ELASTIC MAGNETIC CURVES OF FERROMAGNETIC AND SUPERPARAMAGNETIC MODELS

[1]**Talat Körpınar,** [2,*]**Rıdvan Cem Demirkol,** [3]**Zeliha Körpınar**
[1,2]Department of Mathematics, Muş Alparslan University, Muş, Turkey
[3]Department of Administration, Muş Alparslan University, Muş, Turkey

Abstract. We are interested in defining new energy functionals and solving them by using the variational approach method. That is, we aim to define a new class of elastic curves in the three-dimensional ordinary space. We further improve an alternative method to find critical points of the bending energy functionals acting on a class of magnetic curves. Then, we classify these critical curves as elastic magnetic curves of the Frenet-Serret vector family. Moreover, we investigate the dynamics of the elastic magnetic curves of ferromagnetic and superparamagnetic models and discuss their numerical and analytical analysis.

Keywords: Bending energy functionals; elastic curves; magnetic curves; ferromagnetism; superparamagnetism.
Mathematics Subject Classifications: 53A04, 35B38

1 Introduction

The research of elastic curves connects two traditional subjects; the mechanics of solids and the theory of curves. They were studied by Born, Kirchhoff, Euler, Bernoulli family, Galileo, and many others. Their effort played a crucial role in the improvement of the calculus of variations and elliptic functions.

In 1691, J. Bernoulli attempted to formulate the bending deformation of an ideal elastic rod with uniform density and circular cross-sections. Then, D. Bernoulli showed that the thin elastic rod could be bent along the centerline of the rod unless it has no twist so that the centerline is represented by a simple curve. As a result, they suggested defining elastic curves as the critical points of the bending energy functionals. Later, L. Euler [1] applied the least action principle to the variational problem suggested by Bernoulli's and managed to classify the untwisted planar elastica explicitly. Kirchhoff has made important contributions to the problem by modeling a thin elastic rod subjecting to both twisting and bending. In 1910, Radon [2] investigated the elastica in the three-dimensional space for the first time. In this setting, more recently, Langer and Singer [3] gave a new variational approach of symmetric and uniform Kirchhoff elastic rod in equilibrium. They proved the existence of closed free elastic curves in a compact Riemannian manifold and also present the classification of these curves in a two-dimensional space structure [4] . Singer [5] compiled a series of lectures on the variational problem of the Kirchhoff elastic rod and the Euler elastica in the three-dimensional ordinary space and Riemannian manifolds. In his work, he defined both the Hamiltonian and the Lagrangian characterizations of the rod to test its integrability.

The Bernoulli-Euler-Kirchhoff model of elastica has also emerged as a favorable tool in the research of geodesics and DNA molecule besides other higher dimensional variational problems in the study of submanifolds, fluids, relativity, plasmas, etc $[6-11]$. Moreover, it contributes to investigate a deeper understanding of the physical and geometric model of the Hall effect [12]. It also gives a useful connection between the integrable evolution equations and associated geometric evolution problems including localized induction equation, Betchov-Da Rios equation, binormal motion, Hasimoto evolution, etc. $[13-18]$.

Even though the subject of elastica has been overwhelmingly studied more than two centuries, there still exist some ambiguities for the entire generalization of the concept. For example, equilibrium problems for the large deformation and planar deformation remain not completely figured out. In some cases, the geometric features carried by the configuration of the thin elastic rod are not fully adaptable to the variational problem due to the intuitive or false parametrization.

Based on the recent studies and experiments, in this manuscript, we define a new kind of bending energy functionals and attempt to compute their critical points. To be more specific, we focus on obtaining extremals of the total squared curvature, the sum of the total squared curvature and torsion, the total squared torsion among curves and magnetic curves having the same boundary conditions and length.

The manuscript is organized as follows. In Section 2, we define $\mathbf{n}-$elastic curves and $\mathbf{b}-$elastic curves as the critical points of the normal and binormal bending energy functionals satisfying corresponding Euler-Lagrange equations with suitable boundary conditions in the three-dimensional ordinary space. In Section 3, we develop an alternative method to obtain critical points of the bending energy functionals acting on a class of magnetic curves by revealing the surprising connection between the variational formula of the energy functionals and Lorentz force operator of magnetic curves. As a result, we describe elastic magnetic curves of the Frenet-Serret vector family, which are called as the $\mathbf{t}-$elastic magnetic curves, $\mathbf{n}-$elastic magnetic curves, $\mathbf{b}-$elastic magnetic curves, respectively. We also obtain a new class of Killing vector fields and some fundamental results for both cases. In Section 4, we focus on the dynamics of the elastic magnetic curves of ferromagnetic and superparamagnetic models. In particular, we compute the total energy of the elastic magnetic curves, which are exposed to internal magnetic torque and moment due to the action of the magnetic field. In Section 5, we discuss the numerical and analytical analysis of the elastic magnetic curves and related geometric invariants, which are obtained by solving the associated Euler-Lagrange differential equations. In Section 6, we finalize the manuscript by noting that the model of the elastic magnetic curves has a number of rather intriguing features which may be examined later.

2 Developing a variational approach of a new kind of elasticae

An elastica or elastic curve is the critical point of the bending energy functional

$$\Phi^{\delta}(\gamma) = \int_{\gamma} (\kappa^2 + \delta) ds, \tag{1}$$

where the curvature of the curve γ is represented by κ and the constraint of the length of the curve is given by δ. If $\delta = 0$ critical points of the functional are called elasticae or free elastic curves and Eq. (1) is categorized into a free model. On the other hand, if $\delta \neq 0$ these points are called $\delta-$elasticae and Eq. (1) is categorized into a constrained model. A generalization of this energy functional and associated variational approach has been studied by considering the family

$$\Phi^{\delta}(\gamma) = \int_{\gamma} \mathcal{U}(\kappa)\, ds \tag{2}$$

acting on the curve γ. Here, $\mathcal{U}(\kappa)$ denotes an arbitrary smooth function along γ. For appropriate selections of $\mathcal{U}(\kappa)$ critical points of Eq. (2) contain closed elasticae enclosing a fixed area, elasticae circular at rest, classical elasticae, geodesics, etc. A family of bending energy functionals has astonishing applications in Mathematics to the research of manifolds, where they provide efficient ways to generate new algorithms and submanifolds; in Biophysics to the theory of vesicles and membranes; in Physics to the study of modeling of $p-$branes and relativistic particles [11].

In this section, we are interested in defining new energy functionals and solving them by using the variational approach method. Thus, we aim to build a new kind of elastic curves lying in the three-dimensional ordinary space characterized by the usual metric, which will be denoted by $(\mathbb{E}^3, \ \cdot)$.

In the following, we eventually define energy functionals acting on a certain space of smooth curves with fixed length and boundary conditions in $(\mathbb{E}^3, \ \cdot)$. Prior to these definitions, we firstly describe the space of regular unit speed curves of $(\mathbb{E}^3, \ \cdot)$ in the following manner

$$\phi^{e_o e_1} = \{\vartheta : [0,1] \to \mathbb{E}^3, \ \vartheta(q) = e^q, \ q \in \{0,1\}, \ \ \|\vartheta_s(s)\| = 1, \ \forall s \in [0,1]\}.$$

Then, we can naturally construct the set of unit speed orthonormal Frenet-Serret vectors as follows

$$\phi^{e_o e_1}(u) = \{\mathbf{t}, \mathbf{n}, \mathbf{b} : \vartheta \in \phi^{e_o e_1} \text{ such that } \vartheta_s = \mathbf{t}, \ \frac{1}{\kappa}\vartheta_s = \mathbf{n}, \ \mathbf{t} \times \mathbf{n} = \mathbf{b}\},$$

where $e^q \in \mathbb{E}^3, \ q \in \{0,1\}$, are arbitrary points of $\mathbb{E}^3$ and $\kappa \neq 0 \ \forall s \in [0,1]$. The subscript s denotes differentiation with respect to arc-length. The following equations are known as Frenet-Serret equations and they satisfy

$$\mathbf{t}_s = \kappa \mathbf{n}, \ \mathbf{n}_s = -\kappa \mathbf{t} + \tau \mathbf{b}, \ \mathbf{b}_s = -\tau \mathbf{n}, \tag{3}$$

where κ is the curvature and τ is the torsion function of the curve γ. If we consider a variation of the curve γ, that is $\gamma^{\epsilon}(s) = \gamma(\epsilon, s) : (-\omega, \omega) \times \mathbb{I} \to \mathbb{E}^3$ with $\gamma(0, s) = \gamma(s)$, then associated variational vector field along the curve γ is described by $\mathcal{B}(s) = \mathcal{B}(\epsilon, s) = (\partial\gamma/\partial\epsilon)(0, s)$. We will also take $\mathcal{B} = \mathcal{B}(\epsilon, s)$, $\mathbf{t} = \mathbf{t}(\epsilon, s)$, $\kappa = \kappa(\epsilon, s)$, etc., with trivial meaning.

