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ARTICLE TYPE

Existence and stability of the solution to the system of two diffusion equations in medium with discontinuous characteristics†

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Summary

In this paper, we use the asymptotic analysis (specifically, asymptotic approximations and asymptotic method of differential inequalities) to investigate existence, local uniqueness and stability of a steplike stationary solution of an activator-inhibitor parabolic system with nonlinear discontinuous right-hand side functions. The internal layer of the solution is therefore localized in the vicinity of the simple discontinuities.

KEYWORDS:
system of ordinary differential equations, internal transition layer, small parameter, upper and lower solutions, stable stationary solution, domain of attraction

1 | INTRODUCTION

The article considers a system of two second order nonlinear differential equations with discontinuous functions on the right sides. The aim of the work is to obtain sufficient conditions for the existence, local uniqueness, and asymptotic stability of a stationary solution of a parabolic system with a large gradient in the vicinity of the discontinuity points of the right-hand sides. The area where the function undergoes large gradient is called the internal transition layer.

The authors arrived at this formulation of the problem during the development of the autowave model for the development of megacities [1,2]. This model is based on the activator-inhibitor system of two equations where the urban area acts as the activator, and the inhibitor is determined by environmental or economic factors due to urban planning policies of a country. The presence of barriers that prevent the propagation of the front of the activator, for example, large bodies of water, is taken into account in the model as a jump in the functions on the right-hand sides. Obviously, the numerical solution of such a problem should be preceded by an analytical study of the existence of the mentioned solution, which was done in the present work.

The proof of the existence and asymptotic stability of the stationary solution of the initial-boundary-value problem here is carried out using the asymptotic method of differential inequalities [3,4], based on the method of super- and subsolutions.The latter was extended to problems with a single discontinuity point of the first kind on the right-hand sides of the equations based on a modified proof of the corresponding theorem from [5], where it was carried out for the case of C^2 continuous right-hand sides.

2 | PROBLEM STATEMENT

We consider the following initial-boundary-value problem:

$$\varepsilon^4 y_{xx} - y_t = f(y,z,x,\varepsilon), \quad x \in (0,1),\ t>0, \quad y_x(0,t) = y_x(1,t) = 0, \quad y(x,0) = u^0(x), \tag{1}$$

$$\varepsilon^2 z_{xx} - z_t = g(y,z,x,\varepsilon), \quad x \in (0,1),\ t>0, \quad z_x(0,t) = z_x(1,t) = 0, \quad z(x,0) = v^0(x), \tag{2}$$

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where $u^0(x)$, $v^0(x) \in C([0,1])$ and $u_x^0(0) = u_x^0(1) = v_x^0(0) = v_x^0(1) = 0$, $\varepsilon \in (0, \varepsilon_0]$ is a small parameter.

The functions $f(u, v, x, \varepsilon)$ and $g(u, v, x, \varepsilon)$ have the first kind discontinuities across the surface $\{u \in I_u, v \in I_v,\ x = x_0 \in (0,1)\}$, where I_u and I_v are respectively permissible u and v change intervals:

$$f(u,v,x,\varepsilon) = \begin{cases} f^{(-)}(u,v,x,\varepsilon), \\ f^{(+)}(u,v,x,\varepsilon), \end{cases} \qquad g(u,v,x,\varepsilon) = \begin{cases} g^{(-)}(u,v,x,\varepsilon), & u \in I_u, v \in I_v,\ 0 < x \le x_0\,, \\ g^{(+)}(u,v,x,\varepsilon), & u \in I_u, v \in I_v,\ x_0 < x \le 1\,, \end{cases}$$

$f^{(-)}(u, v, x, \varepsilon)$ and $g^{(-)}(u, v, x, \varepsilon)$ are of class $C^4(I_u \times I_v \times [0, x_0] \times [0, \varepsilon_0])$, $f^{(+)}(u, v, x, \varepsilon)$ and $g^{(+)}(u, v, x, \varepsilon)$ are of class $C^4(I_u \times I_v \times [x_0, 1] \times [0, \varepsilon_0])$.

Denote $D_T := (0,1) \times \mathbb{R}^+$, $D_T^{(-)} := (0, x_0) \times \mathbb{R}^+$, $D_T^{(+)} := (x_0, 1) \times \mathbb{R}^+$.

Definition 1. A pair of functions $(y_\varepsilon(x,t), z_\varepsilon(x,t))$ in $C^{1,0}\left(\overline{D_T}\right) \cap C^{2,1}\left(D_T^{(-)} \cup D_T^{(+)}\right)$ is called the solution to problem (1) if it satisfies equations (1) in $D_T^{(-)} \cup D_T^{(+)}$, the boundary and initial conditions.

Proposition 1. Each of the equations $f^{(\mp)}(u, v, x, 0) = 0$ is solvable with respect to u and the functions $u = \varphi^{(\mp)}(v, x)$ are the isolated solutions to these equations, respectively, in domains $I_v \times [0, x_0]$ and $I_v \times [x_0, 1]$, the inequality $\varphi^{(-)}(v, x_0) < \varphi^{(+)}(v, x_0)$ holds for all $v \in I_v$ and $f_u^{(\mp)}(\varphi^{(\mp)}(v, x), v, x, 0) > 0$ in respective domains.

Denote $h^{(\mp)}(v, x) = g^{(\mp)}(\varphi^{(\mp)}(v, x), v, x, 0)$.

Proposition 2. Each of the equations $h^{(\mp)}(v, x) = 0$ is solvable with respect to v and the functions $v = \psi^{(\mp)}(x)$ are the isolated solutions to these equations, respectively, in the segments $[0, x_0]$ and $[x_0, 1]$, and the inequalities $h_v^{(\mp)}(\psi^{(\mp)}(x), x) > 0$ hold in respective segments.

The main aim of this article is to obtain the existence and stability conditions for the stationary solution to problem1 that is close to functions $(\varphi^{(-)}, \psi^{(-)})$ to the left of point x_0 close to functions $(\varphi^{(+)}, \psi^{(+)})$ to the right of this point and has a large gradient in the vicinity of point x_0, changing rapidly from values $(\varphi^{(-)}, \psi^{(-)})$ to $(\varphi^{(+)}, \psi^{(+)})$. Obviously the stable stationary solution of problem1 is a solution to the following problem

$$\varepsilon^4 u'' = f(u, v, x, \varepsilon), \quad \varepsilon^2 v'' = g(u, v, x, \varepsilon), \quad x \in (0,1), \quad u'(0) = u'(1) = 0, \quad v'(0) = u'(1) = 0. \tag{3}$$

Definition 2. A pair of functions $(u_\varepsilon(x), v_\varepsilon(x))$ in $C^1([0,1]) \cap C^2((0,1)\backslash x_0)$ is called the solution to problem (3) if it satisfies equations (3) in $x \in (0, x_0) \cup (x_0, 1)$ and the boundary conditions.

Proposition 3. (Quasi-monotonicity) Let the inequalities hold: $f_v^{(\mp)}(u, v, x, \varepsilon) > 0$, $g_u^{(\mp)}(u, v, x, \varepsilon) < 0$ for all $(u, v, x) \in I_u \times I_v \times [0, 1]$.

