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RESEARCH ARTICLE

Finding Appell convolution of certain special polynomials.

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Abstract

In this article, the truncated exponential-Gould-Hopper polynomials are taken as base with the Appell polynomials to introduce a hybrid family of truncated exponential-Gould-Hopper-Appell polynomials. These polynomials are framed within the context of monomiality principle and their determinant definition and properties are established. Further, we investigate some members belonging to this family. In addition graphical representation and zeros of these members are demonstrated using computer experiment..

KEYWORDS:

Truncated exponential-Gould-Hopper polynomials; Appell polynomials; Monomiality principle; Operational techniques

1 | INTRODUCTION AND PRELIMINARIES

Generalized and multivariable forms of the special functions of mathematical physics has, in its various forms, been an object of speculation and application during the recent years. Most of the special functions of mathematical physics and their generalizations have been suggested by physical problems. Recently, a systematic study of certain new classes of mixed special polynomials associated to the Appell polynomials sequences is introduced, see for example [1,2,3]. These mixed special polynomials are important due to the fact that they posses important properties such as differential equations, generating functions, series definitions, integral representations etc. We recall the 3-variable truncated exponential based Gould-Hopper polynomials (3VTEGHP), denoted by ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$, defined by means of the following generating function [4]:

$$\frac{\exp(ut+wt^{\mathfrak{s}})}{1-vt^{\mathfrak{r}}} = \sum_{n=0}^{\infty} {}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)\frac{t^n}{n!} \tag{1}$$

and posses the following equivalent forms of series representation in terms of 2 variable truncated exponential polynomials (2VTEP) [5], denoted by $e_n^{(\mathfrak{r})}(u,v)$; Gould-Hopper polynomials (GHP) [6], denoted by $H_n^{(\mathfrak{s})}(u,w)$; and in terms of u, v and w:

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = n!\sum_{k=0}^{[\frac{n}{\mathfrak{s}}]} \frac{w^k\ e_{n-\mathfrak{s}k}^{(\mathfrak{r})}(u,v)}{k!(n-\mathfrak{s}k)!}, \tag{2}$$

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = n!\sum_{m=0}^{[\frac{n}{\mathfrak{r}}]} \frac{v^m\ H_{n-\mathfrak{r}m}^{(\mathfrak{s})}(u,w)}{(n-\mathfrak{r}m)!} \tag{3}$$

and

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = n!\sum_{k,m=0}^{\mathfrak{s}k+\mathfrak{r}m\le n} \frac{u^{n-\mathfrak{s}k-\mathfrak{r}m}v^m w^k}{k!(n-\mathfrak{s}k-\mathfrak{r}m)!}, \tag{4}$$

[0]**Abbreviations:** TEGHAP, truncated exponential-Gould-Hopper based Appell polynomials; 3VTEGHP, 3-variable truncated exponential based Gould-Hopper polynomials; 2VTEP, 2 variable truncated exponential polynomials

respectively.

It is shown in[4], that the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ are quasimonomial[7,8] under the action of the following multiplicative and derivative operators:

$$\hat{M}_{{}_{e^{(\mathfrak{r})}}H^{(\mathfrak{s})}} = u + \mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}-1} + \mathfrak{s}w\partial_u^{\mathfrak{s}-1} \tag{5}$$

and

$$\hat{P}_{{}_{e^{(\mathfrak{r})}}H^{(\mathfrak{s})}} = \partial_u, \tag{6}$$

respectively.

Again since ${}_{e^{(\mathfrak{r})}}H_0^{(\mathfrak{s})}(u,v,w) = 1$, so in view of monomiality principle the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ can be explicitly constructed as:

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = \hat{M}^n_{{}_{e^{(\mathfrak{r})}}H^{(\mathfrak{s})}}\{1\} = \left(u + \mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}-1} + \mathfrak{s}w\partial_u^{\mathfrak{s}-1}\right)^n\{1\}, \tag{7}$$

which yields the series definition (4).

Identity (7) implies that the exponential generating function of the GHP $H_n^{(\mathfrak{s})}(u,v)$ can be cast in the form:

$$\exp(\hat{M}_{{}_{e^{(\mathfrak{r})}}H^{(\mathfrak{s})}}\ t)\{1\} = \sum_{n=0}^{\infty} {}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)\frac{t^n}{n!}, \tag{8}$$

which yields generating function (1).

The operational representation of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ is given by:

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = \exp\left(w\partial_u^{\mathfrak{s}} + v\partial_v v\partial_u^{\mathfrak{r}}\right)\ u^n. \tag{9}$$

The operational representation connecting the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ with the 2VTEP $e_n^{(\mathfrak{r})}(u,v)$ and GHP $H_n^{(\mathfrak{s})}(u,v)$ is given by:

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = \exp\left(w\partial_u^{s}\right)\ e_n^{(\mathfrak{r})}(u,v) \tag{10}$$

and

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = \exp\left(v\partial_v v\partial_u^{\mathfrak{r}}\right)\ H_n^{(\mathfrak{s})}(u,w), \tag{11}$$

respectively.

The integral representation for the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ in terms of 2-iterated Gould-Hopper polynomials (2IGHP)[9] is given by:

$${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = \int_0^{\infty} e^{-x}\ {}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,vx,w)dx. \tag{12}$$

Sequences of polynomials are a topic of interest in enumerative combinatorics, algebraic combinatorics and applied mathematics. They play an important role in numerous branches of sciences. One of the important class of polynomial sequences is the class of Appell polynomial sequences[10]. They are very often found in different applications in pure and applied mathematics. Properties of Appell sequences are naturally handled within the framework of modern classical umbral calculus by Roman[11].

In 1880, Appell[10] introduced and studied sequences of n-degree polynomials $A_n(u),\ n = 0,1,2,\cdots$ satisfying the recurrence relation

$$\frac{d}{du}A_n(u) = nA_{n-1}(u), \quad n = 0,1,2,\cdots. \tag{13}$$

The generating function of the sequence of polynomials $A_n(u)$ is given as:

$$A(t)\exp(ut) = \sum_{n=0}^{\infty} A_n(u)\frac{t^n}{n!}, \tag{14}$$

where $A(t)$ has (at least the formal) expansion:

$$A(t) = \sum_{n=0}^{\infty} A_n\frac{t^n}{n!}, \quad (A_0 \neq 0). \tag{15}$$

Series representation of Appell polynomials is given by:

$$A_n(u) = \sum_{k=0}^{n} {}^nC_k A_k u^{n-k}. \tag{16}$$

The Appell polynomials constitute an important class of polynomials because of their remarkable applications in numerous fields. The Bernoulli polynomials $B_n(u)$ and the Euler polynomials $E_n(u)$ are some of the important polynomials belonging to

TABLE 1 Certain members belonging to the Appell family.

S. No.	A(t)	Name of the Special Polynomial	Generating Function	Series Definition
I	$\frac{t}{e(t)-1}$	Bernoulli polynomials [12]	$\frac{t}{e(t)-1}\exp(ut)=\sum_{n=0}^{\infty}B_n(u)\frac{t^n}{n!}$	$B_n(u)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}B_k u^{n-k}$
II	$\frac{2}{e(t)+1}$	Euler polynomials [12]	$\frac{2}{\exp(t)+1}e(ut)=\sum_{n=0}^{\infty}E_n(u)\frac{t^n}{n!}$	$E_n(u)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_k u^{n-k}$

the class of Appell sequences. These polynomials plays a fundamental job in different extensions and approximations formulae, which are valuable both in classical and numerical analysis and in analytic theory of numbers. By selecting appropriate function $A(t)$, different members of Appell family can be obtained. Notations, names, generating functions and series definitions of certain members belonging to the Appell family are listed in Table 1 .

In this article, a hybrid class of truncated exponential-Gould-Hopper based Appell polynomials is introduced and many important properties of these polynomials are investigated. The generating function, series representation and determinant forms for this hybrid class of polynomials are derived. Further, we study some members belonging to this newly introduced class of special polynomials. In addition, shapes and zeros of this family are shown graphically.

2 | APPELL CONVOLUTION

In this section, a new hybrid class of truncated exponential-Gould-Hopper based Appell polynomials (TEGHAP) denoted by ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ is introduced by convoluting the 3VTEGHP and Appell polynomials by means of generating function.