Case 1 : In the first case, we consider the following bending energy functional described on a family of regular unit speed curves $\phi^{e_o e_1}$ in $\mathbb{E}^3$. Let $\Phi^{\delta} : \phi^{e_o e_1}(u) \to \mathbb{R}$ is given by

$$\Phi^{\delta}(\mathbf{t}) = \frac{1}{2}\int_0^s \|\mathbf{t}_s\|^2 ds = \frac{1}{2}\int_0^s \kappa^2 ds, \tag{4}$$

where $\gamma_s = \mathbf{t}$ and $\gamma \in \phi^{e_o e_1}$. From the Lagrange principle, one can consider the following identity corresponding to Eq. (4) for some pointwise multiplier or Lagrange multiplier Θ

$$\Phi^{\delta}(\mathbf{t}) = \frac{1}{2}\int_0^s (\|\mathbf{t}_s\|^2 + \Theta(\|\mathbf{t}\|^2 - 1))ds. \tag{5}$$

Thus, under the above notation and conditions, we can define the variational vector field $\mathcal{B}$ along the curve γ as follows

$$\partial\Phi^{\delta}\left(\mathcal{B}\right)=\frac{\partial}{\partial\epsilon}\Phi^{\delta}\left(\mathbf{t}+\epsilon\mathcal{B}_{s}\right)|_{\epsilon=0}=0. \tag{6}$$

So, the first variational formula is computed by

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\partial}{\partial\epsilon}\int_{0}^{s}(\|\mathbf{t}_{s}+\epsilon\mathcal{B}_{ss}\|^{2}+\Theta(\|\mathbf{t}+\epsilon\mathcal{B}_{s}\|^{2}-1))ds\,|_{\epsilon=0}\ , \\ 0 &= \int_{0}^{s}(\mathbf{t}_{s}\cdot\mathcal{B}_{ss}+\Theta(\mathbf{t}\cdot\mathcal{B}_{s}))ds. \end{aligned} \tag{7}$$

Then, if we use the integration by parts formula, we have

$$0=[\mathbf{t}_{s}\cdot\mathcal{B}_{s}+(\Theta\mathbf{t}-\mathbf{t}_{ss})\mathcal{B}]_{0}^{s}+\int_{0}^{s}\mathbf{E}\cdot\mathcal{B}ds,$$

where $\mathbf{E}=\mathbf{t}_{sss}-(\Theta\mathbf{t})_{s}$ is called the Euler-Lagrange $(\mathcal{EL})$ equation. Thus, a critical curve of Φ^{δ} is characterized by the $\mathcal{EL}$-equation provided $\mathbf{E}=0$. Solving the $(\mathcal{EL})-$equation is not a trivial task even in the ordinary space. In order to assert that the idea of using Noether's theorem in combination with Killing vector fields along the curve γ has been developed to facilitate the computation of the integral. In particular, we have the following constant vector along the curve γ

$$\mathbf{J}=\mathbf{t}_{ss}-\Theta\mathbf{t}. \tag{8}$$

In order to compute the critical curve of $\Phi^{\delta}\left(\mathbf{t}\right)$ associated with $(\mathcal{EL})-$equation, we differentiate constant vector field $\mathbf{J}$ to get $\mathbf{J}_{s}=0$. In other words, γ is a critical curve iff

$$\kappa_{ss}+\frac{\kappa^{3}}{2}-\kappa\tau^{2}-\frac{\delta\kappa}{2}=0,\ \ \kappa\tau_{s}+2\kappa_{s}\tau=0, \tag{9}$$

where $\Theta=-\frac{3}{2}\kappa^{2}+\frac{\delta}{2}$, for some constant δ [5].

Definition 2.1. A regular unit speed curve $\gamma\in\phi^{e_{o}e_{1}}$ in $\mathbb{E}^{3}$ is called an elastica or $\mathbf{t}-$elastic curve if the $(\mathcal{EL})-$equation (9) is satisfied [5].

Theorem 2.2. If a regular unit speed curve $\gamma\in\phi^{e_{o}e_{1}}$ is an elastica in $\mathbb{E}^{3}$ then the constant vector fields $\mathbf{J}$ and $\mathbf{I}$ defined in Eq. (10) are Killing fields along the curve γ

$$\mathbf{J}=\frac{\kappa^{2}-\delta}{2}\mathbf{t}+\kappa_{s}\mathbf{n}+\kappa\tau\mathbf{b},\ \ \mathbf{I}=\kappa\mathbf{b}=\mathcal{A}+\gamma\times\mathbf{J}, \tag{10}$$

where $\mathcal{A}$ is an arbitrary constant vector field [5].

Corollary 2.3. The associated Killing vector fields $\mathbf{J}$ and $\mathbf{I}$ defined along γ are commutative.

Proof. It is obvious from the fact that $[\mathbf{J}, \mathbf{I}] = 0$.

Case 2 : In the second case, we define the following normal bending energy functional described on a family of regular unit speed curves $\phi^{e_o e_1}$ in $\mathbb{E}^3$. Let $\Phi^{\delta n} : \phi^{e_o e_1}(u) \to \mathbb{R}$ is given by

$$\Phi^{\delta n}(\mathbf{n}) = \frac{1}{2}\int_0^s \|\mathbf{n}_s\|^2 \, ds = \frac{1}{2}\int_0^s (\kappa^2 + \tau^2) ds, \tag{11}$$

$\gamma_s = \mathbf{t}$, $\frac{1}{\kappa}\gamma_s = \mathbf{n}$ and $\gamma \in \phi^{e_o e_1}$. From the Lagrange principle, one can consider the following identity corresponding to Eq. (11) for some normal Lagrange multiplier Θ^n

$$\Phi^{\delta n}(\mathbf{n}) = \frac{1}{2}\int_0^s (\|\mathbf{n}_s\|^2 + \Theta^n(\|\mathbf{n}\|^2 - 1)) ds. \tag{12}$$

Thus, under the above notation and conditions, we can define the variational vector field $\mathcal{B}^n$ along the curve γ as follows

$$\partial\Phi^{\delta n}(\mathcal{B}^n) = \frac{\partial}{\partial \epsilon}\Phi^{\delta n}(\mathbf{n} + \epsilon\mathcal{B}^n_s)\,|_{\epsilon=0} = 0. \tag{13}$$

So, the first variational formula is computed by

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\partial}{\partial\epsilon}\int_0^s (\|\mathbf{n}_s + \epsilon\mathcal{B}^n_{ss}\|^2 + \Theta^n(\|\mathbf{n} + \epsilon\mathcal{B}^n_s\|^2 - 1)) ds\,|_{\epsilon=0}\,, \\ 0 &= \int_0^s (\mathbf{n}_s \cdot \mathcal{B}^n_{ss} + \Theta^n(\mathbf{n} \cdot \mathcal{B}^n_s)) ds. \end{aligned} \tag{14}$$

Then, if we use the integration by parts formula, we have

$$0 = [\mathbf{n}_s \cdot \mathcal{B}^n_s + (\Theta^n\mathbf{n} - \mathbf{n}_{ss})\mathcal{B}^n]_0^s + \int_0^s \mathbf{E}^n \cdot \mathcal{B}^n ds,$$

where $\mathbf{E}^n = \mathbf{n}_{sss} - (\Theta^n\mathbf{n})_s$ is called the normal Euler-Lagrange ($\mathcal{EL}^n$) equation. Thus, a critical curve of $\Phi^{\delta n}$ is characterized by the $\mathcal{EL}^n$-equation provided $\mathbf{E}^n=0$. Solving the ($\mathcal{EL}^n$) $-$equation is not a trivial task even in the ordinary space. In order to assert that the idea of using Noether's theorem in combination with Killing vector fields along the curve γ has been developed to facilitate the computation of the integral. In particular, we have the following constant vector along the curve γ

$$\mathbf{J}^n = \mathbf{n}_{ss} - \Theta^n \mathbf{n}. \tag{15}$$

In order to compute the critical curve of $\Phi^{\delta n}(\mathbf{n})$ associated with $(\mathcal{EL}^n)$ −equation, we differentiate constant vector field $\mathbf{J}^n$ to get $\mathbf{J}^n_s = 0$. In other words, γ is a critical curve iff

$$\kappa_{ss} + \tau_{ss} = (\kappa + \tau)\left(\kappa^2 + \tau^2 + \Theta^n\right), \ \tau\kappa_{ss} - \kappa\tau_{ss} = 0, \tag{16}$$

where $\Theta^n = -\frac{3}{2}\kappa^2 - \frac{3}{2}\tau^2 + \frac{\delta^n}{2}$, for some constant δ^n.