Let's consider so-called associated equations for problem3:

$$\frac{d^2\tilde{v}}{d\tau^2} = h^{(-)}(\tilde{v}, x_0), \quad \tau < 0, \qquad \frac{d^2\tilde{v}}{d\tau^2} = h^{(+)}(\tilde{v}, x_0), \quad \tau > 0, \qquad \tau := \frac{x - x_0}{\varepsilon} \tag{4}$$

$$\frac{d^2\hat{u}}{d\sigma^2} = f^{(-)}\left(\hat{u}, v, x_0, 0\right), \quad \sigma < 0, \qquad \frac{d^2\hat{u}}{d\sigma^2} = f^{(+)}\left(\hat{u}, v, x_0, 0\right), \quad \sigma > 0, \qquad \sigma := \frac{x - x_0}{\varepsilon^2}. \tag{5}$$

Each of the associated equations is equivalent to related associated system

$$\frac{d\tilde{v}}{d\tau} = \Phi^{(\mp)}, \quad \frac{d\Phi^{(\mp)}}{d\tau} = h^{(\mp)}(\tilde{v}, x_0); \qquad \frac{d\hat{u}}{d\sigma} = \Psi^{(\mp)}, \quad \frac{d\Psi^{(\mp)}}{d\sigma} = f^{(\mp)}\left(\hat{u}, v, x_0, 0\right).$$

By Propositions1 and 2 the points $(\psi^{(\mp)}, 0)$ are the saddle-type rest points respectively for the first pair of systems on the phase plain $(\tilde{v}, \Phi)$ and the points $(\varphi^{(\mp)}(v, x_0), 0)$ for each parameter $v \in I_v$ are respectively the saddles of the second pair of associated systems on phase plane $(\hat{u}, \Psi)$.

The functions

$$\Phi^{(\mp)}(v) = \sqrt{2\int\limits_{\psi^{(\mp)}(x_0)}^{v} h^{(\mp)}(s, x_0)ds}\,, \qquad \Psi^{(\mp)}(u, v) = \sqrt{2\int\limits_{\varphi^{(\mp)}(v, x_0)}^{u} f^{(\mp)}(s, v, x_0, 0)ds}$$

are the separatrixes of respective saddle points. If the function $\tilde{v} \to \psi^{(-)}(x_0)$ as $\tau \to -\infty$ and $\tilde{v} \to \psi^{(+)}(x_0)$ as $\tau \to +\infty$ then the separatrixes $\Phi^{(-)}$ and $\Phi^{(+)}$ intersect. If the function $\hat{u} \to \varphi^{(-)}(v, x_0)$ as $\sigma \to -\infty$ and $\hat{u} \to \varphi^{(+)}(v, x_0)$ as $\sigma \to +\infty$ then the separatrixes $\Psi^{(-)}$ and $\Psi^{(+)}$ intersect. We denote

$$H^{v}(\tilde{v}) := \Phi^{(-)}(\tilde{v}) - \Phi^{(+)}(\tilde{v}), \quad H^{u}(\hat{u}, v) := \Psi^{(-)}(\hat{u}, v) - \Psi^{(+)}(\hat{u}, v). \tag{6}$$

Proposition 4. Let there exist the values q_0 in the interval $\left(\psi^{(-)}(x_0), \psi^{(+)}(x_0)\right)$ and p_0 in the interval $\left(\varphi^{(-)}\left(\psi^{(-)}(x_0), x_0\right), \varphi^{(+)}\left(\psi^{(+)}(x_0), x_0\right)\right)$ such that q_0 is the unique solution of the equation $H^{v}(\tilde{v}) = 0$, p_0 is the unique solution of $H^{u}(\hat{u}, q_0) = 0$ in the respective intervals and

$$\frac{dH^{v}}{dv}(q_0) = h^{(-)}(q_0, x_0) - h^{(+)}(q_0, x_0) > 0, \quad \frac{\partial H^{u}}{\partial u}(p_0, q_0) = f^{(-)}(p_0, q_0, x_0, 0) - f^{(+)}(p_0, q_0, x_0, 0) > 0.$$

We introduce the functions

$$v^{(\mp)}(v, x) := g_v^{(\mp)}(\varphi^{(\mp)}(v, x), v, x, 0) + \frac{f_v^{(\mp)}(\varphi^{(\mp)}(v, x), v, x, 0)}{f_u^{(\mp)}(\varphi^{(\mp)}(v, x), v, x, 0)} \cdot g_u^{(\mp)}(\varphi^{(\mp)}(v, x), v, x, 0), \quad \bar{v}^{(\mp)}(x) := v^{(\mp)}(\psi^{(\mp)}(x), x). \tag{7}$$

Proposition 5. In the respective segments $\bar{v}^{(\mp)}(x) > 0$. Also, $\tilde{v}^{(\mp)}(\tilde{v}^{(\mp)}(\tau))$ are such that

$$\int\limits_{\psi^{(-)}(x_0)}^{\tilde{v}} \tilde{v}^{(-)}(s)ds > 0, \quad \tilde{v} \in (\psi^{(-)}(x_0), \psi^{(+)}(x_0)], \qquad \int\limits_{\psi^{(+)}(x_0)}^{\tilde{v}} \tilde{v}^{(+)}(s)ds > 0, \quad \tilde{v} \in [\psi^{(-)}(x_0), \psi^{(+)}(x_0)).$$

3 | ASYMPTOTIC APPROXIMATION

Further to prove the existence and stability theorems we will use the method of differential inequalities[3,4]. The method is valid for problems with internal transition layers and it is based on the method of upper and lower solutions[5]. It implies construction of the upper and lower solutions as modifications of it's asymptotic approximations.

The asymptotic approximation of problem3 here is quite similar to that constructer in paper[4], where a similar system with continuous right-hand sides was considered. We define an asymptotic approximation of (3) as

$$U_1(x, \varepsilon) = \begin{cases} U_1^{(-)}(x, \varepsilon), & 0 \le x \le x_0\,, \\ U_1^{(+)}(x, \varepsilon), & x_0 \le x \le 1\,, \end{cases} \quad V_1(x, \varepsilon) = \begin{cases} V_1^{(-)}(x, \varepsilon), & 0 \le x \le x_0\,, \\ V_1^{(+)}(x, \varepsilon), & x_0 \le x \le 1\,. \end{cases} \tag{8}$$

The functions $U^{(\mp)}$ and $V^{(\mp)}$ are the sums of the following terms:

$$U_1^{(\mp)} = \bar{u}^{(\mp)}(x, \varepsilon) + Q^{(\mp)}u(\tau, \varepsilon) + M^{(\mp)}u(\sigma, \varepsilon) + P_1^{(\mp)}u(\zeta^{(\mp)}, \varepsilon), \quad V_1^{(\mp)} = \bar{v}^{(\mp)}(x, \varepsilon) + Q^{(\mp)}v(\tau, \varepsilon) + P_1^{(\mp)}v(\zeta^{(\mp)}), \tag{9}$$

- $\bar{u}^{(\mp)}(x, \varepsilon) = \bar{u}_0^{(\mp)}(x) + \varepsilon\bar{u}_1^{(\mp)}(x)$, $\bar{v}^{(\mp)}(x, \varepsilon) = \bar{v}_0^{(\mp)}(x) + \varepsilon\bar{v}_1^{(\mp)}(x)$ are the regular part. These functions define the solution behavior far from borders $x = 0$, $x = 1$, $x = x_0$.
- $Q^{(\mp)}u(\tau, \varepsilon) = Q_0^{(\mp)}u(\tau) + \varepsilon Q_1^{(\mp)}u(\tau)$, $Q^{(\mp)}v(\tau, \varepsilon) = Q_0^{(\mp)}v(\tau) + \varepsilon Q_1^{(\mp)}v(\tau)$, $M^{(\mp)}u(\sigma, \varepsilon) = M_0^{(\mp)}u(\sigma) + \varepsilon M_1^{(\mp)}u(\sigma)$ are the functions describing the two-scaled transition layer,
- $P_1^{(\mp)}u(\zeta^{(\mp)})$, $P_1^{(\mp)}v(\zeta^{(\mp)})$ are the boundary layer functions, where $\zeta^{(-)} = x/\varepsilon, \zeta^{(+)} = (1 - x)/\varepsilon$.

The demand the equality holds

$$U_1^{(-)}(x_0, \varepsilon) = U_1^{(+)}(x_0, \varepsilon) = p^*; \quad V_1^{(-)}(x_0, \varepsilon) = V_1^{(+)}(x_0, \varepsilon) = q^*. \tag{10}$$

that provides functions U_1 and V_1 continuity.

