(t)\big)\big(m_2 + x_2(t)\big)}, \\ x_1'(0) = x_2'(0) = 0, \quad x_1(1) = x_2(1) = 1. \end{cases} \tag{54}$$

TABLE 9 Numerical results of solutions for $\alpha_1 = 1$, $\alpha_2 = 2/5$, $\alpha_3 = 1/2$, $\alpha_4 = 1$ of problem 4

t	x_{14}	x_{24}	r_{14}	r_{24}
0.1	0.98217896	2.27658356	1.03E-07	1.75E-07
0.2	0.98253345	2.26782191	1.62E-07	2.72E-07
0.3	0.98314976	2.25327349	2.71E-07	4.52E-07
0.4	0.98406619	2.23301923	6.68E-07	1.10E-06
0.5	0.98533652	2.20717128	8.67E-19	1.39E-06
0.6	0.98703020	2.17587205	1.62E-06	2.63E-06
0.7	0.98923272	2.13929305	1.72E-06	2.76E-06
0.8	0.99204612	2.09763356	3.25E-06	5.16E-06
0.9	0.99558954	2.05111915	1.13E-05	1.78E-05

TABLE 10 Maximal residual error for $\alpha_1 = 1$, $\alpha_2 = 2/5$, $\alpha_3 = 1/2$, $\alpha_4 = 1$ of problem 4

	BCM			SAM[33]	
n	Mr_{1n}	Mr_{2n}	m	MR_{1m}	MR_{2m}
3	8.61E-05	7.40E-05	3	2.40e-1	5.58e-1
4	1.13E-05	1.78E-05	4	4.22e-2	1.11e-1
5	1.05E-06	1.12E-06	5	1.13e-2	2.76e-2
6	3.60E-08	4.70E-08	6	1.73e-3	4.32e-3
7	3.50E-09	2.10E-10	7	3.25e-4	8.18e-4
8	2.13E-10	2.33E-10	8	4.58e-5	1.17e-4
9	7.74E-12	3.50E-12	9	7.11e-6	1.84e-5
10	1.33E-14	3.67E-14	10	9.47e-7	2.48e-6

Its integral form is

$$\begin{cases} x_1(t) = 1 + \int_0^1 G_1(t,s)\, s^{k_1} \left(\mu_2 - \frac{\mu_1\, x_1(s)\, x_2(s)}{\left(l_1+x_1(s)\right)\left(m_1+x_2(s)\right)} - \frac{\mu_3\, x_1(s)\, x_2(s)}{\left(l_2+x_1(s)\right)\left(m_2+x_2(s)\right)} \right) ds, \\ x_2(t) = 1 + \int_0^1 G_2(t,s)\, s^{k_2} \left(\frac{\mu_4\, x_1(s)\, x_2(s)}{\left(l_1+x_1(s)\right)\left(m_1+x_2(s)\right)} - \frac{\mu_2\, x_1(s)\, x_2(s)}{\left(l_2+x_1(s)\right)\left(m_2+x_2(s)\right)} \right) ds, \end{cases} \tag{55}$$

where the Green's kernel functions are

(i) for $k_1 = k_2 = 1$

$$G_1(t,s) = \begin{cases} \ln(s), & t \le s, \\ \ln(t), & s \le t, \end{cases} \quad \text{and} \quad G_2(t,s) = \begin{cases} \ln(s), & t \le s, \\ \ln(t), & s \le t, \end{cases} \tag{56}$$

(ii) for $k_1 = k_2 = 2$

$$G_1(t,s) = \begin{cases} 1 - \frac{1}{s}, & t \le s, \\ 1 - \frac{1}{t}, & s \le t, \end{cases} \quad \text{and } G_2(t,s) = \begin{cases} 1 - \frac{1}{s}, & t \le s, \\ 1 - \frac{1}{t}, & s \le t. \end{cases} \tag{57}$$

The numerical results of the solutions, the residual errors and the maximal residual errors obtained by the BCM are given in Tables 11 and 12 (for $k_1 = k_2 = 1$) and in Tables 13 and 14 (for $k_1 = k_2 = 2$) with the fixed parameters $l_1 = l_2 = m_1 = m_2 = 0.0001$, $\mu_1 = 5$, $\mu_2 = 1$, $\mu_3 = \mu_4 = 0.1$ and $\mu_5 = 0.05$. From these table it can be observed that the convergence rate of the BCM is fast since the maximal residual error decreases rapidly with an increase in the degree of Bernstein polynomial, n.