In view of replacement and operational techniques, replacing u by the multiplicative operator $\hat{M}_{{}_{e^{(\mathfrak{r})}}H^{(\mathfrak{s})}}$ of the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ in the generating function (14) and using equations (1), (5) and (8) and thereafter denoting $A_n(u+\mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}-1}+\mathfrak{s}w\partial_u^{\mathfrak{s}-1})$ by ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$, that is

$$A_n(\hat{M}_{{}_{e^{(\mathfrak{r})}}H^{(\mathfrak{s})}})=A_n(u+\mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}-1}+\mathfrak{s}w\partial_u^{\mathfrak{s}-1})={}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{17}$$

we define the truncated exponential-Gould-Hopper based Appell polynomials as:

Definition 1. The truncated exponential-Gould-Hopper based Appell polynomials are defined by means of the generating function:

$$A(t)\frac{\exp(ut+wt^{\mathfrak{s}})}{1-vt^{\mathfrak{r}}}=\sum_{n=0}^{\infty}{}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!}. \tag{18}$$

Remark 2.1. We remark that equation (17) gives the operational correspondence between the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ and TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$.

Next, replacing u by $\hat{M}_{{}_{e^{(\mathfrak{r})}}H^{(\mathfrak{s})}}$ in the series definition (16) and utilizing equations (7) and (17), we obtain the following series definition of the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$:

Definition 2. The truncated exponential-Gould-Hopper based Appell polynomials are defined by the series:

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=n!\sum_{k=0}^{n}{}^{n}C_k\;A_k\;{}_{e^{(\mathfrak{r})}}H_{n-k}^{(\mathfrak{s})}(u,v,w). \tag{19}$$

Also, we can find the following equivalent forms of the series representation of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$:

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=n!\sum_{k,m=0}^{k+\mathfrak{s}m\le n}\frac{A_k\;w^m\;e_{n-k-\mathfrak{s}m}^{(\mathfrak{r})}(u,v)}{k!m!(n-k-\mathfrak{s}m)!}, \tag{20}$$

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=n!\sum_{k,m=0}^{k+\mathfrak{r}m\le n}\frac{A_k\;v^m\;H_{n-k-\mathfrak{r}m}^{(\mathfrak{s})}(u,w)}{k!(n-k-\mathfrak{r}m)!}, \tag{21}$$

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=n!\sum_{m,p=0}^{\mathfrak{r}m+\mathfrak{s}p\le n}\frac{v^m\;w^p\;A_{n-\mathfrak{r}m-\mathfrak{s}p}(u)}{(n-\mathfrak{r}m-\mathfrak{s}p)!} \tag{22}$$

TABLE 2 Certain members belonging to convoluted Appell family.

S. No.	$\mathbf{A_q(t)}$	Notation and Name of the Resultan Member	Generating Function	Series Definition
I	$A(t)=\frac{t}{e(t)-1}$	${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$:= Truncated exponential-Gould -Hopper-Bernoulli polynomials (TEGHBP)	$\frac{t\exp(ut+wt^{\mathfrak{s}})}{(e^t-1)(1-vt^{\mathfrak{r}})} = \sum_{n=0}^{\infty}{}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!}$	${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = n!\sum_{k=0}^{n}{}^nC_k\ B_k\ {}_{e^{(\mathfrak{r})}}H_{n-k}^{(\mathfrak{s})}(u,v,w)$
II	$A(t)=\frac{2}{e(t)+1}$	${}_{_eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$:= Truncated exponential-Gould -Hopper-Euler polynomials (TEGHEP)	$\frac{2\exp(ut+wt^{\mathfrak{s}})}{(e^t+1)(1-vt^{\mathfrak{r}})} = \sum_{n=0}^{\infty}{}_{_eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!}$	${}_{_eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = n!\sum_{k=0}^{n}{}^nC_k\ E_k\ {}_{e^{(\mathfrak{r})}}H_{n-k}^{(\mathfrak{s})}(u,v,w)$

and

$$ {}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = n!\sum_{k,m,p=0}^{k+\mathfrak{r}m+\mathfrak{s}p\le n}\frac{A_k\ u^{n-k-\mathfrak{r}m-\mathfrak{s}p}\ v^m\ w^p}{k!p!(n-k-\mathfrak{r}m-\mathfrak{s}p)!}. \tag{23} $$

Few members of Appell family are listed in Table 1 . On appropriate selection of function $A(t)$ in generating function (18), we obtain different members belonging to convoluted Appell family. Notations, names, generating functions and series definitions of these members are mentioned in Table 2 .

Over the last few years, there has been increasing interest in a new approach related to special polynomials, that is, determinant approach. Costabile et al.[13] have established a new definition to Bernoulli polynomials based on a determinant approach. Further, this approach has been extended to provide determinant definitions of the Appell polynomials[14]. Recently, Keleshteri and Mahmudov[15] introduce the determinant form of q-Appell polynomials. Because of the importance of determinant forms for applied and computational purposes, the determinant representation of the TEGHAP ${}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ along with few members belonging to this class are obtained.

By following the methodology presented in[3] and in view of equations (7) and (17), the following determinant form for ${}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ is obtained:

Definition 3. The TEGHAP ${}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ of degree n are defined by

$$ {}_{_eH}A_0^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \frac{1}{\beta_0}, $$

$$ {}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} 1 & {}_{e^{(\mathfrak{r})}}H_1^{(\mathfrak{s})}(u,v,w) & {}_{e^{(\mathfrak{r})}}H_2^{(\mathfrak{s})}(u,v,w) & \cdots & {}_{e^{(\mathfrak{r})}}H_{n-1}^{(\mathfrak{s})}(u,v,w) & {}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{24} $$

where $n=1,2,\cdots$, and ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ ($n=1,2,\cdots$ are the 3VTEGHP; $\beta_0\neq 0$ and

$$ \beta_0 = \frac{1}{A_0}, $$

$$ \beta_n = -\frac{1}{A_0}\left(\sum_{k=1}^{n}\binom{n}{k}A_k\beta_{n-k}\right), \quad n=1,2,\cdots. \tag{25} $$

Remark 2.2. Since the TEGHBP ${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ and TEGHEP ${}_{_eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ given in Table 2 are particular members of TEGHAP ${}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$. Thus, by making appropriate selection for the coefficients β_0 and β_i ($i=1,2,\cdots,n$) in determinant representation of TEGHAP ${}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$, the determinant definition of TEGHBP ${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ and TEGHEP ${}_{_eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ can be obtained. For instance, taking $\beta_0=1$ and $\beta_i=\frac{1}{i+1}$, ($i=1,2,\cdots,n$) in equation (24), the following determinant definition of TEGHBP ${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ is obtained:

Definition 4. The TEGHBP ${}_{e H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ of degree n are defined by

$$ {}_{e H}B_0^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = 1, \tag{26} $$

$$ {}_{e H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = (-1)^n \begin{vmatrix} 1 & {}_{e^{(\mathfrak{r})}}H_1^{(\mathfrak{s})}(u,v,w) & {}_{e^{(\mathfrak{r})}}H_2^{(\mathfrak{s})}(u,v,w) & \cdots & {}_{e^{(\mathfrak{r})}}H_{n-1}^{(\mathfrak{s})}(u,v,w) & {}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{n-1} & \binom{n}{1}\frac{1}{n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{n-2} & \binom{n}{2}\frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}, \tag{27} $$

where ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ $(n = 1, 2, \cdots)$ are the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ defined by equation (1).

Further, it has been shown in [14] that for $\beta_0 = 1$ and $\beta_i = \frac{1}{2},\ (i = 1, 2, \cdots, n)$ the determinant definition of Appell polynomials $A_n(u)$ reduces to determinant definition of Euler polynomials $E_n(u)$. Therefore, taking $\beta_0 = 1$ and $\beta_i = \frac{1}{2},\ (i = 1, 2, 3, \cdots, n)$ in equations (24), gives the following determinant form of the TEGHEP ${}_{e H}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$:

Definition 5. The TEGHEP ${}_{e H}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ of degree n are defined by

$$ {}_{e H}E_0^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = 1, \tag{28} $$

$$ {}_{e H}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = (-1)^n \begin{vmatrix} 1 & {}_{e^{(\mathfrak{r})}}H_1^{(\mathfrak{s})}(u,v,w) & {}_{e^{(\mathfrak{r})}}H_2^{(\mathfrak{s})}(u,v,w) & \cdots & {}_{e^{(\mathfrak{r})}}H_{n-1}^{(\mathfrak{s})}(u,v,w) & {}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{1}\frac{1}{2} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2} & \binom{n}{2}\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}, \tag{29} $$

where ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ $(n = 1, 2, \cdots)$ are the 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ defined by equation (1).