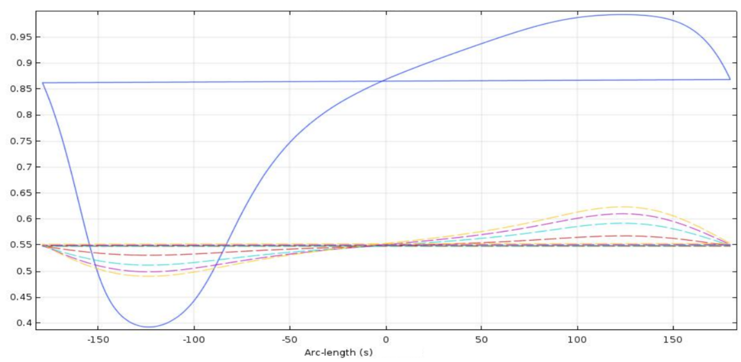

Figure 1. The prototypical evolution of the $\mathbf{n}$−elastic curve.

Figure 1 shows the variation of the evolution of the $\mathbf{n}$−elastic curve with certain values of κ, τ, and δ^n.

Definition 2.4. A regular unit speed curve $\gamma \in \phi^{e_o e_1}$ in $\mathbb{E}^3$ is called a normal elastica or $\mathbf{n}$−elastic curve if the $(\mathcal{EL}^n)$ −equation (16) is satisfied.

Theorem 2.5. If a regular unit speed curve $\gamma \in \phi^{e_o e_1}$ is a normal elastica in $\mathbb{E}^3$ then the constant vector fields $\mathbf{J}^n$ and $\mathbf{I}^n$ defined in Eq. (17) are Killing fields along the curve γ

$$\mathbf{J}^n = -\kappa_s \mathbf{t} + \frac{\kappa^2 + \tau^2 - \delta^n}{2}\mathbf{n} + \tau_s \mathbf{b}, \ \mathbf{I}^n = \tau\mathbf{n} = \mathcal{A}^n - \gamma \times \mathbf{J}^n, \tag{17}$$

where $\mathcal{A}^n$ is an arbitrary constant vector field.

Proof. The proof is obvious if one considers the fact that the variational vector is a symmetry and restriction of a rotation field, in which details can be found in Noether's theorem.

Corollary 2.6. The associated Killing vector fields $\mathbf{J}^n$ and $\mathbf{I}^n$ defined along γ are commutative.

Proof. It is obvious from the fact that $[\mathbf{J}^n, \mathbf{I}^n] = 0$.

Case 3 : In the third case, we define the following binormal bending energy functional described on a family of regular unit speed curves $\phi^{e_o e_1}$ in $\mathbb{E}^3$. Let $\Phi^{\delta b} : \phi^{e_o e_1}(u) \to \mathbb{R}$ is given by

$$\Phi^{\delta b}(\mathbf{b}) = \frac{1}{2}\int_0^s \|\mathbf{b}_s\|^2 ds = \frac{1}{2}\int_0^s \tau^2 ds, \tag{18}$$

$\gamma_s = \mathbf{t}$, $\frac{1}{\kappa}\gamma_s = \mathbf{n}$, $\mathbf{t}\times\mathbf{n} = \mathbf{b}$, and $\gamma \in \phi^{e_o e_1}$. From the Lagrange principle, one can consider the following identity corresponding to Eq. (18) for some binormal Lagrange multiplier Θ^b

$$\Phi^{\delta b}(\mathbf{b}) = \frac{1}{2}\int_0^s (\|\mathbf{b}_s\|^2 + \Theta^b(\|\mathbf{b}\|^2 - 1))ds. \tag{19}$$

Thus, under the above notation and conditions, we can define the variational vector field $\mathcal{B}^b$ along the curve γ as follows

$$\partial\Phi^{\delta b}(\mathcal{B}^b) = \frac{\partial}{\partial\epsilon}\Phi^{\delta b}(\mathbf{b} + \epsilon\mathcal{B}^b_s)\mid_{\epsilon=0} = 0. \tag{20}$$

So, the first variational formula is computed by

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\partial}{\partial\epsilon}\int_0^s (\left\|\mathbf{b}_s + \epsilon\mathcal{B}^b_{ss}\right\|^2 + \Theta^b(\left\|\mathbf{b}+\epsilon\mathcal{B}^b_s\right\|^2 - 1))ds\mid_{\epsilon=0}, \\ 0 &= \int_0^s (\mathbf{b}_s\cdot\mathcal{B}^b_{ss} + \Theta^b(\mathbf{b}\cdot\mathcal{B}^b_s))ds. \end{aligned} \tag{21}$$

Then, if we use the integration by parts formula, we have

$$0 = [\mathbf{b}_s\cdot\mathcal{B}^b_s + (\Theta^b\mathbf{b} - \mathbf{b}_{ss})\mathcal{B}^b]_0^s + \int_0^s \mathbf{E}^b\cdot\mathcal{B}^b ds,$$

where $\mathbf{E}^b = \mathbf{b}_{sss} - (\Theta^b\mathbf{b})_s$ is called the binormal Euler-Lagrange $\left(\mathcal{EL}^b\right)$ equation. Thus, a critical curve of $\Phi^{\delta b}$ is characterized by the $\mathcal{EL}^b$-equation provided $\mathbf{E}^b$=0. Solving the $\left(\mathcal{EL}^b\right)$ −equation is not a trivial task even in the ordinary space. In order to assert that the idea of using Noether's theorem in combination with Killing vector fields along the curve γ has been developed to facilitate the computation of the integral. In particular, we have the following constant vector along the curve γ

$$\mathbf{J}^b = \mathbf{b}_{ss} - \Theta^b \mathbf{b}. \tag{22}$$

In order to compute the critical curve of $\Phi^{\delta b}(\mathbf{b})$ associated with $\left(\mathcal{EL}^b\right)-$equation, we differentiate constant vector field $\mathbf{J}^b$ to get $\mathbf{J}_s^b = 0$. In other words, γ is a critical curve iff

$$\tau_{ss} - \tau(\kappa^2 + \tau^2 + \Theta^b) = 0,\ 2\kappa\tau_s + \kappa_s\tau = 0, \tag{23}$$

where $\Theta^b = -\frac{3}{2}\tau^2 + \frac{\delta^b}{2}$, for some constant δ^b.

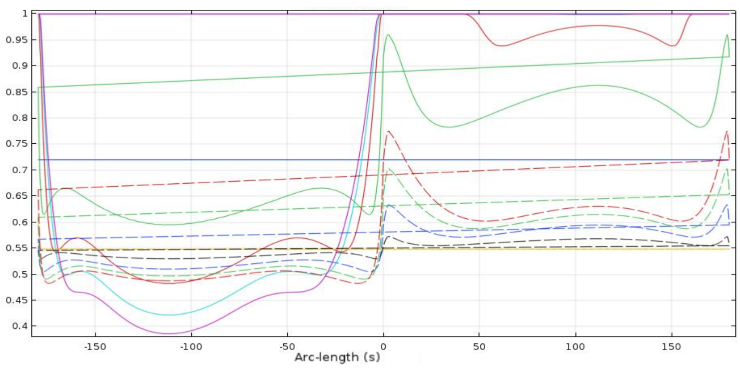

Figure 2. The prototypical evolution of the $\mathbf{b}-$elastic curve.

Figure 2 shows the variation of the evolution of the $\mathbf{b}-$elastic curve with certain values of κ, τ, and δ^b.

Definition 2.7. A regular unit speed curve $\gamma \in \phi^{e_o e_1}$ in $\mathbb{E}^3$ is called a binormal elastica or $\mathbf{b}-$elastic curve if the $\left(\mathcal{EL}^b\right)-$equation (23) is satisfied.