The systems of equations for regular part functions are obtained by aggregating the coefficients with the same ε exponents in Taylor expansion of equalities

$$f^{(\mp)}(\bar{u}^{(\mp)}(x, \varepsilon), \bar{v}^{(\mp)}(x, \varepsilon), x, \varepsilon) - \varepsilon^4 \frac{d\bar{u}^{(\mp)}}{dx}(x, \varepsilon) = 0, \quad g^{(\mp)}(\bar{u}^{(\mp)}(x, \varepsilon), \bar{v}^{(\mp)}(x, \varepsilon), x, \varepsilon) - \varepsilon^2 \frac{d\bar{v}^{(\mp)}}{dx}(x, \varepsilon) = 0.$$

Particularly for the 0-th order we have $\bar{v}_0^{(\mp)}(x) = \psi^{(\mp)}(x)$, $\bar{u}_0^{(\mp)}(x) = \varphi^{(\mp)}(\psi^{(\mp)}(x), x)$.

We obtain the equations for the transitional layer functions by aggregating the coefficients for the same exponents of ε in Taylor expansions of equalities:

$$\varepsilon^4 \frac{d^2 Q^{(\mp)}u}{d\tau^2} = Q^{(\mp)}f, \quad \varepsilon^2 \frac{d^2 Q^{(\mp)}v}{d\tau^2} = Q^{(\mp)}g, \tag{11}$$

where

$$Q^{(\mp)}f(\tau,\varepsilon) := f^{(\mp)}(\bar{u}^{(\mp)}(x_0+\varepsilon\tau,\varepsilon)+Q^{(\mp)}u(\tau,\varepsilon),\bar{v}^{(\mp)}(x_0+\varepsilon\tau,\varepsilon)+Q^{(\mp)}v(\tau,\varepsilon),x_0+\varepsilon\tau,\varepsilon)- \\ -f^{(\mp)}(\bar{u}^{(\mp)}(x_0+\varepsilon\tau,\varepsilon),\bar{v}^{(\mp)}(x_0+\varepsilon\tau,\varepsilon),x_0+\varepsilon\tau,\varepsilon) \tag{12}$$

and $Q^{(\mp)}g(\tau,\varepsilon)$ have similar meaning;

$$\varepsilon^4\frac{d^2M^{(\mp)}u}{d\sigma^2} = M^{(\mp)}f, \tag{13}$$

where

$$M^{(\mp)}f^{(\mp)}(\sigma,\varepsilon) := \\ := f^{(\mp)}(\bar{u}^{(\mp)}(x_0+\varepsilon^2\sigma,\varepsilon)+Q^{(\mp)}u(\varepsilon\sigma,\varepsilon)+M^{(\mp)}u(\sigma,\varepsilon),\bar{v}^{(\mp)}(x_0+\varepsilon^2\sigma,\varepsilon)+Q^{(\mp)}v(\varepsilon\sigma,\varepsilon),x_0+\varepsilon^2\sigma,\varepsilon)- \\ -f^{(\mp)}(\bar{u}^{(\mp)}(x_0+\varepsilon^2\sigma,\varepsilon)+Q^{(\mp)}u(\varepsilon\sigma,\varepsilon),\bar{v}^{(\mp)}(x_0+\varepsilon^2\sigma,\varepsilon)+Q^{(\mp)}v(\varepsilon\sigma,\varepsilon),x_0+\varepsilon^2\sigma,\varepsilon), \tag{14}$$

and $M^{(\mp)}g(\sigma,\varepsilon)$ have similar meaning. Additionally we demand $Q_i^{(\mp)}u(\tau)\to 0, Q_i^{(\mp)}v(\tau)\to 0$ when $\tau\to\mp\infty$, $M_i^{(\mp)}u(\sigma)\to 0$ when $\sigma\to\mp\infty$ for $i=0,1$.

3.1 | 0-th order transition layer functions

Denote

$$\tilde{u}^{(\mp)}(\tau) := \varphi^{(\mp)}(\psi^{(\mp)}(x_0),x_0)+Q_0^{(\mp)}u(\tau),\quad \tilde{v}^{(\mp)}(\tau) := \psi^{(\mp)}(x_0)+Q_0^{(\mp)}u(\tau),\quad \Phi^{(\mp)}(\tau)=\frac{d\tilde{v}}{d\tau}. \tag{15}$$

From equalities11 in 0-th order we obtain equation. $f^{(\mp)}\left(\tilde{u}^{(\mp)}(\tau),\tilde{v}^{(\mp)}(\tau),x_0,0\right) = 0$ from which it comes $\tilde{u}^{(\mp)}(\tau) = \varphi^{(\mp)}(\tilde{v}^{(\mp)}(\tau),x_0)$. Using this from the second equation11 in 0-th order and the joining condition10 we obtain problems to determine functions $\tilde{u}^{(\mp)}(\tau)$

$$\frac{d^2\tilde{v}^{(\mp)}(\tau)}{d\tau^2} = h^{(\mp)}(\tilde{v}^{(\mp)}(\tau),x_0),\ \tilde{v}^{(\mp)}(0)=q^*,\ \tilde{v}^{(\mp)}(\mp\infty)=\psi^{(\mp)}(x_0). \tag{16}$$

These equations are similar to associated equations4 which have solutions with exponential estimates[6]

$$\left|\tilde{v}^{(\mp)}(\tau)-\psi^{(\mp)}(x_0)\right|\le\tilde{C}_0e^{-\kappa_0|\tau|},\quad \left|\tilde{u}^{(\mp)}(\tau)-\varphi^{(\mp)}\left(\psi^{(\mp)}(x_0),x_0\right)\right|\le\tilde{C}_0e^{-\kappa_0|\tau|}.$$

Analogously we denote functions

$$\hat{u}^{(\mp)}(\sigma) := \varphi^{(\mp)}(q^*,x_0)+M_0^{(\mp)}u(\sigma),\quad \Psi(\sigma,q^*)=\frac{d\hat{u}}{d\sigma}. \tag{17}$$

To determine functions $\hat{u}^{(\mp)}(\sigma)$ we obtain problems

$$\frac{d^2\hat{u}^{(\mp)}(\sigma)}{d\sigma^2} = f^{(\mp)}\left(\hat{u}^{(\mp)}(\sigma),q^*,x_0,0\right),\ \hat{u}^{(\mp)}(0)=p^*,\ \hat{u}^{(\mp)}(\mp\infty)=\varphi^{(\mp)}(q^*,x_0). \tag{18}$$

These equations are similar to associated equations5 which have exponentially bounded solutions[6]:

$$\left|\hat{u}^{(\mp)}(\sigma)-\varphi^{(\mp)}\left(q^*,x_0\right)\right|\le\hat{C}_0e^{-K_0|\sigma|}.$$

3.2 | 1-th order transition layer functions

Denote

$$\tilde{f}^{(\mp)}(\tau) := f^{(\mp)}(\varphi^{(\mp)}(\tilde{v}^{(\mp)}(\tau),x_0),\tilde{v}^{(\mp)}(\tau),x_0,0),\quad \tilde{g}^{(\mp)}(\tau) := g^{(\mp)}(\varphi^{(\mp)}(\tilde{v}^{(\mp)}(\tau),x_0),\tilde{v}^{(\mp)}(\tau),x_0,0), \\ \hat{f}^{(\mp)}(\sigma) := f^{(\mp)}\left(\hat{u}^{(\mp)}(\sigma),q_0,x_0,0\right),\quad \hat{g}^{(\mp)}(\sigma) := g^{(\mp)}\left(\hat{u}^{(\mp)}(\sigma),q_0,x_0,0\right), \\ \tilde{\varphi}^{(\mp)}(\tau) := \varphi^{(\mp)}(\tilde{v}^{(\mp)}(\tau),x_0),\quad \tilde{h}^{(\mp)}(\tau) := h^{(\mp)}(\tilde{v}^{(\mp)}(\tau),x_0),$$

and the same meaning have the derivatives.