3 | PROPERTIES

In order to frame the TEGHAP ${}_{e H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ within the context of monomiality principle, we first determine multiplicative and derivative operators:

Theorem 1. The TEGHAP ${}_{e H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ are quasi-monomial under the action of the following multiplicative and derivative operators:

$$ \hat{M}_{{}_{e H}A^{(\mathfrak{r},\mathfrak{s})}} = u + rv\partial_v v\partial_u^{\mathfrak{r}-1} + \mathfrak{s}w\partial_u^{\mathfrak{s}-1} + \frac{A'(\partial_u)}{A(\partial_u)} \tag{30} $$

and

$$ \hat{P}_{{}_{e H}A^{(\mathfrak{r},\mathfrak{s})}} = \partial_u\ , \tag{31} $$

respectively.

Proof. Consider the identity

$$ \partial_u \left(\frac{A(t)\exp(ut + wt^s)}{1 - vt^{\mathfrak{r}}} \right) = t \left(\frac{A(t)\exp(ut + wt^s)}{1 - vt^{\mathfrak{r}}} \right). \tag{32} $$

Replacing u by the multiplicative operator $\hat{M}_{e^{(\mathfrak{r})}H^{(\mathfrak{s})}}$ in the generating function (14), we get

$$ A(t)\exp(\hat{M}_{e^{(\mathfrak{r})}H^{(\mathfrak{s})}}\ t) = \sum_{n=0}^{\infty} A_n(\hat{M}_{e^{(\mathfrak{r})}H^{(\mathfrak{s})}})\frac{t^n}{n!}. \tag{33} $$

Next, differentiating equation (33) partially with respect to t, we find

$$ \left(\hat{M}_{e^{(\mathfrak{r})}H^{(\mathfrak{s})}} + \frac{A'(t)}{A(t)} \right) A(t)\exp(\hat{M}_{e^{(\mathfrak{r})}H^{(\mathfrak{s})}}\ t) = \sum_{n=0}^{\infty} A_n(\hat{M}_{e^{(\mathfrak{r})}H^{(\mathfrak{s})}})\frac{t^{n-1}}{(n-1)!}\ . \tag{34} $$

Using equation (33) on l.h.s. and then using relation (17) on both sides of equation (34), we obtain

$$ \left(\hat{M}_{e^{(\mathfrak{r})}H^{(\mathfrak{s})}} + \frac{A'(t)}{A(t)} \right) \sum_{n=0}^{\infty} {}_{e H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{e H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^{n-1}}{(n-1)!}. \tag{35} $$

Now, putting the value of multiplicative operator of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ from (5) and using generating function of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ (18) in the l.h.s. of above equation, we get

$$\left(u+\mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}-1}+\mathfrak{s}w\partial_u^{\mathfrak{s}-1}+\frac{A'(t)}{A(t)}\right)\frac{A(t)\exp(ut+wt^{\mathfrak{s}})}{1-vt^{\mathfrak{r}}}=\sum_{n=0}^{\infty}{}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^{n-1}}{(n-1)!} \tag{36}$$

which on using identity (32) and then using generating function of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ (18) in the l.h.s. gives

$$\left(u+\mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}-1}+\mathfrak{s}w\partial_u^{\mathfrak{s}-1}+\frac{A'(\partial_u)}{A(\partial_u)}\right)\sum_{n=0}^{\infty}{}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{(n)!}=\sum_{n=0}^{\infty}{}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^{n-1}}{(n-1)!}. \tag{37}$$

Equating coefficients of the same powers of t gives

$$\left(u+\mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}-1}+\mathfrak{s}w\partial_u^{\mathfrak{s}-1}+\frac{A'(\partial_u)}{A(\partial_u)}\right){}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)={}_{eH}A_{n+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) \tag{38}$$

which in view of monomiality principle yields assertion (30).

In order to prove assertion (31), we use generating function of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ (18) in both sides of the identity (32) and then equating coefficients of the same powers of t in both sides of the resultant equation, we find

$$\partial_u\left\{{}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\}=n\ {}_{eH}A_{n-1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{39}$$

which in view of monomiality principle yields assertion (31). □

Remark 3.1. We remark that equations (38) and (39) are the differential recurrence relations satisfied by the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$.

To derive the differential equation for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$, we prove the following result:

Theorem 2. The TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ satisfy the following differential equation:

$$\left(u\partial_u+\mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}}+\mathfrak{s}w\partial_u^{\mathfrak{s}}+\partial_u\frac{A'(\partial_u)}{A(\partial_u)}-n\right){}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0. \tag{40}$$

Proof. Using expressions (30) and (31) and in view of monomiality principle, we get assertion (40). □

Now, we derive some operational representations for TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$. First we will prove the following operational rule:

Theorem 3. The following operational representation connecting TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ and Appell polynomials $A_n(u)$ holds true:

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=\exp\left(w\partial_u^{\mathfrak{s}}+v\partial_v v\partial_u^{\mathfrak{r}}\right)A_n(u). \tag{41}$$

Proof. Using operational representation (9) of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ in the r.h.s of the series definition (19) of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$, we get

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=n!\sum_{k=0}^{n}{}^nC_k\ A_k\ \exp\left(w\partial_u^{\mathfrak{s}}+v\partial_v v\partial_u^{\mathfrak{r}}\right)\ u^{n-k}. \tag{42}$$

which on using the series representation (16) of Appell polynomials $A_n(u)$ on the r.h.s, gives assertion (41). □

Theorem 4. The following operational representation connecting TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ and 2-variable truncated exponential-Appell polynomials (2VTEAP)[2] denoted by ${}_{e^{(\mathfrak{r})}}A_n(u,v)$ holds true:

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=\exp\left(w\partial_u^{s}\right){}_{e^{(\mathfrak{r})}}A_n(u,v). \tag{43}$$

Proof. Using operational representation (10) of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ in the r.h.s of the series definition (19) of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$, we get

$${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=n!\sum_{k=0}^{n}{}^nC_k\ A_k\ \exp\left(w\partial_u^{s}\right)\ e_n^{(\mathfrak{r})}(u,v). \tag{44}$$

As 2VTEP $e_n^{(\mathfrak{r})}(u,v)$ is quasi-monomial, so by using monomiality principle and series representation (16) of Appell polynomials $A_n(u)$ on the r.h.s, gives assertion (43). □

Theorem 5. The following operational representation connecting TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ and Gould-Hopper-Appell polynomials (GHAP)[1] denoted by ${}_{H^{(\mathfrak{s})}}A_n(u,v)$ holds true:

$$ {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \exp\left(v\partial_v v\partial_u^{\mathfrak{r}}\right)\, {}_{H^{(\mathfrak{s})}}A_n(u,w). \tag{45}$$

Proof. Using operational representation (11) of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ in the r.h.s of the series definition (19) of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$, we get

$$ {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = n!\sum_{k=0}^{n} {}^{n}C_k\ A_k\ \exp\left(v\partial_v v\partial_u^{\mathfrak{r}}\right)\ H_n^{(\mathfrak{s})}(u,w). \tag{46}$$

As GHP $H_n^{(\mathfrak{s})}(u,w)$ is quasi-monomial, so by using monomiality principle and series representation (16) of Appell polynomials $A_n(u)$ on the r.h.s, gives assertion (45). □

Recall that 2-iterated Gould-Hopper polynomials (2IGHP) ${}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$[9] is defined by the following generating function:

$$ \exp(ut + vt^{\mathfrak{r}} + wt^{\mathfrak{s}}) = \sum_{n=0}^{\infty} {}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)\frac{t^n}{n!}. \tag{47}$$

It is shown in[9], that 2IGHP ${}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ are quasimonomial under the action of the following multiplicative and derivative operators:

$$ \hat{M}_{{}_{H^{(\mathfrak{r})}}H^{(\mathfrak{s})}} = u + \mathfrak{r}v\partial_u + \mathfrak{s}w\partial_u^{\mathfrak{s}-1} \tag{48}$$

and

$$ \hat{P}_{{}_{H^{(\mathfrak{r})}}H^{(\mathfrak{s})}} = \partial_u, \tag{49}$$

respectively.