Theorem 2.8. If a regular unit speed curve $\gamma \in \phi^{e_o e_1}$ is a binormal elastica in $\mathbb{E}^3$ then the constant vector fields $\mathbf{J}^b$ and $\mathbf{I}^b$ defined in Eq. (24) are Killing fields along the curve γ

$$\mathbf{J}^b = \kappa\tau\mathbf{t} + \tau_s\mathbf{n} + \frac{\tau^2 - \delta}{2}\mathbf{b},\ \mathbf{I}^b = \tau\mathbf{b} = \mathcal{A}^b - \gamma \times \mathbf{J}^b, \tag{24}$$

where $\mathcal{A}^b$ is an arbitrary constant vector field.

Proof. The proof is obvious if one considers the fact that variational vector is a symmetry and restriction of a rotation field, in which details can be found in Noether's theorem.

Corollary 2.9. The associated Killing vector fields $\mathbf{J}^b$ and $\mathbf{I}^b$ defined along γ are commutative.

Proof. It is obvious from the fact that $\left[\mathbf{J}^b, \mathbf{I}^b\right] = 0$.

3 Characterizing a geometry of elasticae of magnetic curves

In this section, we create a geometric model of the elastic magnetic curves and classify corresponding Euler-Lagrange equations satisfied by these curves. The model is based on solving the critical curves belonging to special bending energy functional paying close attention to magnetic curves and associated magnetic fields in $(\mathbb{E}^3,\ \cdot)$. To sum up, we not only construct a theoretical model to describe a new example of elasticae, but we also develop a mathematical tool for investigating physical parameters on the dynamics of a magnetically driven particle having an elastic feature.

In physics, the characterization of magnetic curves is given by the motion of a point charged particle of the associated magnetic field $\mathcal{F}^*$ in $(\mathbb{E}^3,\ \cdot)$. That is, a regular unit speed curve γ in the three-dimensional ordinary space with the usual metric is called a magnetic curve of $(\mathbb{E}^3,\ \cdot,\ \mathcal{F}^*)$ if the tangent vector of γ satisfies the following second-order non-linear Lorentz equation

$$\nabla_{\gamma_s}\gamma_s = \sigma(\gamma_s), \tag{25}$$

where ∇ is the Levi-Civita connection of the given metric and σ is the skew-symmetric Lorentz force operator. On the other hand, in geometry, magnetic curves are considered as local critical points of the following functional

$$\mathcal{L}_\gamma = \int_\gamma (\gamma_s \cdot \gamma_s + \epsilon\gamma_s)ds, \tag{26}$$

where $\gamma \in \phi^{e_o e_1}$, $\mathcal{F}^* = d\epsilon$. Based on these two distinct approaches, it has been investigated three important classes of magnetic curves for a given magnetic field in $(\mathbb{E}^3,\ \cdot)$ [12].

Definition 3.1. Let $\gamma \in \phi^{e_o e_1}$, $\mathbf{t} \in \phi^{e_o e_1}(u)$ and $\mathcal{F}$ be a magnetic field in $(\mathbb{E}^3,\ \cdot)$. γ is called a magnetic curve if it satisfies

$$\nabla_{\gamma_s}\mathbf{t} = \sigma(\mathbf{t}) = \mathcal{F}\times\mathbf{t}, \tag{27}$$

and $\mathcal{F} = \varsigma\mathbf{t} + \kappa\mathbf{b}$ such that

$$\sigma(\mathbf{t}) = \kappa\mathbf{n},\ \sigma(\mathbf{n}) = -\kappa\mathbf{t}+\varsigma\mathbf{b},\ \sigma(\mathbf{b}) = -\varsigma\mathbf{n},$$

where ς is an arbitrary non-zero smooth function defined along γ [12].

Definition 3.2. Let $\gamma \in \phi^{e_o e_1}$, $\mathbf{n} \in \phi^{e_o e_1}(u)$ and $\mathcal{F}^n$ be a magnetic field in $(\mathbb{E}^3, \cdot)$. γ is called an $\mathbf{n}-$magnetic curve if it satisfies

$$\nabla_{\gamma_s}\mathbf{n} = \sigma(\mathbf{n}) = \mathcal{F}^n \times \mathbf{n}, \tag{28}$$

and $\mathcal{F}^n = \tau\mathbf{t} - \alpha\mathbf{n} + \kappa\mathbf{b}$ such that

$$\sigma(\mathbf{t}) = \kappa\mathbf{n}+\alpha\mathbf{b},\ \sigma(\mathbf{n}) = -\kappa\mathbf{t}+\tau\mathbf{b},\ \sigma(\mathbf{b}) = -\alpha\mathbf{t}-\tau\mathbf{n},$$

where α is an arbitrary non-zero smooth function defined along γ [19].

Definition 3.3. Let $\gamma \in \phi^{e_o e_1}$, $\mathbf{b} \in \phi^{e_o e_1}(u)$ and $\mathcal{F}^b$ be a magnetic field in $(\mathbb{E}^3, \cdot)$. γ is called a $\mathbf{b}-$magnetic curve if it satisfies

$$\nabla_{\gamma_s}\mathbf{b} = \sigma(\mathbf{b}) = \mathcal{F}^b \times \mathbf{b}, \tag{29}$$

and $\mathcal{F}^b = \tau\mathbf{t} + \upsilon\mathbf{b}$ such that

$$\sigma(\mathbf{t}) = \upsilon\mathbf{n},\ \sigma(\mathbf{n}) = -\upsilon\mathbf{t}+\tau\mathbf{b},\ \sigma(\mathbf{b}) = -\tau\mathbf{n},$$

where υ is an arbitrary non-zero smooth function defined along γ [19].

Below, we present a new method to find the minimizer of the bending energy functionals acting on a class of magnetic curves. Even though the traditional approach is applicable for that case, we will not prefer to use it for a couple of reasons. As is known, in the traditional approach, the solution of the critical curves acting on the given integral is computed by involving integration by parts, which leads to classical $\mathcal{EL}$-equation. In most cases, however, this equation looks intimidating and solving the equation requires a heavy workload and hard work even for the least complicated space and metric structures. However, the new method introduces a nice trick to manipulate the bending energy functionals and finally allows investigating the critical curves together with their intrinsic properties in a simpler manner. According to this method, the choice of the associated variational vector field along the magnetic curve is not arbitrary. So, this selection establishes a remarkable correlation between the $\mathcal{EL}$-equation and Lorentz force operator of magnetic curves. All in all, we determine the minimizer of the bending energy functionals on a class of magnetic curves explicitly by expending less effort and producing more efficient output.

Case 1 : In the first case, we suppose that $\gamma \in \phi^{e_o e_1}$ is a magnetic curve of the magnetic field $\mathcal{F}$ in $(\mathbb{E}^3, \ \cdot)$. Now, let us consider the bending energy functional given by Eqs. $(4,5)$

$$\Phi^{\delta}\left(\mathbf{t}\right) = \frac{1}{2}\int_0^s (\|\mathbf{t}_s\|^2 + \Theta(\|\mathbf{t}\|^2 - 1))ds. \tag{30}$$

If magnetic vector field $\mathcal{F} = \mathcal{F}\left(\epsilon, s\right)$ is chosen as the variational vector field, then we obtain the following equation of the variation of the magnetic curve γ

$$0 = \int_0^s ((\sigma(\mathbf{t}) \cdot \mathcal{F}_{ss} + \Theta(\mathbf{t} \cdot \mathcal{F}_s))ds. \tag{31}$$

If we use the definition of magnetic curves then Eq. (31) is written by

$$0 = \int_0^s \mathbf{t}(\mathcal{F}_{ss} \times \mathcal{F} + \Theta\mathcal{F}_s)ds. \tag{32}$$

Thus, Eq. (32) holds when the Lagrange multiplier satisfies the following identity

$$\Theta = \frac{\mathcal{F}_s}{\|\mathcal{F}_s\|}(\mathcal{F} \times \mathcal{F}_{ss}). \tag{33}$$

Then, by a straightforward computation, it is obtained that

$$\begin{aligned} \Theta &= \frac{1}{\sqrt{(\varsigma_s)^2 + \kappa^2(\varsigma - \tau)^2 + (\kappa_s)^2}}((2\varsigma_s\kappa + \kappa_s\varsigma - 2\kappa_s\tau \\ &\quad -\tau_s\kappa)(\kappa_s\varsigma - \varsigma_s\kappa) + \kappa(\varsigma - \tau)(\varsigma_{ss}\kappa - \kappa(\varsigma - \tau)(\kappa^2 - \varsigma\tau) + \kappa_{ss}\varsigma)). \end{aligned} \tag{34}$$

Theorem 3.4. A magnetic curve γ of the magnetic field $\mathcal{F}$ in $(\mathbb{E}^3, \ \cdot)$ is called as a $\mathbf{t}-$elastic magnetic curve if the following Euler-Lagrange equation is satisfied

$$\Theta(\varsigma_s + \kappa_s) = (\varsigma - \kappa)\left(2\varsigma_s\kappa + \kappa_s\varsigma - 2\kappa_s\tau - \tau_s\kappa\right),$$
$$\varsigma_{ss}\kappa - \kappa_{ss}\varsigma = \kappa(\varsigma - \tau)(\Theta + \kappa^2 + \varsigma\tau),$$

where Θ is given by Eq. (34).