For the functions $Q_1^{(\mp)}u(\tau)$ and $Q_1^{(\mp)}v(\tau)$ we obtain the following systems of equations from11 with boundary conditions from10:

$$0=\tilde{f}_u^{(\mp)}(\tau)Q_1^{(\mp)}u(\tau)+\tilde{f}_v^{(\mp)}(\tau)Q_1^{(\mp)}v(\tau)+Q_1^{(\mp)}\tilde{f}(\tau), \\ \frac{d^2Q_1^{(\mp)}v}{d\tau^2}=\tilde{h}_v^{(\mp)}(\tau)Q_1^{(\mp)}v(\tau)+Q_1^{(\mp)}\tilde{g}(\tau),\ Q_1^{(\mp)}v(0)=-\bar{v}_1^{(\mp)}(x_0),\ Q_1^{(\mp)}v(\mp\infty)=0. \tag{19}$$

For the functions $M_1^{(\mp)}u(\sigma)$ we obtain the following problems from13 with boundary conditions from10:

$$\frac{d^2 M_1^{(\mp)}u}{d\sigma^2} = \hat{f}_u^{(\mp)}(\sigma)M_1^{(\mp)}u(\sigma) + M_1^{(\mp)}\hat{f}(\sigma), \quad M_1^{(\mp)}u(0) = -\bar{u}_1^{(\mp)}(x_0) - Q_1^{(\mp)}u(0), \quad M_1^{(\mp)}u(\mp\infty) = 0, \tag{20}$$

The functions $Q_1^{(\mp)}\tilde{f}(\tau)$, $Q_1^{(\mp)}\tilde{g}(\tau)$ and $M_1^{(\mp)}\hat{f}(\sigma)$ in19 and 20 are known and they exponentially decrease to zero as $\tau \to \mp\infty$, $\sigma \to \mp\infty$ respectively. The problems19 and 20 are linear and thus solvable and the following exponential estimates are valid:

$$\left|Q_1^{(\mp)}u(\tau)\right| \le \tilde{C}_1 e^{-\kappa_1|\tau|}, \quad \left|Q_1^{(\mp)}v(\tau)\right| \le \tilde{C}_1 e^{-\kappa_1|\tau|}, \quad \left|M_1^{(\mp)}u(\sigma)\right| \le \hat{C}_1 e^{-K_1|\sigma|}.$$

3.3 | The higher order functions

As it is mentioned in[4] to construct the upper and lower solutions we have to define the boundary layer functions $P_i^{(\mp)}u(\zeta^{(\mp)})$ and $P_i^{(\mp)}v(\zeta^{(\mp)})$ for $i = 1,2$ $R_3^{(\mp)}u(\eta^{(\mp)}$, where $\eta^{(-)} = x/\varepsilon^2$, $\eta^{(+)} = (x-1)/\varepsilon^2$, and also the transition layer functions $M_i^{(\mp)}v(\sigma)$ for $i = 2,3$.

The boundary layer functions are standardly defined as in[7].

The problems for functions $M_i^{(\mp)}v(\sigma)$ for $i = 2,3$. can be determined analogously to problems for functions $M_i^{(\mp)}u(\sigma)$ from equalities $\varepsilon^2 \frac{d^2 M^{(\mp)}v}{d\sigma^2} = M^{(\mp)}g$, where $M^{(\mp)}g$ has the similar sense as14.

3.4 | The derivatives joining condition

Let's assume the following conditions for derivatives

$$\left(\frac{dM_0^{(-)}u}{d\sigma} - \frac{dM_0^{(+)}u}{d\sigma}\right)\Bigg|_{\sigma=0} + \varepsilon\left[\left(\frac{dQ_0^{(-)}u}{d\tau} - \frac{dQ_0^{(+)}u}{d\tau}\right)\Bigg|_{\tau=0} + \left(\frac{dM_1^{(-)}u}{d\sigma} - \frac{dM_1^{(+)}u}{d\sigma}\right)\Bigg|_{\sigma=0}\right] + O(\varepsilon^2) = 0$$

$$\left(\frac{dQ_0^{(-)}v}{d\tau} - \frac{dQ_0^{(+)}v}{d\tau}\right)\Bigg|_{\tau=0} + \varepsilon\left[\left(\frac{dQ_1^{(-)}v}{d\tau} - \frac{dQ_1^{(+)}v}{d\tau}\right)\Bigg|_{\tau=0} + \right.$$
$$\left. + \left(\frac{dM_2^{(-)}v}{d\sigma} - \frac{dM_2^{(+)}v}{d\sigma}\right)\Bigg|_{\sigma=0} + \left(\frac{d\psi^{(-)}}{dx} - \frac{d\psi^{(+)}}{dx}\right)\Bigg|_{x=x_0}\right] + O(\varepsilon^2) = 0. \tag{21}$$

We also assume the following representation for values q^* and p^* that are parameters of $Q-$ and $M-$ functions (see settlements 16 and 18: $q^* = q_0 + \varepsilon q_1$ and $p^* = p_0 + \varepsilon p_1$.

Zeroth order in ε exponents of21 yields $H^v(q_0) = 0$, $H^u(p_0, q_0) = 0$, (see notations6,15 and17). The values q_0 and p_0 exist due to Proposition4. Denote $\Phi(0) := \Phi^{(-)}(q_0) = \Phi^{(+)}(q_0)$, $\Psi(0) := \Psi^{(-)}(p_0, q_0) = \Psi^{(+)}(p_0, q_0)$. First order, in21, yields

$$\frac{1}{\Phi(0)}\frac{dH^v}{dv}(q_0) \cdot q_1 = H_1^v(p_0, q_0), \quad \frac{1}{\Psi(0)}\frac{\partial H^u}{\partial u}(p_0, q_0) \cdot p_1 = H_1^u(p_0, q_0),$$

where $H_1^v(p_0, q_0)$, $H_1^u(p_0, q_0)$ are known functions.

4 | UPPER AND LOWER SOLUTIONS

Denote

$$L_{u,\varepsilon}(u, v) := \varepsilon^4 \frac{d^2u}{dx^2} - f(u, v, x, \varepsilon), \quad L_{v,\varepsilon}(u, v) := \varepsilon^2 \frac{d^2v}{dx^2} - g(u, v, x, \varepsilon).$$

Definition 3. Pairs of functions $\left(\bar{U}, \bar{V}\right)$ and $\left(\tilde{U}, \tilde{V}\right)$ in $C([0,1]) \cap C^2((0,1)\backslash x_0)$ are called respectively upper and lower solutions of the problem (3) if

(A_1). $\tilde{U}(x) \le \bar{U}(x)$, $\tilde{V}(x) \le \bar{V}(x)$, $x \in [0,1]$;

(A_2). $L_{1,\varepsilon}(\bar{U}, v) \le 0 \le L_{1,\varepsilon}(\tilde{U}, v)$ $\tilde{V} \le v \le \bar{V}$, $x \in (0,1)\backslash x_0$, $L_{2,\varepsilon}(u, \bar{V}) \le 0 \le L_{1,\varepsilon}(v, \tilde{V})$ $\tilde{U} \le u \le \bar{U}$, $x \in (0,1)\backslash x_0$;

(A_3). $\bar{U}_x(0) \le 0 \le \tilde{U}_x(0)$, $\bar{U}_x(1) \ge 0 \ge \tilde{U}_x(1)$, $\bar{V}_x(0) \le 0 \le \tilde{V}_x(0)$, $\bar{V}_x(1) \ge 0 \ge \tilde{V}_x(1)$;

(A$_4$). $\left(\bar{U}_x^{(-)} - \bar{U}_x^{(+)}\right)\Big|_{x=x_0} \geq 0,\ \left(\tilde{U}_x^{(-)} - \tilde{U}_x^{(+)}\right)\Big|_{x=x_0} \leq 0,\ \left(\bar{V}_x^{(-)} - \bar{V}_x^{(+)}\right)\Big|_{x=x_0} \geq 0,\ \left(\tilde{V}_x^{(-)} - \tilde{V}_x^{(+)}\right)\Big|_{x=x_0} \leq 0.$