From monomiality principle, the exponential generating function of the 2IGHP ${}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ can be cast in the form:

$$ \exp(\hat{M}_{{}_{H^{(\mathfrak{r})}}H^{(\mathfrak{s})}}\ t)\{1\} = \sum_{n=0}^{\infty} {}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)\frac{t^n}{n!}, \tag{50}$$

Replacing u by the multiplicative operator $\hat{M}_{{}_{H^{(\mathfrak{r})}}H^{(\mathfrak{s})}}$ of the 2IGHP ${}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ in the generating function (14) of Appell polynomials $A_n(u)$, we get

$$ A(t)\exp(\hat{M}_{{}_{H^{(\mathfrak{r})}}H^{(\mathfrak{s})}}\ t) = \sum_{n=0}^{\infty} A_n(\hat{M}_{{}_{H^{(\mathfrak{r})}}H^{(\mathfrak{s})}})\frac{t^n}{n!}. \tag{51}$$

Now using equation (50) in l.h.s. and denoting the resultant in the r.h.s. by ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)$, we find

$$ A(t)\sum_{n=0}^{\infty} {}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)\frac{t^n}{n!}, \tag{52}$$

which on using generating function (47) of 2IGHP ${}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$, in the l.h.s. gives the generating function for the new family of polynomials called 2-iterated Gould-Hopper-Appell polynomials (2IGHAP), denoted by ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)$:

$$ A(t)\exp(ut + vt^{\mathfrak{r}} + wt^{\mathfrak{s}}) = \sum_{n=0}^{\infty} {}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)\frac{t^n}{n!}. \tag{53}$$

Now, we will establish an integral representation for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ in terms of 2IGHAP ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)$:

Theorem 6. The following integral representation for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ in terms of 2IGHAP holds true:

$$ {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \int_0^{\infty} e^{-x}\, {}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,vx,w)dx. \tag{54}$$

Proof. First, we recall the following integral representation of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$[4]:

$$ {}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w) = \int_0^{\infty} e^{-x}\, {}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,vx,w)dx. \tag{55}$$

Using generating function (1) of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ in the generating function (18) of TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$, we obtain

$$\sum_{n=0}^{\infty} {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = A(t)\sum_{n=0}^{\infty} {}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)\frac{t^n}{n!}, \tag{56}$$

which on using integral representation (55) of 3VTEGHP ${}_{e^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$ gives

$$\sum_{n=0}^{\infty} {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = A(t)\sum_{n=0}^{\infty}\left(\int_0^{\infty} e^{-x}\, {}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,vx,w)dx\right)\frac{t^n}{n!}. \tag{57}$$

Making use of generating function (47) of 2IGHP ${}_{H^{(\mathfrak{r})}}H_n^{(\mathfrak{s})}(u,v,w)$, we get

$$\sum_{n=0}^{\infty} {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = \int_0^{\infty} A(t)\exp(-x+ut+vxt^{\mathfrak{r}}+wt^{\mathfrak{s}})dx. \tag{58}$$

Finally, using generating function (53) of 2IGHAP ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)$ gives

$$\sum_{n=0}^{\infty} {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = \sum_{n=o}^{\infty}\left(\int_0^{\infty} e^{-x}\, {}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,vx,w)dx\right)\frac{t^n}{n!}, \tag{59}$$

which on equating the coefficients of the same powers of t yields assertion (54). □

Further, corresponding results for the above properties established for members belonging to the truncated exponential Gould Hopper Appell family are derived and mentioned in Table 3 .

Where, ${}_{e^{(\mathfrak{r})}}B_n(u,v)$ is truncated exponential-Bernoulli polynomials[2], ${}_{H^{(\mathfrak{s})}}B_n(u,w)$ is Gould-Hopper-Bernoulli polynomials[1, Table 2 (I)] and ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}B_n(u,v,w)$ is 2-iterated Gould-Hopper-Bernoulli polynomials which can be obtained by reducing Appell polynomials to Bernoulli polynomials by taking $A(t) = \frac{t}{e^t-1}$ in the generating function definition (53) of 2IGHAP ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)$.

Further in Table 3 , ${}_{e^{(\mathfrak{r})}}E_n(u,v)$ is truncated exponential-Euler polynomials[2], ${}_{H^{(\mathfrak{s})}}E_n(u,w)$ is Gould-Hopper-Euler polynomials[1, Table 2 (II)] and ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}E_n(u,v,w)$ is 2-iterated Gould-Hopper-Euler polynomials which can be obtained by reducing Appell polynomials to Euler polynomials by taking $A(t) = \frac{2}{e^t+1}$ in the generating function definition (53) of 2IGHAP ${}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}A_n(u,v,w)$.

4 | RECURRENCE RELATIONS, SHIFT OPERATORS AND DIFFERENTIAL EQUATIONS

In this section, we derive the recurrence relations and shift operators for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$. Then using shift operators we derive the differential, integro-differential and partial differential equations for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$. First we derive the recurrence relation for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ by proving the following result:

Theorem 7. The TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ satisfy the following recurrence relations:

$$\begin{aligned} {}_{eH}A_{n+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = (u+\alpha_0)\,{}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) + \sum_{k=0}^{n-1} {}^nC_k\alpha_{n-k}\,{}_{eH}A_k^{(\mathfrak{r},\mathfrak{s})}(u,v,w) + \frac{n!}{(n-\mathfrak{s}+1)!}\mathfrak{s}w\, {}_{eH}A_{n-\mathfrak{s}+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) \\ + \sum_{k=0}^{n-\mathfrak{r}-1}\frac{n!}{k!(n-\mathfrak{r}-k+1)!}rv\, e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)\, {}_{eH}A_k^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \end{aligned} \tag{60}$$

where the coefficients $\{\alpha_k\}_{k\in\mathbb{N}_0}$ are given by the expansions

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty}\alpha_n\frac{t^n}{n!} \tag{61}$$

and $e_n^{(\mathfrak{r})}(u,v)$ are the truncated exponential polynomials defined by the generating function:

$$\frac{e^{ut}}{1-vt^{\mathfrak{r}}} = \sum_{n=0}^{\infty} e_n^{(\mathfrak{r})}(u,v)\frac{t^n}{n!}. \tag{62}$$

TABLE 3 Results for the TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ and TEGHEP ${}_{eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$.

S.No	Special Polynomials	Results	Expressions
I	TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$	Multiplicative and derivative operators	$\hat{M}_{{}_{eH}B^{(\mathfrak{r},\mathfrak{s})}} = u + rv\partial_v v\partial_u^{\mathfrak{r}-1} + \mathfrak{s}w\partial_u^{\mathfrak{s}-1} + \frac{exp(\partial_u)(1-\partial_u)-1}{\partial_u(\exp(\partial_u)-1)}$ $\hat{P}_{{}_{eH}B^{(\mathfrak{r},\mathfrak{s})}} = \partial_u$
		Differential equation	$\left(u\partial_u + \mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}} + \mathfrak{s}w\partial_u^{\mathfrak{s}} + \left(\frac{exp(\partial_u)(1-\partial_u)-1}{\partial_u(\exp(\partial_u)-1)}\right)\partial_u - n\right)$ ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = 0$
		Operational rules	${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \exp\left(w\partial_u^{\mathfrak{s}} + v\partial_v v\partial_u^{\mathfrak{r}}\right) B_n(u)$ ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \exp\left(w\partial_u^{\mathfrak{s}}\right) {}_{e^{(\mathfrak{r})}}B_n(u,v)$ ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \exp\left(v\partial_v v\partial_u^{\mathfrak{r}}\right) {}_{H^{(\mathfrak{s})}}B_n(u,w)$
		Integral representation	${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \int_0^\infty e^{-x} {}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}B_n(u,vx,w)dx$
II	TEGHEP ${}_{eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$	Multiplicative and derivative operators	$\hat{M}_{{}_{eH}E^{(\mathfrak{r},\mathfrak{s})}} = u + rv\partial_v v\partial_u^{\mathfrak{r}-1} + \mathfrak{s}w\partial_u^{\mathfrak{s}-1} - \frac{\exp(\partial_u)}{\exp(\partial_u)+1}$ $\hat{P}_{{}_{eH}E^{(\mathfrak{r},\mathfrak{s})}} = \partial_u$
		Differential equation	$\left(u\partial_u + \mathfrak{r}v\partial_v v\partial_u^{\mathfrak{r}} + \mathfrak{s}w\partial_u^{\mathfrak{s}} - \left(\frac{\exp(\partial_u)}{\exp(\partial_u)+1}\right)\partial_u - n\right)$ ${}_{eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = 0$
		Operational rules	${}_{eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \exp\left(w\partial_u^{\mathfrak{s}} + v\partial_v v\partial_u^{\mathfrak{r}}\right) E_n(u)$ ${}_{eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \exp\left(w\partial_u^{s}\right) {}_{e^{(\mathfrak{r})}}E_n(u,v)$ ${}_{eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \exp\left(v\partial_v v\partial_u^{\mathfrak{r}}\right) {}_{H^{(\mathfrak{s})}}E_n(u)$
		Integral representation	${}_{eH}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \int_0^\infty e^{-x} {}_{H^{(\mathfrak{r})}H^{(\mathfrak{s})}}E_n(u,vx,w)dx$