Proof. It can be proved by direct observation of Eqs. $(30-34)$.

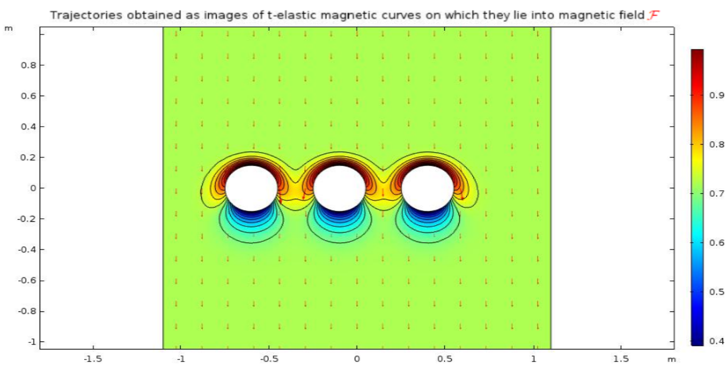

Figure 3. The shape of the $\mathbf{t}-$elastic magnetic curve.

Figure 3 shows the shapes of the $\mathbf{t}-$elastic magnetic curve for appropriate values of the κ, τ, and ς.

Proposition 3.5. If γ is a $\mathbf{t}-$elastic magnetic curve of the magnetic field $\mathcal{F}$ in $(\mathbb{E}^3, \cdot)$ then followings are Killing magnetic fields

$$\mathbf{J}^m = \left(-\kappa^2 - \Theta\right)\mathbf{t} + \kappa_s \mathbf{n} + \kappa\tau \mathbf{b},\ \mathbf{I}^m = \kappa \mathbf{b} = \mathcal{A}^m + \gamma \times \mathbf{J}^m, \tag{35}$$

where $\mathcal{A}^m$ is an arbitrary constant vector field and Θ is given by Eq. (34).

Proof. The proof is obvious if one considers the fact that the magnetic variational vector is a symmetry and restriction of a rotation field.

Corollary 3.6. The associated magnetic Killing vector fields $\mathbf{J}^m$ and $\mathbf{I}^m$ defined along the $\mathbf{t}-$elastic magnetic curve γ are commutative.

Proof. It is obvious from the fact that $[\mathbf{J}^m, \mathbf{I}^m] = 0$.

Case 2 : In the second case, we suppose that $\gamma \in \phi^{e_o e_1}$ is an $\mathbf{n}-$magnetic curve of the magnetic field $\mathcal{F}^n$ in $(\mathbb{E}^3, \cdot)$. Now, let us consider the normal bending energy functional given by Eqs. $(11, 12)$

$$\Phi^{\delta n}(\mathbf{n}) = \frac{1}{2}\int_0^s (\|\mathbf{n}_s\|^2 + \Theta^n(\|\mathbf{n}\|^2 - 1))ds. \tag{36}$$

If magnetic vector field $\mathcal{F}^n = \mathcal{F}^n(\epsilon, s)$ is chosen as the variational vector field, then we obtain the following equation of the variation of the $\mathbf{n}-$magnetic curve

$$0 = \int_0^s ((\sigma(\mathbf{n}) \cdot \mathcal{F}^n_{ss} + \Theta^n(\mathbf{n} \cdot \mathcal{F}^n_s))ds. \tag{37}$$

If we use the definition of $\mathbf{n}-$magnetic curves then Eq. (37) is written by

$$0=\int_0^s \mathbf{n}(\mathcal{F}_{ss}^n\times\mathcal{F}^n+\Theta^n\mathcal{F}_s^n)ds. \tag{38}$$

Thus, Eq. (38) holds when the normal Lagrange multiplier satisfies the following identity

$$\Theta^n=\frac{\mathcal{F}_s^n}{\|\mathcal{F}_s^n\|}(\mathcal{F}^n\times\mathcal{F}_{ss}^n). \tag{39}$$

Then, by a straightforward computation, it is obtained that

$$\begin{aligned}\Theta^n \;=\; & \frac{1}{\sqrt{(\tau_s+\alpha\kappa)^2+(\alpha_s)^2+(-\alpha\tau+\kappa_s)^2}}((\tau_s+\alpha\kappa)(2\alpha_s\alpha\tau\\ &+\tau_s\alpha^2-\kappa_{ss}\alpha-\tau_s\kappa^2-\alpha\kappa^3+\alpha_{ss}\kappa-\alpha\kappa\tau^2+\kappa_s\kappa\tau)\\ &-\alpha_s(2\alpha_s\tau^2+\tau_s\alpha\tau-\kappa_{ss}\tau+\tau_{ss}\kappa+2\alpha_s\kappa^2+\kappa_s\kappa\alpha)\\ &+(-\alpha\tau+\kappa_s)(\tau_s\kappa\tau+\alpha\kappa^2\tau-\alpha_{ss}\tau+\alpha\tau^3-\kappa_s\tau^2\\ &+\tau_{ss}\alpha+2\alpha_s\alpha\kappa+\kappa_s\alpha^2)).\end{aligned} \tag{40}$$

Theorem 3.7. An $\mathbf{n}-$magnetic curve γ of the magnetic field $\mathcal{F}^n$ in $(\mathbb{E}^3,\ \cdot)$ is called as an $\mathbf{n}-$elastic magnetic curve if the following Euler-Lagrange equation is satisfied

$$\begin{gathered}\Theta^n(\kappa_s+\tau_s+\alpha(\kappa-\tau))=(\kappa_s+\tau_s)\left(\alpha^2+\kappa\tau\right)+(\kappa-\tau)\left(\alpha_{ss}+\alpha\kappa\tau\right.\\ \left.-\alpha(\kappa^2+\kappa\tau+\tau^2)\right)+2\alpha_s\alpha(\kappa+\tau)-\alpha\left(\kappa_{ss}-\tau_{ss}\right)-\kappa_s\tau^2-\tau_s\kappa^2,\\ \kappa_{ss}\tau-\tau_{ss}\kappa-\alpha_s\Theta^n=2\alpha_s(\kappa^2+\tau^2)+\alpha(\kappa_s\kappa+\tau_s\tau),\end{gathered}$$

where Θ^n is given by Eq. (40).

Proof. It can be proved by direct observation of Eqs. $(36-40)$.

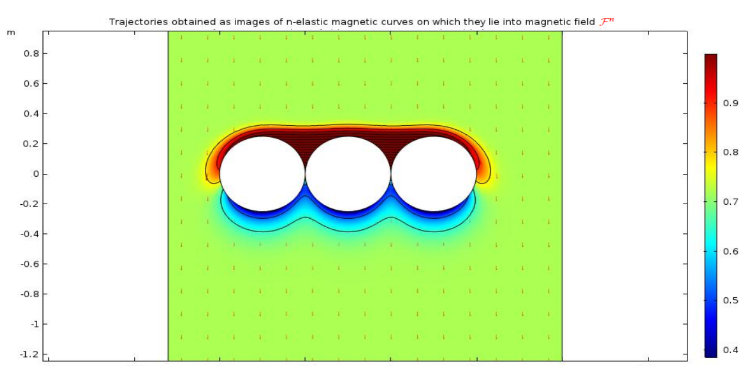

Figure 4. The shape of the $\mathbf{n}-$elastic magnetic curve.