In case of Proposition3 the inequality (A$_2$) will hold if

$$L_{u,\varepsilon}(\bar{U},\tilde{V}) < 0 < L_{u,\varepsilon}(\tilde{U},\bar{V}), \quad L_{v,\varepsilon}(\bar{U},\bar{V}) < 0 < L_{v,\varepsilon}(\tilde{U},\tilde{V}), \quad x \in (0,1)\backslash x_0 \tag{22}$$

In this article the upper and lower solutions are of the analogous structure to (8)-(9) with two separate parts — left and right relative to x_0. Functions $\bar{U}^{(\mp)}$, $\tilde{U}^{(\mp)}$, $\bar{V}^{(\mp)}$, $\tilde{V}^{(\mp)}$ are the modifications of asymptotic approximation of the solution in respective regions:

$$\begin{aligned}
\bar{U}^{(\mp)} &= U_1^{(\mp)} + \varepsilon\left(\alpha^{(\mp)}(x) + q^{(\mp)}U(\tau) + m^{(\mp)}U(\sigma)\right) + \varepsilon\bar{\Omega}_u(x,\varepsilon),\\
\tilde{U}^{(\mp)} &= U_1^{(\mp)} - \varepsilon\left(\alpha^{(\mp)}(x) + q^{(\mp)}U(\tau) + m^{(\mp)}U(\sigma)\right) + \varepsilon\tilde{\Omega}_u(x,\varepsilon),\\
\bar{V}^{(\mp)} &= V_1^{(\mp)} + \varepsilon\left(\beta^{(\mp)}(x) + q^{(\mp)}V(\tau)\right) + \varepsilon\bar{\Omega}_v(x,\varepsilon),\\
\tilde{V}^{(\mp)} &= V_1^{(\mp)} - \varepsilon\left(\beta^{(\mp)}(x) + q^{(\mp)}V(\tau)\right) + \varepsilon\tilde{\Omega}_v(x,\varepsilon),
\end{aligned}$$

where functions $\bar{\Omega}_u$, $\tilde{\Omega}_u$, $\bar{\Omega}_v$, $\tilde{\Omega}_v$ are additions to provide the inequalities (A$_3$) (A$_2$) in the vicinities of boundary points $x = 0$ and $x = 1$ and are similar to[4]. The continuity condition in x_0 yields

$$\bar{U}^{(-)}(x_0,\varepsilon) = \bar{U}^{(+)}(x_0,\varepsilon),\ \tilde{U}^{(-)}(x_0,\varepsilon) = \tilde{U}^{(+)}(x_0,\varepsilon),\ \ \bar{V}^{(-)}(x_0,\varepsilon) = \bar{V}^{(+)}(x_0,\varepsilon),\ \tilde{V}^{(-)}(x_0,\varepsilon) = \tilde{V}^{(+)}(x_0,\varepsilon).$$

Assuming A, B to be arbitrary positive constants we define the systems for $\alpha^{(\mp)}(x)$ and $\beta^{(\mp)}(x)$ as

$$\bar{f}_u^{(\mp)}(x)\alpha^{(\mp)}(x) - \bar{f}_v^{(\mp)}(x)\beta^{(\mp)}(x) = A, \quad \bar{g}_u^{(\mp)}(x)\alpha^{(\mp)}(x) + \bar{g}_v^{(\mp)}(x)\beta^{(\mp)}(x) = B, \tag{23}$$

The systems are solvable due to Propositions1-3 and5 and the functions $\alpha^{(\mp)}(x)$ and $\beta^{(\mp)}(x)$ have positive values for sufficient A and B.

Moving forward, for sufficiently big values d^u and D^u and also sufficiently small values k^u and K^u we determine the following problems for functions $qU^{(\mp)}(\tau)$ and $qV^{(\mp)}(\tau)$.

$$\tilde{f}_u^{(\mp)}(\tau)qU^{(\mp)}(\tau) - \tilde{f}_v^{(\mp)}(x)qV^{(\mp)}(\tau) + \left[\tilde{f}_u^{(\mp)}(\tau) - \bar{f}_u^{(\mp)}(x_0)\right]\alpha^{(\mp)}(x_0) - \left[\tilde{f}_v^{(\mp)}(\tau) - \bar{f}_v^{(\mp)}(x_0)\right]\beta^{(\mp)}(x_0) = d^u e^{-k^u|\tau|}, \tag{24}$$

$$\frac{d^2 qV^{(\mp)}(\tau)}{d\tau^2} = \tilde{v}^{(\mp)}(\tau)\cdot q^{(\mp)}V(\tau) + \tilde{G}^{(\mp)}(\tau), \ \ q^{(\mp)}V(0) = \delta_v - \beta^{(\mp)}(x_0), \ \ q^{(\mp)}V(\mp\infty) = 0. \tag{25}$$

where functions $\tilde{v}^{(\mp)}(\tau)$ are defined in7,

$$\begin{aligned}
\tilde{G}^{(\mp)}(\tau) :=& \left[\left(\tilde{g}_u^{(\mp)}(\tau) - \bar{g}_u^{(\mp)}(x_0)\right) - \frac{\tilde{g}_u^{(\mp)}(\tau)}{\tilde{f}_u^{(\mp)}(\tau)}\left(\tilde{f}_u^{(\mp)}(\tau) - \bar{f}_u^{(\mp)}(x_0)\right)\right]\alpha^{(\mp)}(x_0) +\\
&+ \left[\left(\tilde{g}_v^{(\mp)}(\tau) - \bar{g}_v^{(\mp)}(x_0)\right) + \frac{\tilde{g}_u^{(\mp)}(\tau)}{\tilde{f}_u^{(\mp)}(\tau)}\left(\tilde{f}_v^{(\mp)}(\tau) - \bar{f}_v^{(\mp)}(x_0)\right)\right]\beta^{(\mp)}(x_0) - \frac{\tilde{g}_u^{(\mp)}(\tau)}{\tilde{f}_u^{(\mp)}(\tau)} d^u e^{-k^u|\tau|} - D^v e^{-K^v|\tau|}.
\end{aligned}$$

The following lemma is true:

Lemma 1. Assume $\tilde{v}^{(\mp)}(\tilde{v}^{(\mp)}(\tau))$ satisfy Proposition5. Then on $\mathbb{R}_\mp$ for the equations $W_{\tau\tau}^{(\mp)}(\tau) - \tilde{v}^{(\mp)}(\tilde{v}^{(\mp)}(\tau))W^{(\mp)}(\tau) = 0$ there exist positive fundamental solutions $W^{(\mp)}(\tau)$ such that $W^{(\mp)}(\tau) \leq C_\gamma e^{-\gamma|\tau|}$ and $W_\tau^{(-)}(0,q_0)/W^{(-)}(0,q_0) > 0$, $W_\tau^{(+)}(0,q_0)/W^{(+)}(0,q_0) < 0$.

The proof of Lemma 1 is included in Appendix.