Proof. Differentiating both sides of generating function (18) with respect to t, we have

$$\sum_{n=0}^{\infty} {}_{eH}A_{n+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = \left(\frac{A'(t)}{A(t)} + u + w\mathfrak{s}t^{\mathfrak{s}-1} + vrt^{\mathfrak{r}-1}\frac{1}{1-vt^{\mathfrak{r}}}\right)A(t)\frac{\exp(ut+wt^{\mathfrak{s}})}{1-vt^{\mathfrak{r}}}. \tag{63}$$

Using equation (18), (61) and (62) in the right hand side of the above equation we get

$$\sum_{n=0}^{\infty} {}_{eH}A_{n+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{n!} + u + w\mathfrak{s}t^{\mathfrak{s}-1} + vrt^{\mathfrak{r}-1}\sum_{n=0}^{\infty}e_n^{(\mathfrak{r})}(0,v)\frac{t^n}{n!}\right)\sum_{n=0}^{\infty} {}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\frac{t^n}{n!}. \tag{64}$$

Applying Cauchy product rule and comparing the coefficients of similar powers of t gives assertion (60). □

Now, we proceed to explore the shift operators for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ by proving following two results:

Theorem 8. The lowering operators for the TEGHAP ${}_{eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ are given by:

$${}_u\mathcal{L}_n := \frac{1}{n}D_u, \tag{65}$$

$${}_v\mathcal{L}_n := \frac{1}{n}\frac{D_u^{-(\mathfrak{r}-1)}D_v}{\exp(D_u^{\mathfrak{r}}\, vD_v v)}, \tag{66}$$

$$ {}_{w}\mathcal{L}_{n} := \frac{1}{n} D_{u}^{-(\mathfrak{s}-1)} D_{w}, \tag{67}$$

where

$$ D_{u} := \frac{\partial}{\partial u},\ D_{v} := \frac{\partial}{\partial v},\ D_{w} := \frac{\partial}{\partial w}\ and\ D_{u}^{-1} := \int_{0}^{u} f(\xi) d\xi\ . $$

Proof. Differentiating generating function (18) with respect to u and then equating the coefficients of similar powers of t on both sides of the resultant equation gives

$$ D_{u}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = n\ {}_{eH}A_{n-1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{68}$$

Consequently, we have

$$ {}_{u}\mathcal{L}_{n}\ \left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = \frac{1}{n} D_{u}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = {}_{eH}A_{n-1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{69}$$

hence, assertion (65) follows.

Differentiating generating function (18) with respect to v and then equating the coefficients of similar powers of t on both sides of the resultant equation, it follows that

$$ D_{v}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = \frac{n!}{(n-\mathfrak{r})!}\ \exp(D_{u}^{\mathfrak{r}}\ vD_{v}v)\ {}_{eH}A_{n-\mathfrak{r}}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{70}$$

which in view of (68) can be rewritten as

$$ D_{v}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = n\exp(D_{u}^{\mathfrak{r}}\ vD_{v}v)\ D_{u}^{\mathfrak{r}-1}\ {}_{eH}A_{n-\mathfrak{r}-1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{71}$$

which finally gives

$$ {}_{v}\mathcal{L}_{n}\ \left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = \frac{D_{v}}{n\exp(D_{u}^{\mathfrak{r}}\ vD_{v}v)D_{u}^{\mathfrak{r}-1}}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = {}_{eH}A_{n-1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{72}$$

Thus assertion (66) is proved.

Again, differentiating generating function (18) with respect to w and then equating the coefficients of similar powers of t on both sides of the resultant equation yields

$$ D_{w}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = \frac{n!}{(n-\mathfrak{s})!}\ {}_{eH}A_{n-\mathfrak{s}}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{73}$$

which in view of (68) can be rewritten as

$$ D_{w}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = nD_{u}^{\mathfrak{s}-1}\ {}_{eH}A_{n-1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{74}$$

Consequently, we have

$$ {}_{w}\mathcal{L}_{n}\ \left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = \frac{D_{w}}{nD_{u}^{\mathfrak{s}-1}}\left\{{}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\} = {}_{eH}A_{n-1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{75}$$

which proves assertion (67). □

Theorem 9. The raising operators for the TEGHAP ${}_{eH}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ are given by

$$ {}_{u}\mathcal{R}_{n} := u + \alpha_0 + \mathfrak{s}wD_{u}^{\mathfrak{s}-1} + \sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}D_{u}^{n-k} + \mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}D_{u}^{n-k}, \tag{76}$$

$$ {}_{v}\mathcal{R}_{n} := u+\alpha_0+\mathfrak{s}w\frac{D_{u}^{-(\mathfrak{r}-1)(\mathfrak{s}-1)}D_{v}^{\mathfrak{s}-1}}{[\exp(D_{u}^{\mathfrak{r}}\ vD_{v}v)]^{\mathfrak{s}-1}}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}\frac{D_{u}^{-(\mathfrak{r}-1)(n-k)}D_{v}^{n-k}}{[\exp(D_{u}^{\mathfrak{r}}\ vD_{v}v)]^{n-k}}+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}\frac{D_{u}^{-(\mathfrak{r}-1)(n-k)}D_{v}^{n-k}}{[\exp(D_{u}^{\mathfrak{r}}\ vD_{v}v)]^{n-k}}, \tag{77}$$

$$ {}_{w}\mathcal{R}_{n} := u + \alpha_0 + \mathfrak{s}wD_{u}^{-(\mathfrak{s}-1)^2}D_{w}^{\mathfrak{s}-1} + \sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}D_{u}^{-(\mathfrak{s}-1)(n-k)}D_{w}^{n-k} + \mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}D_{u}^{-(\mathfrak{s}-1)(n-k)}D_{w}^{n-k}, \tag{78}$$

where

$$ D_{u} := \frac{\partial}{\partial u},\ D_{v} := \frac{\partial}{\partial v},\ D_{w} := \frac{\partial}{\partial w}\ and\ D_{u}^{-1} := \int_{0}^{u} f(\xi) d\xi\ . $$

Proof. In order to derive the expression for raising operator (76), the following relation is used:

$$ {}_{e}H A_{k}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = {}_{u}\mathcal{L}_{k+1}\, {}_{u}\mathcal{L}_{k+2}\, \cdots\, {}_{u}\mathcal{L}_{n-1}\, {}_{u}\mathcal{L}_{n} \left\{ {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) \right\}, \tag{79} $$

which in view of (69) can be simplified as

$$ {}_{e}H A_{k}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \frac{k!}{n!} D_{u}^{n-k}\, {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{80} $$

Making use of equation (80) in recurrence relation (60), we find

$$ {}_{e}H A_{n+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \left(u + \alpha_0 + \mathfrak{s} w D_{u}^{\mathfrak{s}-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_{u}^{n-k} + \mathfrak{r} v \sum_{k=0}^{n-\mathfrak{r}-1} \frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!} D_{u}^{n-k} \right) {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{81} $$

which yields expression (76) of raising operator ${}_{u}\mathcal{R}_{n}$.

Next, to obtain the raising operator ${}_{v}\mathcal{R}_{n}$, the following relation is used:

$$ {}_{e}H A_{k}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = {}_{v}\mathcal{L}_{k+1}\, {}_{v}\mathcal{L}_{k+2}\, \cdots\, {}_{v}\mathcal{L}_{n-1}\, {}_{v}\mathcal{L}_{n} \left\{ {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) \right\}, \tag{82} $$

which in view of (72) can be simplified as

$$ {}_{e}H A_{k}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \frac{k!}{n!} \frac{D_{u}^{-(\mathfrak{r}-1)(n-k)} D_{v}^{n-k}}{[\exp(D_{u}^{\mathfrak{r}}\, v D_{v} v)]^{n-k}}\, {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{83} $$

Making use of equation (83) in recurrence relation (60), we find

$$ \begin{aligned} {}_{e}H A_{n+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \Bigg(& u + \alpha_0 + \mathfrak{s} w \frac{D_{u}^{-(\mathfrak{r}-1)(\mathfrak{s}-1)} D_{v}^{\mathfrak{s}-1}}{[\exp(D_{u}^{\mathfrak{r}}\, v D_{v} v)]^{\mathfrak{s}-1}} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} \frac{D_{u}^{-(\mathfrak{r}-1)(n-k)} D_{v}^{n-k}}{[\exp(D_{u}^{\mathfrak{r}}\, v D_{v} v)]^{n-k}} \\ & + \mathfrak{r} v \sum_{k=0}^{n-\mathfrak{r}-1} \frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!} \frac{D_{u}^{-(\mathfrak{r}-1)(n-k)} D_{v}^{n-k}}{[\exp(D_{u}^{\mathfrak{r}}\, v D_{v} v)]^{n-k}} \Bigg) {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \end{aligned} \tag{84} $$

which yields expression (77) of raising operator ${}_{v}\mathcal{R}_{n}$.