Figure 4 shows the shapes of the $\mathbf{n}-$elastic magnetic curve for appropriate values of the $\kappa,\ \tau$, and α.

Proposition 3.8. If γ is an $\mathbf{n}-$elastic magnetic curve of the magnetic field $\mathcal{F}^n$ in $(\mathbb{E}^3,\ \cdot)$ then followings are Killing magnetic fields

$$\mathbf{J}^{nm}{=}-\kappa_s\mathbf{t}{-}(\kappa^2+\tau^2+\Theta^n)\mathbf{n}{+}\tau_s\mathbf{b},\ \mathbf{I}^{nm}{=}\kappa\mathbf{b}={\mathcal{A}}^{nm}{+}\gamma{\times}\mathbf{J}^{nm}, \tag{41}$$

where $\mathcal{A}^{nm}$ is an arbitrary constant vector field and Θ^n is given by Eq. (40).

Proof. The proof is obvious if one considers the fact that the magnetic variational vector is a symmetry and restriction of a rotation field.

Corollary 3.9. The associated magnetic Killing vector fields $\mathbf{J}^{nm}$ and $\mathbf{I}^{nm}$ defined along the $\mathbf{n}-$elastic magnetic curve γ are commutative.

Proof. It is obvious from the fact that $[\mathbf{J}^{nm},\mathbf{I}^{nm}]=0$.

Case 3 : In the third case, we suppose that $\gamma\in\phi^{e_o e_1}$ is a $\mathbf{b}-$magnetic curve of the magnetic field $\mathcal{F}^b$ in $(\mathbb{E}^3,\ \cdot)$. Now, let us consider the binormal bending energy functional given by Eqs. (18, 19)

$$\Phi^{\delta b}\left(\mathbf{b}\right)=\frac{1}{2}\int_0^s(\|\mathbf{b}_s\|^2+\Theta^b(\|\mathbf{b}\|^2-1))ds. \tag{42}$$

If magnetic vector field $\mathcal{F}^b{=}\ \mathcal{F}^b\left(\epsilon,s\right)$ is chosen as the variational vector field, then we obtain the following equation of the variation of the $\mathbf{b}-$magnetic curve

$$0=\int_0^s((\sigma(\mathbf{b})\cdot\mathcal{F}^b_{ss}+\Theta^b(\mathbf{b}\cdot\mathcal{F}^b_s))ds. \tag{43}$$

If we use the definition of $\mathbf{b}-$magnetic curves then Eq. (43) is written by

$$0=\int_0^s\mathbf{b}(\mathcal{F}^b_{ss}\times\mathcal{F}^b+\Theta^b\mathcal{F}^b_s)ds. \tag{44}$$

Thus, Eq. (44) holds when the binormal Lagrange multiplier satisfies the following identity

$$\Theta^b=\frac{\mathcal{F}^b_s}{\|\mathcal{F}^b_s\|}(\mathcal{F}^b\times\mathcal{F}^b_{ss}). \tag{45}$$

Then, by a straightforward computation, it is obtained that

$$\begin{aligned}\Theta^b\ =\ &\frac{1}{\sqrt{(\tau_s)^2+\tau^2(\kappa-\upsilon)^2+(\upsilon_s)^2}}((2\tau_s\kappa+\kappa_s\tau-\tau_s\upsilon \qquad (46)\\ &-2\upsilon_s\tau)(\upsilon_s\tau-\tau_s\upsilon)+\tau\left(\kappa-\upsilon\right)\left(\tau_{ss}\upsilon-\tau\left(\kappa-\upsilon\right)\left(\tau^2+\upsilon\kappa\right)-\tau\upsilon_{ss}\right).\end{aligned}$$

Theorem 3.10. A $\mathbf{b}-$magnetic curve γ of the magnetic field $\mathcal{F}^b$ in $(\mathbb{E}^3,\ \cdot)$ is called as a $\mathbf{b}-$elastic magnetic curve if the following Euler-Lagrange equation is satisfied

$$\begin{aligned}\Theta^b(\upsilon_s+\tau_s) &= (\tau-\upsilon)\left(2\tau_s\kappa+\kappa_s\tau-\tau_s\upsilon-2\upsilon_s\tau\right),\\ \tau_{ss}\upsilon-\upsilon_{ss}\tau &= \tau\left(\kappa-\upsilon\right)\left(\Theta^b+\tau^2+\upsilon\kappa\right),\end{aligned}$$

where Θ^b is given by Eq. (46).

Proof. It can be proved by direct observation of Eqs. $(42-46)$.

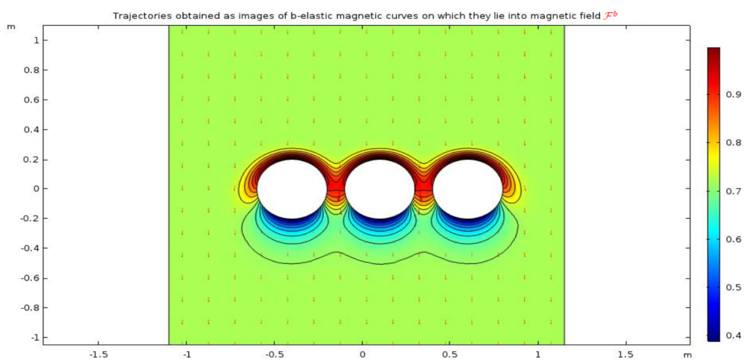

Figure 5. The shape of the $\mathbf{b}-$elastic magnetic curve.

Figure 5 shows the shapes of the $\mathbf{b}-$elastic magnetic curve for appropriate values of the $\kappa,\ \tau$, and υ.

Proposition 3.11. If γ is a $\mathbf{b}-$elastic magnetic curve of the magnetic field $\mathcal{F}^b$ in $(\mathbb{E}^3,\ \cdot)$ then followings are Killing magnetic fields

$$\mathbf{J}^{bm}{=}\kappa\tau\mathbf{t}{-}\tau_s\mathbf{n}-(\Theta^b+\tau^2)\mathbf{b},\ \mathbf{I}^{bm}{=}\kappa\mathbf{b}={\mathcal{A}}^{bm}{+}\gamma{\times}\mathbf{J}^{bm}, \tag{47}$$

where $\mathcal{A}^{bm}$ is an arbitrary constant vector field and Θ^b is given by Eq. (46).

Proof. The proof is obvious if one considers the fact that the magnetic variational vector is a symmetry and restriction of a rotation field.

Corollary 3.12. The associated magnetic Killing vector fields $\mathbf{J}^{bm}$ and $\mathbf{I}^{bm}$ defined along the $\mathbf{b}-$elastic magnetic curve γ are commutative.

Proof. It is obvious from the fact that $\left[\mathbf{J}^{bm},\mathbf{I}^{bm}\right]=0$.

Before going on, it should be noted that in this section, we show that Th. (3.4), Th. (3.7), and Th. (3.10) can be both considered as definitions of $\mathbf{t}-$elastic magnetic curves, $\mathbf{n}-$elastic magnetic curves, $\mathbf{b}-$elastic magnetic curves or examples of critical curves of bending energy functional, normal bending energy functional, and binormal bending energy functional given by Eq. (4), Eq. (11), and Eq. (18), respectively. In this section, we also obtain new Killing magnetic fields associated with each elastic magnetic curve. These fields may contain significant features to deal with the corresponding Landau-Hall problem and Hall-Killing effect since both concepts could be explained by Euler-Lagrange equations and variational approach. Later on, we will discuss the numerical and analytical analysis of Euler-Lagrange equations of elastic magnetic curves by considering the geometric invariants of each curve satisfying certain differential equation systems given by Th. (3.4), Th. (3.7), and Th. (3.10) up to certain parameters.

4 Investigating the dynamics of elastic magnetic curves

The model of elastic magnetic curves has attracted great interest in recent years due to its expected impact on many research areas such as mechanical properties of the interior of cells, micromechanical sensors, magnetic microdevices, viscous fluid, and elastic linkers. This model also allows one to define various physical concepts including the self-propelling features of the curve in magnetic field, nonsynchronous and synchronous regimes of the curve motion in the nonrotating or rotating field relying on the dynamics of curves on several physical and mathematical parameters such as coefficient of friction, frequency of magnetic field, magnetoelastic number, etc.