TABLE 11 Numerical results of solutions for $k_1 = k_2 = 1$ of problem 5

t	x_{14}	x_{24}	r_{14}	r_{24}
0.1	2.01455441	1.03711925	2.52E-07	7.42E-09
0.2	1.98381003	1.03599441	6.64E-07	1.95E-08
0.3	1.93256937	1.03411969	6.31E-07	1.86E-08
0.4	1.86083252	1.03149507	2.44E-07	7.17E-09
0.5	1.76859959	1.02812058	8.72E-07	2.57E-08
0.6	1.65587071	1.02399620	3.31E-07	9.73E-09
0.7	1.52264603	1.01912194	1.19E-06	3.50E-08
0.8	1.36892577	1.01349782	1.86E-06	5.48E-08
0.9	1.19471026	1.00712383	1.29E-06	3.80E-08

TABLE 12 Maximal residual error for $k_1 = k_2 = 1$ of problem 5

n	3	4	5	6	7	8	9	10
Mr_{1n}	3.30E-05	9.10E-06	2.89E-06	9.85E-07	3.51E-07	1.29E-07	4.83E-08	1.84E-08
Mr_{2n}	9.69E-07	2.68E-07	8.51E-08	2.90E-08	1.03E-08	3.79E-09	1.42E-09	5.41E-10

TABLE 13 Numerical results of solutions for $k_1 = k_2 = 2$ of problem 5

t	x_{14}	x_{24}	r_{14}	r_{24}
0.1	1.67635958	1.02474587	5.35E-09	1.57E-10
0.2	1.65586359	1.02399599	8.66E-09	2.55E-10
0.3	1.62170364	1.02274619	1.51E-08	4.44E-10
0.4	1.57387977	1.02099646	3.89E-08	1.14E-09
0.5	1.51239204	1.01874682	9.71E-17	2.71E-19
0.6	1.43724054	1.01599727	1.06E-07	3.12E-09
0.7	1.34842540	1.01274781	1.21E-07	3.56E-09
0.8	1.24594676	1.00899843	2.50E-07	7.35E-09
0.9	1.12980486	1.00474916	9.68E-07	2.85E-08

TABLE 14 Maximal residual error for $k_1 = k_2 = 2$ of problem 5

n	3	4	5	6	7	8	9	10
Mr_{1n}	3.49E-06	9.68E-07	2.55E-07	6.05E-08	1.31E-08	2.41E-09	3.23E-10	1.12E-12
Mr_{2n}	1.03E-07	2.85E-08	7.51E-09	1.78E-09	3.84E-10	7.08E-11	9.51E-12	1.47E-14

6 | CONCLUSION

The singular boundary value problem arises in various fields of applied mathematics, physical and chemical sciences like in dusty fluid model[7], in excess sludge production from water treatment plants[8]. In this work, the Bernstein collocation method has been applied for numerical approximation of the coupled Lane-Emden problem with Neumann-Robin boundary conditions. The equivalent integral form of the coupled Lane-Emden equation has been considered. Unlike the HAM[32] and the SAM[33], the proposed method requires less computational work which can be seen from Table 10 . The error analysis of the the Bernstein collocation method has been established under quite general conditions. The accuracy and efficiency of the present method has been checked by evaluating the maximum absolute error L_∞, the L_2 error and the residual error of several numerical examples. The obtained numerical results demonstrated that the present method is very efficient to use and has high accuracy.

Acknowledgement

One of the author, Julee Shahni would like to acknowledge the financial assistance provided by Department of Science and Technology (DST) under the scheme of INSPIRE Fellowship, New Delhi, India.