Further, to obtain the raising operator ${}_{w}\mathcal{R}_{n}$, the following relation is used:

$$ {}_{e}H A_{k}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = {}_{w}\mathcal{L}_{k+1}\, {}_{w}\mathcal{L}_{k+2}\, \cdots\, {}_{w}\mathcal{L}_{n-1}\, {}_{w}\mathcal{L}_{n} \left\{ {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) \right\}, \tag{85} $$

which in view of (75) can be simplified as

$$ {}_{e}H A_{k}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \frac{k!}{n!} D_{u}^{-(\mathfrak{s}-1)(n-k)} D_{w}^{n-k}\, {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{86} $$

Making use of equation (86) in recurrence relation (60), we find

$$ \begin{aligned} {}_{e}H A_{n+1}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = \Bigg(& u + \alpha_0 + \mathfrak{s} w D_{u}^{-(\mathfrak{s}-1)^2} D_{w}^{\mathfrak{s}-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_{u}^{-(\mathfrak{s}-1)(n-k)} D_{w}^{n-k} \\ & + \mathfrak{r} v \sum_{k=0}^{n-\mathfrak{r}-1} \frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!} D_{u}^{-(\mathfrak{s}-1)(n-k)} D_{w}^{n-k} \Bigg) {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \end{aligned} \tag{87} $$

which yields expression (78) of raising operator ${}_{w}\mathcal{R}_{n}$. □

Next, the differential and integro-differential equations for the TEGHAP ${}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ are derived by proving the following results.

Theorem 10. The TEGHAP ${}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ satisfy the following differential equation:

$$ \left((u+\alpha_0) D_{u} + \mathfrak{s} w D_{u}^{\mathfrak{s}} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_{u}^{n-k+1} + \mathfrak{r} v \sum_{k=0}^{n-\mathfrak{r}-1} \frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!} D_{u}^{n-k+1} - (n+1) \right) {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) = 0. \tag{88} $$

Proof. Consider the following factorization relation:

$$ {}_{u}\mathcal{L}_{n+1}\, {}_{u}\mathcal{R}_{n} \left\{ {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w) \right\} = {}_{e}H A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w). \tag{89} $$

which on using expressions (65) and (76) of the shift operators in the above equation, we get assertion (88). □

Theorem 11. The TEGHAP ${}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ satisfy the following integro-differential equations:

$$\Bigg((u+\alpha_0)\frac{D_v}{\exp(D_u^{\mathfrak{r}}\,vD_v v)}+\mathfrak{s}w\frac{D_u^{-(\mathfrak{r}-1)(\mathfrak{s}-1)}D_v^{\mathfrak{s}}}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{\mathfrak{s}}}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}\frac{D_u^{-(\mathfrak{r}-1)(n-k)}D_v^{n-k+1}}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{n-k+1}}$$
$$+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}\frac{D_u^{-(\mathfrak{r}-1)(n-k)}D_v^{n-k+1}}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{n-k+1}}-(n+1)D_u^{\mathfrak{r}-1}\Bigg){}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0, \tag{90}$$

$$\Bigg((u+\alpha_0)D_w+\mathfrak{s}wD_u^{-(\mathfrak{s}-1)^2}D_w^{\mathfrak{s}}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}D_u^{-(\mathfrak{s}-1)(n-k)}D_w^{n-k+1}$$
$$+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}D_u^{-(\mathfrak{s}-1)(n-k)}D_w^{n-k+1}-(n+1)D_u^{\mathfrak{s}-1}\Bigg){}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0, \tag{91}$$

$$\Bigg((u+\alpha_0)D_v+\mathfrak{s}wD_u^{-(\mathfrak{s}-1)^2}D_w^{\mathfrak{s}-1}D_v+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}D_u^{-(\mathfrak{s}-1)(n-k)}D_w^{n-k}D_v$$
$$+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}D_u^{-(\mathfrak{s}-1)(n-k)}D_w^{n-k}D_v-(n+1)\exp(D_u^{\mathfrak{r}}\,vD_v v)D_u^{\mathfrak{r}-1}\Bigg){}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0, \tag{92}$$

$$\Bigg((u+\alpha_0)D_w+\mathfrak{s}w\frac{D_u^{-(\mathfrak{r}-1)(\mathfrak{s}-1)}D_v^{\mathfrak{s}-1}D_w}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{\mathfrak{s}-1}}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}\frac{D_u^{-(\mathfrak{r}-1)(n-k)}D_v^{n-k}D_w}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{n-k}}+$$
$$\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}\frac{D_u^{-(\mathfrak{r}-1)(n-k)}D_v^{n-k}D_w}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{n-k}}-(n+1)D_u^{\mathfrak{s}-1}\Bigg){}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0. \tag{93}$$

Proof. Use of expressions (66) and (77) of the shift operators in the following factorization relation:

$${}_{v}\mathcal{L}_{n+1}\ {}_{v}\mathcal{R}_{n}\left\{{}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\}={}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{94}$$

yields assertion (90).

Use of expressions (67) and (78) of the shift operators in the following factorization relation:

$${}_{w}\mathcal{L}_{n+1}\ {}_{w}\mathcal{R}_{n}\left\{{}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\}={}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{95}$$

yields assertion (91).

Use of expressions (66) and (78) of the shift operators in the following factorization relation:

$${}_{v}\mathcal{L}_{n+1}\ {}_{w}\mathcal{R}_{n}\left\{{}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\}={}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{96}$$

yields assertion (92).

Use of expressions (67) and (77) of the shift operators in the following factorization relation:

$${}_{w}\mathcal{L}_{n+1}\ {}_{v}\mathcal{R}_{n}\left\{{}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)\right\}={}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w), \tag{97}$$

yields assertion (93). □

Remark 4.1. The partial differential equations for the TEGHAP ${}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ is deduced as the following consequence of Theorem 11:

Corollary 1. The TEGHAP ${}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ satisfy the following partial differential equations:

$$\Bigg((u+\alpha_0)\frac{D_u^{n(\mathfrak{r}-1)}D_v}{\exp(D_u^{\mathfrak{r}}\,vD_v v)}+\mathfrak{s}w\frac{D_u^{(n-\mathfrak{s}+1)(\mathfrak{r}-1)}D_v^{\mathfrak{s}}}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{\mathfrak{s}}}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}\frac{D_u^{k(\mathfrak{r}-1)}D_v^{n-k+1}}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{n-k+1}}$$
$$+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}\frac{D_u^{k(\mathfrak{r}-1)}D_v^{n-k+1}}{[\exp(D_u^{\mathfrak{r}}\,vD_v v)]^{n-k+1}}-(n+1)D_u^{(n+1)(\mathfrak{r}-1)}\Bigg){}_{e}{}_{H}A_{n}^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0, \tag{98}$$

$$\Bigg((u+\alpha_0)D_u^{n(\mathfrak{s}-1)}D_w+\mathfrak{s}wD_u^{(n-\mathfrak{s}+1)(\mathfrak{s}-1)}D_w^{\mathfrak{s}}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}D_u^{k(\mathfrak{s}-1)}D_w^{n-k+1}$$

$$+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}D_u^{k(\mathfrak{s}-1)}D_w^{n-k+1}-(n+1)D_u^{(n+1)(\mathfrak{s}-1)}\Bigg)\,{}_{e}{}_{H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0, \tag{99}$$

$$\Bigg((u+\alpha_0)D_u^{n(\mathfrak{s}-1)}D_v+\mathfrak{s}wD_u^{(n-\mathfrak{s}+1)(\mathfrak{s}-1)}D_w^{\mathfrak{s}-1}D_v+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}D_u^{k(\mathfrak{s}-1)}D_w^{n-k}D_v+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}$$
$$\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}D_u^{k(\mathfrak{s}-1)}D_w^{n-k}D_v-(n+1)\exp(D_u^{\mathfrak{r}}\,vD_vv)D_u^{n(\mathfrak{s}-1)+(\mathfrak{r}-1)}\Bigg)\,{}_{e}{}_{H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0, \tag{100}$$

$$\Bigg((u+\alpha_0)D_u^{n(\mathfrak{r}-1)}D_w+\mathfrak{s}w\frac{D_u^{(n-\mathfrak{s}+1)(\mathfrak{r}-1)}D_v^{\mathfrak{s}-1}D_w}{[\exp(D_u^{\mathfrak{r}}\,vD_vv)]^{\mathfrak{s}-1}}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}\frac{D_u^{k(\mathfrak{r}-1)}D_v^{n-k}D_w}{[\exp(D_u^{\mathfrak{r}}\,vD_vv)]^{n-k}}+\mathfrak{r}v\sum_{k=0}^{n-\mathfrak{r}-1}$$
$$\frac{e_{n-k-\mathfrak{r}+1}^{(\mathfrak{r})}(0,v)}{(n-k-\mathfrak{r}+1)!}\frac{D_u^{k(\mathfrak{r}-1)}D_v^{n-k}D_w}{[\exp(D_u^{\mathfrak{r}}\,vD_vv)]^{n-k}}-(n+1)D_u^{n(\mathfrak{r}-1)+(\mathfrak{s}-1)}\Bigg)\,{}_{e}{}_{H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0. \tag{101}$$