Elastic magnetic curves are mostly produced in laboratories while they can be created theoretically. In this paper, as outlined in the previous section, the characterization of a new kind of elastic magnetic curves has been constructed successfully based on the extension of Euler-Lagrange equations by including magnetic vector field terms into corresponding bending energy functionals. In this section, we focus on the dynamics and behavior of the elastic magnetic curves under the action of the elastic and magnetic energy together with magnetic torque and moment in the ordinary space.

Case 1 : In the first case, let $\gamma \in \phi^{e_o e_1}$ be a $\mathbf{t}-$elastic magnetic curve of the magnetic field $\mathcal{F}$ in $(\mathbb{E}^3, \ \cdot)$ such that $\mathcal{F} = \varsigma \mathbf{t} + \kappa \mathbf{b}$ and

$$\sigma(\mathbf{t}) = \kappa \mathbf{n}, \ \sigma(\mathbf{n}) = -\kappa \mathbf{t} + \varsigma \mathbf{b}, \ \sigma(\mathbf{b}) = -\varsigma \mathbf{n},$$

where σ is the skew-symmetric Lorentz force operator. In the Frenet frame, the bending elastic energy of the $\mathbf{t}-$elastic magnetic γ curve is computed by

$$\Phi^{\delta}(\mathbf{t}) = \mathcal{E}_e = \frac{1}{2} \int_0^s \kappa^2 ds, \tag{48}$$

and the magnetic energy of the $\mathbf{t}-$elastic magnetic curve γ is computed by

$$\mathcal{E}_r = - \|\mathcal{M}\| \int_0^s \mathbf{t} \cdot \mathcal{F} ds. \tag{49}$$

The above definitions of energy functionals are valid when γ is assumed to be inextensible. Therefore, the $\mathbf{t}-$elastic magnetic curve is assumed to satisfy the following functional in most applications

$$\mathcal{E}_{\Sigma} = \frac{1}{2} \int_0^s \Sigma(s) ds. \tag{50}$$

Moreover, in Eq. (49), the magnetic energy functional of the $\mathbf{t}-$elastic magnetic curve holds when γ is ferromagnetic. This definition can also be extended to the following formula

$$\mathcal{E}_{rp} = - \int_0^s \pi^2 \chi^2 \varrho^2 (1 + 2\pi\varrho)^{-1} (\mathbf{t} \cdot \mathcal{F})^2 ds + c, \tag{51}$$

when γ is assumed to be a superparamagnetic [20]. Here, $\mathcal{M}$ is the magnetic moment of γ, χ is the radius of the curvature of γ, ϱ is the magnetic susceptibility, and c is a constant term. The total energy of the $\mathbf{t}-$elastic magnetic curve is given by $\mathcal{E} = \mathcal{E}_e + \mathcal{E}_r + \mathcal{E}_{\Sigma}$. Now, we aim to obtain magnetic force and torque acting on the $\mathbf{t}-$elastic magnetic curve by considering the energy functionals given by Eqs. $(48 - 51)$.

In the case of the ferromagnetic model of the $\mathbf{t}-$elastic magnetic curve, the torque is calculated by the following formula

$$\mathcal{T} = -\frac{m}{2} (\sigma(\mathbf{t}) \times \nabla_{\gamma_s} \sigma(\mathbf{t})) \times \mathcal{F}, \tag{52}$$

where m is any positive scalar. A straightforward calculation yields that

$$\mathcal{T} = \frac{m}{2}\kappa^3(\tau - \varsigma)\mathbf{n}. \tag{53}$$

The torque acting on the local element of the $\mathbf{t}-$elastic magnetic curve is also obtained as follows

$$\mathcal{T} = \mathcal{M} \times \mathcal{F}. \tag{54}$$

Hence, we can obtain the magnetic moment $\mathcal{M}$ along the $\mathbf{t}-$elastic magnetic curve by considering Eqs. $(53, 54)$ in the following manner

$$\mathcal{M} = \frac{m}{2}\kappa^3(\frac{\varsigma}{\kappa}\mathbf{t} + \frac{\tau}{\varsigma}\mathbf{b}), \tag{55}$$

where both $\kappa \neq 0$ and $\varsigma \neq 0$ along γ. Now, if we take into account the well-known formula of the magnetic force acting on the $\mathbf{t}-$elastic magnetic curve then the force $\mathcal{G}$ is computed by

$$\mathcal{G} = -q\kappa\mathbf{n}, \tag{56}$$

where q is a non-zero scalar. Finally, let us consider the following identity

$$\mathcal{G} = -\nabla^2_{\gamma_s}\mathbf{t} + \Sigma\mathbf{t} + \mathcal{G}_r, \tag{57}$$

where $\mathcal{G}_r = -\left\|\mathcal{M}\right\|\mathcal{F}$. If we consider Eqs. $(52 - 56)$ and equate the respective coefficients of vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ in Eq. (57) then it is obtained that

$$\Sigma = -\kappa^2 + \frac{m}{2}\kappa^3\varsigma\sqrt{\frac{\varsigma^2}{\kappa^2} + \frac{\tau^2}{\varsigma^2}},\ q = \frac{\kappa_s}{\kappa},\ \frac{1}{m} = -\frac{\kappa^3}{2\tau}\sqrt{\frac{\varsigma^2}{\kappa^2} + \frac{\tau^2}{\varsigma^2}}. \tag{58}$$

Thus, we not only compute the inextensibility coefficient Σ but we also determine the scalars m and q in terms of the curvature and torsion functions of the $\mathbf{t}-$elastic magnetic curve.

In the case of the superparamagnetic model of the $\mathbf{t}-$elastic magnetic curve we have

$$\mathcal{G}_r = -\frac{2\pi^2\chi^2\varrho^2}{1 + 2\pi\varrho}(\mathbf{t} \cdot \mathcal{F}) \cdot \mathcal{F}. \tag{59}$$

From Eqs. $(52 - 57, 59)$ we have following equalities

$$\Sigma = -\kappa^2 + \frac{2\pi^2\chi^2\varrho^2}{1 + 2\pi\varrho}\varsigma^2,\ q = \frac{\kappa_s}{\kappa},\ \frac{\tau}{\varsigma} = -\frac{2\pi^2\chi^2\varrho^2}{1 + 2\pi\varrho}, \tag{60}$$

where $1 + 2\pi\varrho \neq 0$, $\kappa \neq 0$, and $\varsigma \neq 0$.

Teorem 4.1. Let $\gamma \in \phi^{e_o e_1}$ be a $\mathbf{t}-$elastic magnetic curve of the magnetic field $\mathcal{F}$ in $(\mathbb{E}^3, \ \cdot)$. In the case of the ferromagnetic model, the total energy of the $\mathbf{t}-$elastic magnetic curve is given by

$$\mathcal{E} = -\frac{m}{2}\kappa^3\sqrt{\frac{\varsigma^2}{\kappa^2}+\frac{\tau^2}{\varsigma^2}}\int_0^s \varsigma ds + \frac{m}{4}\int_0^s \kappa^3\varsigma\sqrt{\frac{\varsigma^2}{\kappa^2}+\frac{\tau^2}{\varsigma^2}}ds, \tag{61}$$

where $q = \frac{\kappa_s}{\kappa}, \ \frac{1}{m} = -\frac{\kappa^3}{2\tau}\sqrt{\frac{\varsigma^2}{\kappa^2}+\frac{\tau^2}{\varsigma^2}}, \ \kappa \neq 0,$ and $\varsigma \neq 0.$

Proof. It is obvious from Eqs. $(48-50, 52-58)$.

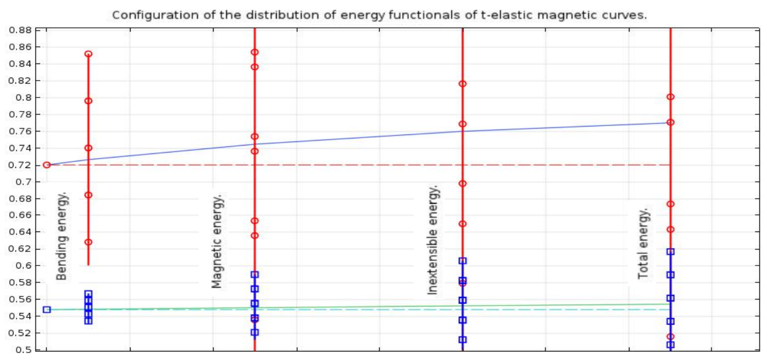

Figure 6. The energy of the $\mathbf{t}-$elastic magnetic curve.