References

1. J. H. Lane, On the theoretical temperature of the sun; under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases as known to terrestrial experiment, American Journal of Science and Arts (1820-1879) 50 (148) (1870) 57.
2. R. Emden, Gaskugeln: Anwendungen der mechanischen Wärmetheorie auf kosmologische und meteorologische Probleme, B. Teubner, 1907.
3. S. Chandrasekhar, An introduction to the study of stellar structure, Ciel et Terre 55 (1939) 412.
4. D. McElwain, A re-examination of oxygen diffusion in a spherical cell with Michaelis-Menten oxygen uptake kinetics, Journal of Theoretical Biology 71 (1978) 255–263.
5. R. A. Van Gorder, Exact first integrals for a Lane-Emden equation of the second kind modeling a thermal explosion in a rectangular slab, New Astronomy 16 (8) (2011) 492–497.
6. K. Reger, R. Van Gorder, Lane-Emden equations of second kind modelling thermal explosion in infinite cylinder and sphere, Applied Mathematics and Mechanics 34 (12) (2013) 1439–1452.
7. D. Flockerzi, K. Sundmacher, On coupled Lane-Emden equations arising in dusty fluid models, Journal of Physics 268 (2011) 012006.
8. S. Muthukumar, M. Veeramuni, R. Lakshmanan, Analytical expression of concentration of substrate and oxygen in excess sludge production using Adomian decomposition method, Indian J. Appl. Res 4 (2014) 387–391.
9. M. Chawla, C. Katti, Finite difference methods and their convergence for a class of singular two point boundary value problems, Numerische Mathematik 39 (3) (1982) 341–350.
10. R. Pandey, On the convergence of a finite difference method for a class of singular two point boundary value problems, International Journal of Computer Mathematics 42 (3-4) (1992) 237–241.
11. M. Kumar, A three-point finite difference method for a class of singular two-point boundary value problems, Journal of Computational and Applied Mathematics 145 (1) (2002) 89–97.
12. S. Iyengar, P. Jain, Spline finite difference methods for singular two point boundary value problems, Numerische Mathematik 50 (3) (1986) 363–376.
13. J. Rashidinia, Z. Mahmoodi, M. Ghasemi, Parametric spline method for a class of singular two-point boundary value problems, Applied Mathematics and Computation 188 (1) (2007) 58–63.
14. A. R. Kanth, Cubic spline polynomial for non-linear singular two-point boundary value problems, Applied Mathematics and Computation 189 (2) (2007) 2017–2022.
15. A. Taghavi, S. Pearce, A solution to the Lane-Emden equation in the theory of stellar structure utilizing the Tau method, Mathematical Methods in the Applied Sciences 36 (10) (2013) 1240–1247.
16. M. Lakestani, M. Dehghan, Four techniques based on the B-spline expansion and the collocation approach for the numerical solution of the Lane-Emden equation, Mathematical Methods in the Applied Sciences 36 (16) (2013) 2243–2253.
17. R. Singh, J. Kumar, G. Nelakanti, Numerical solution of singular boundary value problems using Green's function and improved decomposition method, Journal of Applied Mathematics and Computing 43 (1-2) (2013) 409–425.

18. R. Singh, J. Kumar, An efficient numerical technique for the solution of nonlinear singular boundary value problems, Computer Physics Communications 185 (4) (2014) 1282–1289.

19. R. Singh, J. Kumar, G. Nelakanti, Approximate series solution of singular boundary value problems with derivative dependence using Green's function technique, Computational and Applied Mathematics 33 (2) (2014) 451–467.

20. R. Singh, J. Kumar, The Adomian decomposition method with Green's function for solving nonlinear singular boundary value problems, Journal of Applied Mathematics and Computing 44 (1-2) (2014) 397–416.

21. F. Zhou, X. Xu, Numerical solutions for the linear and nonlinear singular boundary value problems using Laguerre wavelets, Advances in Difference Equations 2016 (1) (2016) 17.

22. R. Singh, H. Garg, V. Guleria, Haar wavelet collocation method for Lane-Emden equations with Dirichlet, Neumann and Neumann-Robin boundary conditions, Journal of Computational and Applied Mathematics 346 (2019) 150–161.