Proof. Differentiating integro-differential equation (90) $n(\mathfrak{r}-1)$ times with respect to u, we get the partial differential equation (98). Similarly, by taking the derivatives of the integro-differential equation (91) $n(\mathfrak{s}-1)$ times with respect to u, we get the partial differential equation (99). In the same way the partial differential equation (100) can be obtained by taking the derivatives of the integro-differential equation (92) $n(\mathfrak{s}-1)$ times with respect to u and the partial differential equation (101) can be obtained by taking the derivatives of the integro-differential equation (93) $n(\mathfrak{r}-1)$ times with respect to u. □

Remark 4.2. For $A(t)=\frac{t}{e(t)-1}$, TEGHAP ${}_{e}{}_{H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ reduces to TEGHBP ${}_{e}{}_{H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ (Table 2 (I)). So in view of equation (61), corresponding results for the differential, integro-differential and partial differential equations derived above can be obtained for TEGHBP ${}_{e}{}_{H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ by putting

$$\alpha_n=-\frac{B_{n+1}(1)}{n+1}\quad\text{and}\quad\alpha_0=-\frac{1}{2}. \tag{102}$$

Remark 4.3. For $A(t)=\frac{2}{e(t)+1}$, TEGHAP ${}_{e}{}_{H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ reduces to TEGHEP ${}_{e}{}_{H}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ (Table 2 (II)). So in view of equation (61), corresponding results for the differential, integro-differential and partial differential equations derived above can be obtained for TEGHEP ${}_{e}{}_{H}E_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ by putting

$$\alpha_n=-\frac{E_n(1)}{2}\quad\text{and}\quad\alpha_0=-\frac{1}{2}. \tag{103}$$

5 | CONCLUDING REMARKS

Over the years, there has been rise in interest in solving physical and mathematical problems with the help of computers. By using computers, we can understand concepts much more easily and in less time than in the past. The ability to manipulate and create figures on the screen of computer enables us to produce and visualize several problems, demonstrate the properties of figures and examine the patterns. This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the TEGHAP ${}_{e}{}_{H}A_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$. The TEGHBP ${}_{e}{}_{H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ can be determined explicitly. A few of them for $\mathfrak{r}=\mathfrak{s}=1$ are:

$$\begin{aligned}
{}_{e}{}_{H}B_0^{(\mathfrak{r},\mathfrak{s})}(u,v,w)&=1,\\
{}_{e}{}_{H}B_1^{(\mathfrak{r},\mathfrak{s})}(u,v,w)&=-(1/2)+u+v+w,\\
{}_{e}{}_{H}B_2^{(\mathfrak{r},\mathfrak{s})}(u,v,w)&=2(1/2(1/6-u+u^2)+(-(1/2)+u)v+v^2+(-(1/2)+u)w+vw+w^2),\\
{}_{e}{}_{H}B_3^{(\mathfrak{r},\mathfrak{s})}(u,v,w)&=6(1/6(u/2-(3u^2)/2+u^3)+1/2(1/6-u+u^2)v+(-(1/2)+u)v^2+v^3\\
&\quad+1/2(1/6-u+u^2)w+(-(1/2)+u)vw+v^2w+(-(1/2)+u)w^2+vw^2+w^3).
\end{aligned}$$

We display the shapes of the TEGHBP ${}_{e}{}_{H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ and investigate its zeros. We plot the graph of TEGHBP ${}_{e}{}_{H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ for $n=1,2,3,\cdots,10$ combined together. The shape of TEGHBP ${}_{e}{}_{H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ for $\mathfrak{r}=1,\ \mathfrak{s}=2,\ v=1,\ w=-1$ and $-6\le u\le 6$ are displayed in Figure 1 .

The surface plot of TEGHBP ${}_{e}{}_{H}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ for $\mathfrak{r}=16,\ \mathfrak{s}=24$ and $n=20$ are displayed in Figure 2 .

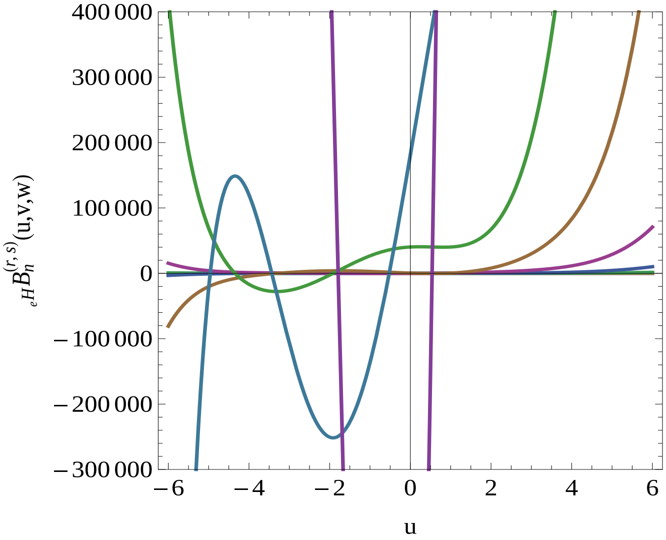

FIGURE 1 Curve of TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$.

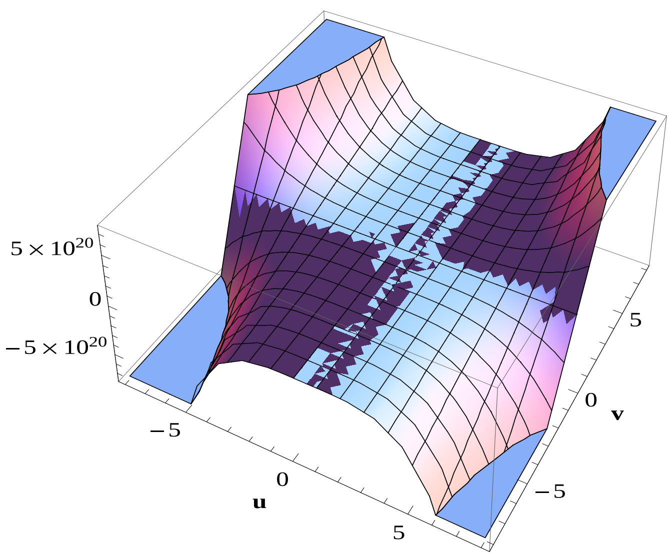

FIGURE 2 Surface plot of TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$.

Our numerical results for number of real and complex zeros of the TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)$ for $\mathfrak{r}=1,\ \mathfrak{s}=2,\ v=1$ and $w=-1$ are listed in Table 4 .

Next, we have calculated an approximate solution satisfying the TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0$ for $\mathfrak{r}=1,\ \mathfrak{s}=2,\ v=1$ and $w=-1$. The results are given in Table 5 .

Further, we investigate the beautiful zeros of the TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0$ by using computer. The zeros of the TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0$ for $\mathfrak{r}=1,\ \mathfrak{s}=2,\ v=1,\ w=-1$ and $u\in\mathbb{C}$ are plotted in Figure 3 .

In Figure 3 , we choose $n=10$ (top-left), $n=20$ (top-right), $n=30$ (bottom-left) and $n=40$ (bottom-right).

Using computers it has been checked for several values of n that for $b,\ d\in\mathbb{R}$ and $u\in\mathbb{C}$, ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,b,d)$ has $Im(u)=0$ reflection symmetry. However, ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,b,d)$ has not $Re(u)=a$ reflection symmetry (see Figure 4). But, it still remains unknown whether this is true or not for all values n.

Next, we plot the real zeros of the TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u,v,w)=0$ for $\mathfrak{r}=1,\ \mathfrak{s}=2,\ v=1,\ w=-1,\ u\in\mathbb{R}$ and $1\le n\le 20$ in Figure 5 .