If we choose the value D^v sufficiently large then the functions $\tilde{G}^{(\mp)}(\tau)$ are negative and explicit solutions to problems25 are positive:

$$q^{(\mp)}V(\tau) = \left(\delta_v - \beta^{(\mp)}(x_0)\right)\frac{W^{(\mp)}(\tau,q_0)}{W^{(\mp)}(0,q_0)} + W^{(\mp)}(\tau,q_0)\int\limits_0^\tau \frac{d\tau_1}{\left[W^{(\mp)}(\tau_1,q_0)\right]^2}\int\limits_{\mp\infty}^{\tau_1} W^{(\mp)}(\tau_2,q_0)\tilde{G}^{(\mp)}(\tau_2)d\tau_2.$$

The functions $m^{(\mp)}U(\sigma)$ we define as solutions to problems

$$\frac{d^2 m^{(\mp)}U}{d\sigma^2} = \hat{f}_v^{(\mp)}(\sigma)\cdot m^{(\mp)}U(\sigma) + \hat{F}^{(\mp)}(\sigma), \ \ m^{(\mp)}U(0) = \delta_u - \alpha^{(\mp)}(x_0) - q^{(\mp)}U(0), \ \ m^{(\mp)}U(\mp\infty) = 0.$$

where

$$\hat{F}^{(\mp)}(\sigma) := \left[\hat{f}_u^{(\mp)}(\sigma) - \tilde{f}_u^{(\mp)}(0)\right]\left(\alpha^{(\mp)}(x_0) + q^{(\mp)}U(0)\right) + \left[\hat{f}_v^{(\mp)}(\sigma) - \tilde{f}_v^{(\mp)}(0)\right]\delta_v - D^u e^{-K^u|\sigma|}.$$

For sufficiently large coefficient D^u these problems have positive solutions

$$mU^{(\mp)}(\sigma) = \left(\delta_u - \alpha^{(\mp)}(x_0) - q^{(\mp)}U(0)\right) \frac{\Psi^{(\mp)}\left(\hat{u}^{(\mp)}(\sigma), q_0\right)}{\Psi^{(\mp)}(p_0, q_0)} +$$
$$+ \Psi^{(\mp)}\left(\hat{u}^{(\mp)}(\sigma), q_0\right) \int_0^{\sigma} \frac{d\sigma_1}{\left[\Psi^{(\mp)}\left(\hat{u}^{(\mp)}(\sigma_1), q_0\right)\right]^2} \int_{\mp\infty}^{\sigma_1} \Psi^{(\mp)}\left(\hat{u}^{(\mp)}(\sigma_2), q_0\right) \underbrace{\hat{F}^{(\mp)}(\sigma_2)}_{<0} d\sigma_2.$$

Due to the choice of functions $\alpha^{(\mp)}(x)$, $\beta^{(\mp)}(x)$, $q^{(\mp)}U(\tau)$, $q^{(\mp)}V(\tau)$, and $m^{(\mp)}U(\sigma)$ the inequalities (A_1) and (A_2) are satisfied.

To satisfy conditions in x_0 we consider for the upper solutions the following equations (in case of lower solutions right-hand sides have negative sign):

$$\left.\left(\frac{d\hat{U}^{(-)}}{dx} - \frac{d\hat{U}^{(+)}}{dx}\right)\right|_{x=x_0} = \varepsilon^2 \left.\left(\frac{dm^{(-)}U}{d\sigma} - \frac{dm^{(+)}U}{d\sigma}\right)\right|_{\sigma=0} + O(\varepsilon^3), \tag{26}$$
$$\left.\left(\frac{d\hat{V}^{(-)}}{dx} - \frac{d\hat{V}^{(+)}}{dx}\right)\right|_{x=x_0} = \varepsilon \left.\left(\frac{dq^{(-)}V}{d\tau} - \frac{dq^{(+)}V^{(+)}}{d\tau}\right)\right|_{\xi=0} + O(\varepsilon^2).$$

As

$$\left.\left(\frac{dq^{(-)}V}{d\sigma} - \frac{dq^{(+)}V}{d\tau}\right)\right|_{\tau=0} = \left(\frac{W_\tau^{(-)}(0,q_0)}{W^{(-)}(0,q_0)} - \frac{W_\tau^{(+)}(0,q_0)}{W^{(+)}(0,q_0)}\right) \cdot \delta_v - \frac{W_\tau^{(-)}(0,q_0)}{W^{(-)}(0,q_0)} \cdot \beta^{(-)}(x_0) + \frac{W_\tau^{(+)}(0,q_0)}{W^{(+)}(0,q_0)} \cdot \beta^{(+)}(x_0) +$$
$$+ \frac{1}{W^{(-)}(0,q_0)} \int_{-\infty}^{0} W^{(-)}(\tau,q_0)\tilde{G}^{(-)}(\tau)d\tau + \frac{1}{W^{(+)}(0,q_0)} \int_{0}^{+\infty} W^{(+)}(\tau,q_0)\tilde{G}^{(+)}(\tau)d\tau.$$

and

$$\left.\left(\frac{dm^{(-)}U}{d\sigma} - \frac{dm^{(+)}U}{d\sigma}\right)\right|_{\sigma=0} =$$
$$= \frac{1}{\Psi(0)} \left(\frac{\partial H^u}{\partial u}(q_0,p_0) \cdot \delta_u - \left[\alpha^{(-)}(x_0) + q^{(-)}U(0)\right] \cdot f^{(-)}\left(q_0,p_0,x_0,0\right) + \left[\alpha^{(+)}(x_0) + q^{(+)}U(0)\right] \cdot f^{(+)}\left(q_0,p_0,x_0,0\right)\right) +$$
$$+ \frac{1}{\Psi(0)} \left(\int_{-\infty}^{0} \Psi^{(-)}\left(\hat{u}^{(-)}(\sigma),q_0\right)\hat{F}^{(-)}(\sigma)d\sigma - \int_{+\infty}^{0} \Psi^{(+)}\left(\hat{u}^{(+)}(\sigma),q_0\right)\hat{F}^{(+)}(\sigma)d\sigma\right).$$

The right-hand side of26 can be made positive by choosing sufficiently big δ_v, δ_u and sufficiently small ε due to Lemma 1 and Proposition4.

5 | THE EXISTENCE OF STATIONARY SOLUTION

Theorem 1. Suppose Propositions1-5 hold. Then for sufficiently small $\varepsilon > 0$ there exists a solution $(u_\varepsilon(x), u_\varepsilon(x))$ of the problem (3), for which the pair of functions $(U(x,\varepsilon), V(x,\varepsilon))$ is a uniform asymptotic approximation with the accuracy of $O(\varepsilon^2)$, that is, for all $x \in [0,1]$, the inequality holds

$$|U(x,\varepsilon) - u_\varepsilon(x)| + |V(x,\varepsilon) - v_\varepsilon(x)| \le C\varepsilon^2, \ x \in [0,1],$$

where C is positive constant independent on ε.

Proof of the theorem is based on the proof of Pao[5] with slight modifications concerning presence of simple discontinuity in x_0. We define the iterative process as

$$-\varepsilon^4 \frac{d^2u^{(k)}}{dx^2} + cu^{(k)} = \mathcal{F}_1^{(k-1)}(x), \quad -\varepsilon^2 \frac{d^2v^{(k)}}{dx^2} + cv^{(k)} = \mathcal{F}_2^{(k-1)}(x), \quad \left.\frac{du^{(k)}}{dx}\right|_{x=0} = \left.\frac{du^{(k)}}{dx}\right|_{x=1} = \left.\frac{dv^{(k)}}{dx}\right|_{x=0} = \left.\frac{dv^{(k)}}{dx}\right|_{x=1} = 0, \tag{27}$$

where $c > 0$ is a sufficiently big constant and

$$\mathcal{F}_1^{(k-1)}(u^{(k-1)}, v^{(k-1)}, x) := -f(u^{(k-1)}, v^{(k-1)}, x, \varepsilon) + cu^{(k-1)}, \quad \mathcal{F}_2^{(k-1)}(u^{(k-1)}, v^{(k-1)}, x) := -g(u^{(k-1)}, v^{(k-1)}, x, \varepsilon) + cv^{(k-1)}.$$

The solutions to (27) can be expressed explicitly as

$$\hat{u}^{(k)}(x) = \int_0^1 G_1(x,s)\mathcal{F}_1^{(k-1)}(u^{(k-1)}, v^{(k-1)}, s)ds, \quad \hat{v}^{(k)}(x) = \int_0^1 G_2(x,s)\mathcal{F}_2^{(k-1)}(u^{(k-1)}, v^{(k-1)}, s)ds \tag{28}$$

and are of $C^1[0,1] \cap C^2((0,1)\backslash x_0)$ for $u^{(0)}, v^{(0)} \in C([0,1])$ class [8].