23. R. Singh, J. Shahni, H. Garg, A. Garg, Haar wavelet collocation approach for Lane-Emden equations arising in mathematical physics and astrophysics, The European Physical Journal Plus 134 (11) (2019) 548.

24. R. Mohammadzadeh, M. Lakestani, M. Dehghan, Collocation method for the numerical solutions of Lane-Emden type equations using cubic Hermite spline functions, Mathematical Methods in the Applied Sciences 37 (9) (2014) 1303–1717.

25. A. M. Wazwaz, R. Rach, J. S. Duan, A study on the systems of the Volterra integral forms of the Lane-Emden equations by the Adomian decomposition method, Mathematical Methods in the Applied Sciences 37 (1) (2014) 10–19.

26. Y. Öztürk, An efficient numerical algorithm for solving system of Lane-Emden type equations arising in engineering, Nonlinear Engineering 8 (1) (2019) 429–437.

27. L. J. Xie, C. l. Zhou, S. Xu, Solving the systems of equations of Lane-Emden type by differential transform method coupled with Adomian polynomials, Mathematics 7 (4) (2019) 377.

28. J. H. He, F. Y. Ji, Taylor series solution for Lane-Emden equation, Journal of Mathematical Chemistry 57 (8) (2019) 1932–1934.

29. R. Rach, J. S. Duan, A. M. Wazwaz, Solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method, Journal of Mathematical Chemistry 52 (1) (2014) 255–267.

30. J. S. Duan, R. Rach, A. M. Wazwaz, Oxygen and carbon substrate concentrations in microbial floc particles by the Adomian decomposition method, MATCH Commun. Math. Comput. Chem 73 (2015) 785–796.

31. A. M. Wazwaz, R. Rach, J. S. Duan, Variational iteration method for solving oxygen and carbon substrate concentrations in microbial floc particles, MATCH Commun. Math. Comput. Chem 76 (2016) 511–523.

32. R. Singh, A. M. Wazwaz, An efficient algorithm for solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions: The homotopy analysis method, MATCH Commun. Math. Comput. Chem 81 (2019) 785-800.

33. T. C. Hao, F. Z. Cong, Y. F. Shang, An efficient method for solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions and error estimate, Journal of Mathematical Chemistry 56 (9) (2018) 2691–2706.

34. H. Madduri, P. Roul, A fast-converging iterative scheme for solving a system of Lane-Emden equations arising in catalytic diffusion reactions, Journal of Mathematical Chemistry 57 (2) (2019) 570–582.

35. D. D. Bhatta, M. I. Bhatti, Numerical solution of Kdv equation using modified Bernstein polynomials, Applied Mathematics and Computation 174 (2) (2006) 1255–1268.

36. M. I. Bhatti, P. Bracken, Solutions of differential equations in a Bernstein polynomial basis, Journal of Computational and Applied Mathematics 205 (1) (2007) 272–280.

37. B. N. Mandal, S. Bhattacharya, Numerical solution of some classes of integral equations using Bernstein polynomials, Applied Mathematics and Computation 190 (2) (2007) 1707–1716.

38. K. Maleknejad, E. Hashemizadeh, R. Ezzati, A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation, Communications in Nonlinear Science and Numerical Simulation 16 (2) (2011) 647–655.

39. Ş. Yüzbaşı, A numerical approach for solving a class of the nonlinear Lane-Emden type equations arising in astrophysics, Mathematical Methods in the Applied Sciences 34 (18) (2011) 2218–2230.

40. O. R. Isik, M. Sezer, Bernstein series solution of a class of Lane-Emden type equations, Mathematical Problems in Engineering 2013.

41. A. Jafarian, S. A. M. Nia, A. K. Golmankhaneh, D. Baleanu, Numerical solution of linear integral equations system using the Bernstein collocation method, Advances in Difference Equations 2013 (1) (2013) 123.

42. M. J. D. Powell, Approximation theory and methods, Cambridge University Press; 1981.

43. G. Lorentz, R. DeVore, Constructive approximation, polynomials and splines approximation, Berlin: Springer-Verlag, 1993.

□