TABLE 4 Numbers of real and complex zeros of ${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w) = 0$.

Degree n	Number of Real Zeros	Number of Complex Zeros
1	1	0
2	2	0
3	3	0
4	2	2
5	1	4
6	2	4
7	3	4

TABLE 5 Approximate solutions of ${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w) = 0$

Degree n	Real Roots	Complex Roots
1	-0.5	–
2	$-1.5408,\ 0.5408$	–
3	$-2.5551,\ 0.8149,\ 0.24014$	–
4	$-3.0015,\ -1.9976$	$1.4996 - 1.3223\,i,\quad 1.4996 + 1.3223\,i$
5	-0.5218	$-3.3166 - 0.8748\,i,\quad -3.31662 + 0.8748\,i,$ $2.3275 - 2.0808\,i,\quad 2.3275 + 2.0808\,i$
6	$-1.7873,\ 0.5408$	$-3.9641 - 1.4325\,i,\quad -3.9641 + 1.4325\,i,$ $3.0873 - 2.8173\,i,\quad 3.0873 + 2.8173\,i$
7	$-3.3141,\ 0.2401,\ 0.8149$	$-4.4455 - 2.0174\,i,\quad -4.4455 + 2.0174\,i,$ $3.8250 - 3.5428\,i,\quad 3.8250 + 3.5428\,i$

Stacks of zeros of TEGHBP ${}_{_eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w) = 0$ for $\mathfrak{r} = 1,\ \mathfrak{s} = 2,\ v = 1,\ w = -1$ and $1 \leq n \leq 20$ form a 3-D structure and are presented in Figure 6 .

We expect that the research in this direction will be a new approach using numerical computations for the study of the TEGHAP ${}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w)$. The figures presented here gives an unrestricted ability to carry out visual mathematical examinations of the behaviour of TEGHAP ${}_{_eH}A_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w)$. The methodology presented in this research work is general and opens new prospect to deal with other convoluted class of special polynomials. The results established in this research work may find several applications in solving the existing as well as new emerging problems of certain branches of mathematics, physics and engineering.

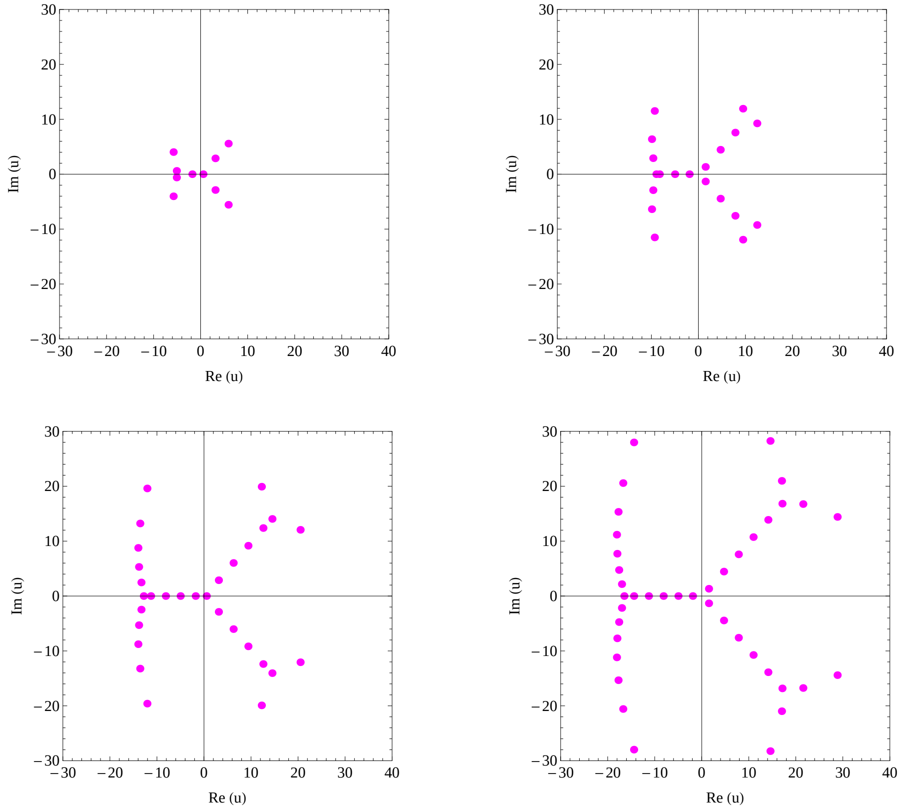

FIGURE 3 Zeros of TEGHBP ${}_{e}H B_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w) = 0$.

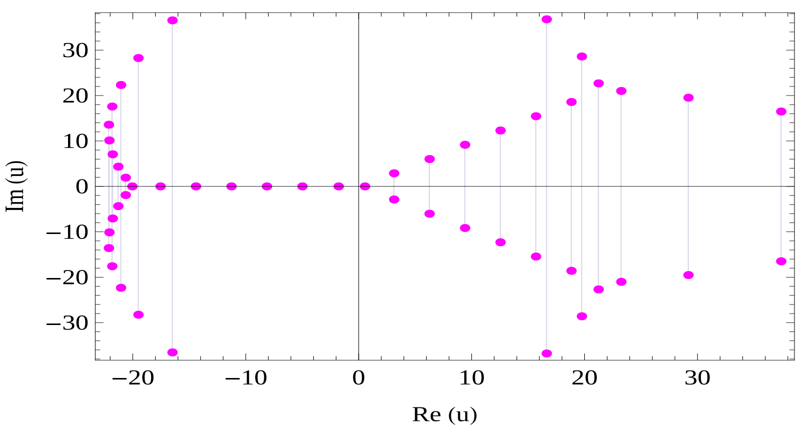

FIGURE 4 ${}_{e}H B_{50}^{(\mathfrak{r},\mathfrak{s})}(u, b, d)$ has $Im(u) = 0$ reflection symmetry.

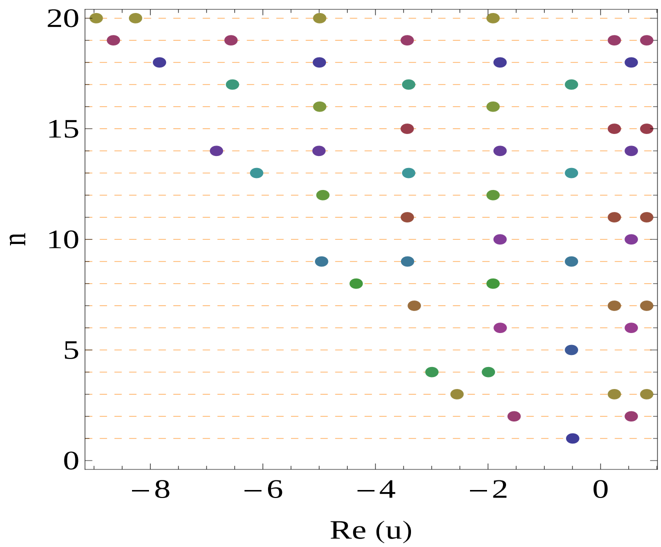

FIGURE 5 Real zeros of ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w) = 0$.

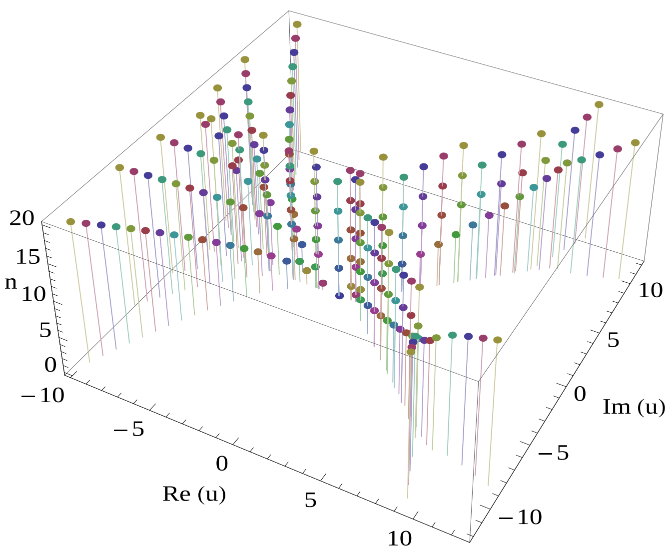

FIGURE 6 Stacks of zeros of TEGHBP ${}_{eH}B_n^{(\mathfrak{r},\mathfrak{s})}(u, v, w) = 0$.

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