Following Pao [5] we consider monotone sequences

$$\tilde{U} \le \tilde{u}^{(k-1)} \le \tilde{u}^{(k)} \le \bar{u}^{(k)} \le \bar{u}^{(k-1)} \le \bar{U}, \quad \tilde{V} \le \tilde{v}^{(k-1)} \le \tilde{v}^{(k)} \le \bar{v}^{(k)} \le \bar{v}^{(k-1)} \le \bar{V}.$$

It is worth noting that in [5] monotonicity is proven for C^2 functions on the basis of maximum principle, in our case we use **Lemma 2**. Assume $w(x) \in C^1[0,1] \cap C^2((0,1)\backslash x_0)$ for some continuous $c(x) > 0$ satisfies

$$-w''(x) + c(x)w(x) \ge 0,\ x \in (0,1)\backslash x_0, \quad w'(x_0 - 0) \ge w'(x_0 + 0), \quad w'(0) \le 0 \le w'(1), \tag{29}$$

then $w(x) \ge 0,\ x \in [0,1]$.

From now on we consider (27) for the supersequence (for the subsequence reasoning is analogous). In (28) there exist limits of the left-hand sides therefore there exist limits of the left-hand sides.

Utilizing Levi's Theorem from explicit form (28)

$$\bar{u}_\varepsilon(x) = \int_0^1 G_1(x,s)\bar{\mathcal{F}}_1(s)ds, \quad \bar{v}_\varepsilon(x) = \int_0^1 G_2(x,s)\bar{\mathcal{F}}_2(s)ds, \tag{30}$$

$$\bar{\mathcal{F}}_1(s) = \lim_{k\to\infty} \mathcal{F}_1^{(k)}(u^{(k)}, v^{(k)}, s) \ \left(= \mathcal{F}_1^{(k)}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, s)\right), \quad \bar{\mathcal{F}}_2(s) = \lim_{k\to\infty} \mathcal{F}_2^{(k)}(u^{(k)}, v^{(k)}, s) \ \left(= \mathcal{F}_2^{(k)}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, s)\right)$$

we conclude continuity of the limits which coupled with expressions in parentheses allows then to conclude from (28) that $\bar{u}_\varepsilon(x),\ \bar{v}_\varepsilon(x) \in C^1[0,1] \cap C^2((0,1)\backslash x_0)$ and are indeed solutions of the stationary problem (3) in the sense of Definition 1.

6 | THE LOCALLY UNIQUENESS AND STABILITY OF THE STATIONARY SOLUTION

Theorem 2. Suppose Proposition1-5 hold. Then for sufficiently small $\varepsilon > 0$ there exists locally unique and asymptotically stable in the sense of Lyapunov solution $(u_\varepsilon(x), v_\varepsilon(x))$ to the problem (1) having the internal transition layer in the vicinity of the point x_0 with the domain of attraction not less than $[\tilde{U}(x,\varepsilon), \bar{U}(x,\varepsilon)] \times [\tilde{V}(x,\varepsilon), \bar{V}(x,\varepsilon)]$.

Proof of this theorem is based on sub-supersolutions method. Introduce in $\overline{D_T}$ functions:

$$\tilde{U}_T(x,t,\varepsilon) = u_\varepsilon(x) + \left(\tilde{U} - u_\varepsilon(x)\right)e^{-\varepsilon\lambda t}, \tilde{V}_T(x,t,\varepsilon) = u_\varepsilon(x) + \left(\tilde{V} - v_\varepsilon(x)\right)e^{-\varepsilon\lambda t}, \tag{31}$$

$$\bar{U}_T(x,t,\varepsilon) = u_\varepsilon(x) + \left(\bar{U} - u_\varepsilon(x)\right)e^{-\varepsilon\lambda t}, \bar{V}_T(x,t,\varepsilon) = u_\varepsilon(x) + \left(\bar{V} - v_\varepsilon(x)\right)e^{-\varepsilon\lambda t}, \tag{32}$$

where $\left(u_\varepsilon(x), v_\varepsilon(x)\right)$ - any solution of (3). Also for the initial functions in (1) we demand for all $x \in [0,1]$

$$\tilde{U}_T(x,0,\varepsilon) = \tilde{U}(x,\varepsilon) \le u^0(x) \le \bar{U}(x,\varepsilon) = \bar{U}_T(x,0,\varepsilon), \quad \tilde{V}_T(x,0,\varepsilon) = \tilde{V}(x,\varepsilon) \le v^0(x) \le \bar{V}(x,\varepsilon) = \bar{V}_T(x,0,\varepsilon).$$

These functions (31) are indeed the lower and the upper solutions as defined and proven in [9] .

We define the iterative process for the initial boundary value problem as

$$\begin{aligned} &\frac{\partial y^{(k)}}{\partial t} - \varepsilon^4 \frac{\partial^2 y^{(k)}}{\partial x^2} + c y^{(k)} = \mathcal{F}_1^{(k-1)}(y^{(k-1)}, z^{(k-1)}, x, t), \quad y_x^{(k)}(0,t) = y_x^{(k)}(1,t) = 0, \quad y^{(k)}(x,0) = u^0(x), \\ &\frac{\partial z^{(k)}}{\partial t} - \varepsilon^2 \frac{\partial^2 z^{(k)}}{\partial x^2} + c z^{(k)} = \mathcal{F}_2^{(k-1)}(y^{(k-1)}, z^{(k-1)}, x, t), \quad z_x^{(k)}(0,t) = z_x^{(k)}(1,t) = 0, \quad z^{(k)}(x,0) = v^0(x). \end{aligned} \tag{33}$$

where

$$\mathcal{F}_1^{(k)}(y^{(k)}, z^{(k)}, x, t) := -f(y^{(k)}, z^{(k)}, x, \varepsilon) + c y^{(k)}, \quad \mathcal{F}_2^{(k)}(y^{(k)}, z^{(k)}, x, t) := -g(y^{(k)}, z^{(k)}, x, \varepsilon) + c z^{(k)}.$$

Following Pao [5] and using a proposition analogous to Lemma 1 for parabolic systems [10] we obtain monotone sequences

$$\tilde{U}_T \le \tilde{y}^{(k-1)} \le \tilde{y}^{(k)} \le \bar{y}^{(k)} \le \bar{y}^{(k-1)} \le \bar{U}_T, \quad \tilde{V}_T \le \tilde{z}^{(k-1)} \le \tilde{z}^{(k)} \le \bar{z}^{(k)} \le \bar{z}^{(k-1)} \le \bar{V}_T.$$

Denote

$$\varkappa_1^{(k)}(t) := f^{(-)}(y^{(k)}(x_0,t), z^{(k)}(x_0,t), x_0, \varepsilon) - f^{(+)}(y^{(k)}(x_0,t), z^{(k)}(x_0,t), x_0, \varepsilon),$$
$$f_0^{(k)}(x,t) := f(y^{(k)}(x,t), z^{(k)}(x,t), x, \varepsilon) + \Theta(x-x_0)\varkappa_{1,sub}^{(k)}(t),$$
$$\varkappa_2^{(k)}(t) := g^{(-)}(y^{(k)}(x_0,t), z^{(k)}(x_0,t), x_0, \varepsilon) - g^{(+)}(y^{(k)}(x_0,t), z^{(k)}(x_0,t), x_0, \varepsilon),$$
$$g_0^{(k)}(x,t) := g(y^{(k)}(x,t), z^{(k)}(x,t), x, \varepsilon) + \Theta(x-x_0)\varkappa_2^{(k)}(t),$$

therefore

$$f(y^{(k)}(x,t), z^{(k)}(x,t), x, \varepsilon) \equiv f_0^{(k)}(x,t) + \Theta(x-x_0)\varkappa_1^{(k)}(t), \quad g(y^{(k)}(x,t), z^{(k)}(x,t), x, \varepsilon) \equiv g_0^{(k)}(x,t) + \Theta(x-x_0)\varkappa_2^{(k)}(t),$$
$$\mathcal{F}_1^{(k-1)}(y^{(k-1)}, z^{(k-1)}, x, t) \equiv \mathcal{F}_{1,0}^{(k-1)}(x,t) + \Theta(x-x_0)\varkappa_1^{(k)}(t), \quad \mathcal{F}_2^{(k-1)}(y^{(k-1)}, z^{(k-1)}, x, t) \equiv \mathcal{F}_{2,0}^{(k-1)}(x,t) + \Theta(x-x_0)\varkappa_2^{(k)}(t).$$

Here we introduce

$$Y^{(k)}(x,t) := \int_0^1 G_1(x,s,t)u^0(s)ds + \int_0^t d\tau \int_0^1 G_1(x,s,t-\tau)\mathcal{F}_{1,0}^{(k-1)}(s,\tau)ds, \quad U^{(k)}(x,t) := \int_0^t d\tau \int_{x_0}^1 G_1(x,s,t-\tau)\varkappa_1^{(k)}(\tau)ds$$

The function $Y^{(k)}(x,t)$ is obviously a classical solution to the problem

$$\frac{\partial Y^{(k)}}{\partial t} - \varepsilon^4 \frac{\partial^2 Y^{(k)}}{\partial x^2} + cY^{(k)} = \mathcal{F}_{1,0}^{(k-1)}(x,t), \quad Y_x^{(k)}(0,t) = Y_x^{(k)}(1,t) = 0, \quad Y^{(k)}(x,0) = u^0(x). \tag{34}$$

Considering $U^{(k)}(x,t)$ as a function defined on $\overline{D_T^a}$, utilizing the reasoning of Friedman [11] and the estimates [12] it can be proven that $U^{(k)}(x,t) \in C\left(\overline{D_T^a}\right)$ and

$$\tilde{U}_x^{(k)}(x,t) := \int_0^t d\tau \int_{x_0}^1 \frac{\partial}{\partial x} G_1(x,s,t-\tau)\varkappa_1^{(k)}(\tau)ds \in C\left(\overline{D_T^a}\right),$$

moreover, $U_x^{(k)}(x,t) \equiv \tilde{U}_x^{(k)}(x,t)$ for $x \in D_T^a$. Lagrange's Theorem provides that $U_x^{(k)}(x_0,t) = \tilde{U}_x^{(k)}(x_0,t)$, $U_x^{(k)}(1,t) = \tilde{U}_x^{(k)}(1,t)$ for $t \in (0,T]$ and due to [12] or [13] $U_x^{(k)}(x,t) \equiv \tilde{U}_x^{(k)}(x,t) \rightrightarrows x \in [-a, 1+a]t \to +00 \equiv U_x^{(k)}(x,0)$, hence $U_x^{(k)}(x,t) \in C\left(\overline{D_T}\right) \subset C\left(\overline{D_T^a}\right)$. Furthermore, with regard to [11,12] the equation $\tilde{U}_t^{(k)} - \varepsilon^4 \tilde{U}_{xx}^{(k)} + c\tilde{U}^{(k)} = \Theta(x-x_0)\varkappa_1^{(k)}(t)$, $(x,t) \in D_T^{(-)} \cup D_T^{(+)}$ is being satisfied in the classical sense, therefore it concludes in $y^{(k)}(x,t) = Y^{(k)}(x,t) + U^{(k)}(x,t)$ being the solution to (34) in the sense of Definition 1 and having an explicit form

$$y^{(k)}(x,t) = \int_0^1 G_1(x,s,t)u^0(s)ds + \int_0^t d\tau \int_0^1 G_1(x,s,t-\tau)\mathcal{F}_1^{(k-1)}(y^{(k-1)}, z^{(k-1)}, s, \tau)ds,.$$

Same for $z^{(k)}(x,t)$.

Now we consider, for example, supersequence — using the same steps as in the previous paragraph it can be proven firstly that $\left(\bar{y}^{(k)}(x,t), \bar{z}^{(k)}(x,t)\right) \rightrightarrows \overline{D_T} k \to \infty \left(\bar{y}_\varepsilon(x,t), \bar{z}_\varepsilon(x,t)\right) \in C\left(\overline{D_T}\right)$, then that the pair

$$\bar{y}_\varepsilon(x,t) = \int_0^1 G_1(x,s,t)u^0(s)ds + \int_0^t d\tau \int_0^1 G_1(x,s,t-\tau)\bar{\mathcal{F}}_1(s,\tau)ds,$$
$$\bar{z}_\varepsilon(x,t) = \int_0^1 G_2(x,s,t)v^0(s)ds + \int_0^t d\tau \int_0^1 G_2(x,s,t-\tau)\bar{\mathcal{F}}_2(s,\tau)ds,$$
$$\bar{\mathcal{F}}_1(s,\tau) = \lim_{k\to\infty} \mathcal{F}_1^{(k)}(y^{(k)}, z^{(k)}, s, \tau) \ \left(= \mathcal{F}_1(\bar{y}_\varepsilon, \bar{z}_\varepsilon, s, \tau)\right), \quad \bar{\mathcal{F}}_2(s,\tau) = \lim_{k\to\infty} \mathcal{F}_2^{(k)}(y^{(k)}, z^{(k)}, s, \tau) \ \left(= \mathcal{F}_2(\bar{y}_\varepsilon, \bar{z}_\varepsilon, s, \tau)\right)$$

is indeed the solution to (1) in the sense of Definition (1).

Finally, due to uniqueness of the solution [5] to the initial boundary value problem we have $y_\varepsilon(x,t) := \bar{y}_\varepsilon(x,t) = \tilde{y}_\varepsilon(x,t)$, $z_\varepsilon(x,t) := \bar{z}_\varepsilon(x,t) = \tilde{z}_\varepsilon(x,t)$. and from (31) follows that

$$\lim_{t\to+\infty} \|y_\varepsilon(x,t) - u_\varepsilon(x)\|_{C[0,1]} = 0, \quad \lim_{t\to+\infty} \|z_\varepsilon(x,t) - v_\varepsilon(x)\|_{C[0,1]} = 0.$$

7 | CONCLUSION

Though only the one-dimensional problems are considered in this paper it is already sufficient for development and justification of various models in physics especially when numerical experiments are preferable or simply unavoidable. Moreover, as a natural step forward our approach with several slight adjustments can be extended to the 2D problems which are proven to be extremely prolific for modelling.

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APPENDIX

A THE PROOF OF LEMMA 1

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Proof. The existence of the fundamental exponentially bounded solution immediately follows from the fact that $\tilde{v}^{(\mp)}(\tilde{v}^{(\mp)}(\tau))$ are bounded continuous functions on $\mathbb{R}_{\mp}$ and the book [14]. The inequalities for derivatives and positivity are the result of linearity of the equation in question and functions

$$\overline{W}^{(\mp)}(\tau) := \exp\left(-\int\limits_{q^*}^{\tilde{v}^{(\mp)}(\tau)} \frac{ds_1}{\left(\Phi^{(\mp)}(s_1)\right)^2} \int\limits_{s_1}^{\psi^{(\mp)}(x_0)} \tilde{v}^{(\mp)}(s_2)ds_2\right), \quad \underline{W}^{(\mp)}(\tau) = \frac{\Phi^{(\mp)}(\tilde{v}^{(\mp)}(\tau))}{\Phi^{(\mp)}(q^*)}$$

being respectively super- and subsolutions to problems

$$W^{(\mp)}_{\tau\tau}(\tau) - \tilde{v}^{(\mp)}(\tilde{v}^{(\mp)}(\tau))W^{(\mp)}(\tau) = 0, \quad \tau \in (0, T^{(\mp)}), \quad W^{(\mp)}(0) = 1, \quad W^{(\mp)}(T^{(\mp)}) = \delta^{(\mp)}$$

for any $T^{(+)} > 0$, $T^{(-)} < 0$ and $\delta^{(\mp)}$ such that $\underline{W}^{(\mp)}(T^{(\mp)}) \le \delta^{(\mp)} \le \overline{W}^{(\mp)}(T^{(\mp)})$. □

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