Figure 6 shows the energy variation of the $\mathbf{t}-$elastic magnetic curve for appropriate values of the $\kappa, \ \tau, \ m,$ and ς.

Teorem 4.2. Let $\gamma \in \phi^{e_o e_1}$ be a $\mathbf{t}-$elastic magnetic curve of the magnetic field $\mathcal{F}$ in $(\mathbb{E}^3, \ \cdot)$. In the case of the superparamagnetic model, the total energy of the elastic magnetic curve is given by

$$\mathcal{E} = c, \tag{62}$$

where $q = \frac{\kappa_s}{\kappa}, \ \frac{\tau}{\varsigma} = -\frac{2\pi^2\chi^2\varrho^2}{1+2\pi\varrho}, \ \kappa \neq 0, \varsigma \neq 0,$ and c is a constant.

Proof. It is obvious from Eqs. $(48, 50-57, 59, 60)$.

Case 2 : In the second case, let $\gamma \in \phi^{e_o e_1}$ be an $\mathbf{n}-$elastic magnetic curve of the magnetic field $\mathcal{F}^n$ in $(\mathbb{E}^3, \ \cdot)$ such that $\mathcal{F}^n = \tau\mathbf{t} - \alpha\mathbf{n} + \kappa\mathbf{b}$ and

$$\sigma(\mathbf{t}) = \kappa\mathbf{n} + \alpha\mathbf{b}, \ \sigma(\mathbf{n}) = -\kappa\mathbf{t} + \tau\mathbf{b}, \ \sigma(\mathbf{b}) = -\alpha\mathbf{t} - \tau\mathbf{n},$$

where σ is the skew-symmetric Lorentz force operator. In the Frenet frame, the normal bending elastic energy of the $\mathbf{n}-$elastic magnetic curve γ is computed by

$$\Phi^{\delta n}(\mathbf{n}) = \mathcal{E}_e^n = \frac{1}{2}\int_0^s (\kappa^2 + \tau^2) ds, \tag{63}$$

and the normal magnetic energy of the $\mathbf{n}-$elastic magnetic curve γ is computed by

$$\mathcal{E}_r^n = -\|\mathcal{M}^n\| \int_0^s \mathbf{n} \cdot \mathcal{F}^n ds. \tag{64}$$

The above definitions of energy functionals are valid when γ is assumed to be inextensible. Therefore, the $\mathbf{n}-$elastic magnetic curve is assumed to satisfy the following functional in most applications

$$\mathcal{E}_\Sigma^n = \frac{1}{2}\int_0^s \Sigma^n(s) ds. \tag{65}$$

Moreover, in Eq. (64), normal magnetic energy functional of the $\mathbf{n}-$elastic magnetic curve holds when γ is a normal ferromagnetic. This definition can be extended to the following formula

$$\mathcal{E}_{rp}^n = -\int_0^s \pi^2 \chi^2 \varrho^2 (1 + 2\pi\varrho)^{-1} (\mathbf{n} \cdot \mathcal{F}^n)^2 ds + c^n, \tag{66}$$

when γ is assumed to be a normal superparamagnetic. Here, $\mathcal{M}^n$ is the normal magnetic moment of γ, χ is the radius of the curvature of γ, ϱ is the magnetic susceptibility, and c^n is a constant term. The total energy of the $\mathbf{n}-$elastic magnetic curve is given by $\mathcal{E}^n = \mathcal{E}_e^n + \mathcal{E}_r^n + \mathcal{E}_\Sigma^n$. Now, we aim to obtain normal magnetic force and torque acting on the $\mathbf{n}-$elastic magnetic curve by considering the energy functionals given by Eqs. $(63-66)$.

In the case of the normal ferromagnetic model of the $\mathbf{n}-$elastic magnetic curve, the normal torque is calculated by the following formula

$$\mathcal{T}^n = -\frac{m^n}{2} (\sigma(\mathbf{n}) \times \nabla_{\gamma_s} \sigma(\mathbf{n})) \times \mathcal{F}^n, \tag{67}$$

where m^n is any positive scalar. A straightforward calculation yields that

$$\mathcal{T}^n = \frac{m^n}{2} (\alpha\kappa(\kappa^2 + \tau^2) - \kappa(\kappa\tau)_s)\mathbf{t} + \frac{m^n}{2} (-\alpha\tau(\kappa^2 + \tau^2) + \tau(\kappa\tau)_s)\mathbf{b}. \tag{68}$$

The normal torque acting on the local element of the $\mathbf{n}-$elastic magnetic curve is also obtained as follows

$$\mathcal{T}^n = \mathcal{M}^n \times \mathcal{F}^n. \tag{69}$$

Hence, we can obtain the normal magnetic moment $\mathcal{M}^n$ along the $\mathbf{n}-$elastic magnetic curve by considering Eqs. (68, 69) in the following manner

$$\mathcal{M}^n = \frac{m^n}{2}\tau((\kappa^2+\tau^2) - \frac{(\kappa\tau)_s}{\alpha})\mathbf{t} + \frac{m^n}{2}\kappa((\kappa^2+\tau^2) - \frac{(\kappa\tau)_s}{\alpha})\mathbf{b}, \tag{70}$$

where $\alpha \neq 0$ along γ. Now, if we take into account the well-known formula of the normal magnetic force acting on the $\mathbf{n}-$elastic magnetic curve then the force $\mathcal{G}^n$ is computed by

$$\mathcal{G}^n = q^n(\kappa\mathbf{t} - \tau\mathbf{b}), \tag{71}$$

where q^n is a non-zero scalar. Finally, let us consider the following identity

$$\mathcal{G}^n = -\nabla^2_{\gamma_s}\mathbf{n} + \Sigma^n\mathbf{n} + \mathcal{G}^n_r, \tag{72}$$

where $\mathcal{G}^n_r = -\left\|\mathcal{M}^n\right\|\mathcal{F}^n$. If we consider Eqs. $(67-71)$ and equate the respective coefficients of vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ in Eq. (72) then it is obtained that

$$\begin{aligned}
\Sigma^n &= -(\kappa^2+\tau^2) - \frac{m^n}{2}\alpha(\kappa^2+\tau^2 - \frac{(\kappa\tau)_s}{\alpha})\sqrt{\kappa^2+\tau^2}, \\
q^n &= \frac{\kappa_s}{\kappa} - \frac{m^n}{2}\frac{\tau}{\kappa}(\kappa^2+\tau^2 - \frac{(\kappa\tau)_s}{\alpha})\sqrt{\kappa^2+\tau^2}, \\
q^n &= \frac{\tau_s}{\tau} - \frac{m^n}{2}\frac{\kappa}{\tau}(\kappa^2+\tau^2 - \frac{(\kappa\tau)_s}{\alpha})\sqrt{\kappa^2+\tau^2}, \\
m^n &= \frac{2}{(\kappa^2+\tau^2 - \frac{(\kappa\tau)_s}{\alpha})}\frac{(\kappa_s\tau - \kappa\tau_s)}{(\tau-\kappa)(\kappa+\tau)^3},
\end{aligned} \tag{73}$$

where $\tau \neq \pm\kappa,\ \kappa \neq 0,\ \tau \neq 0,\ \kappa^2+\tau^2 \neq \frac{(\kappa\tau)_s}{\alpha}$, and $\alpha \neq 0$. Thus, we not only compute the inextensibility coefficient Σ^n but we also determine the scalars m^n and q^n in terms of the curvature and torsion functions of the $\mathbf{n}-$elastic magnetic curve.

In the case of the normal superparamagnetic model of the $\mathbf{n}-$elastic magnetic curve, we have

$$\mathcal{G}^n_r = -\frac{2\pi^2\chi^2\varrho^2}{1+2\pi\varrho}(\mathbf{n}\cdot\mathcal{F}^n)\cdot\mathcal{F}^n. \tag{74}$$

From Eqs. $(67-72,74)$ we have following equalities

$$\Sigma^{n}=-(\kappa^{2}+\tau^{2})+\frac{2\pi^{2}\chi^{2}\varrho^{2}}{1+2\pi\varrho}\alpha^{2},\ q^{n}(\kappa^{2}+\tau^{2})=(\kappa\tau)_{s}\,. \tag{75}$$

where $1+2\pi\varrho\neq 0$.

Teorem 4.3. Let $\gamma\in\phi^{e_{o}e_{1}}$ be an $\mathbf{n}-$elastic magnetic curve of the magnetic field $\mathcal{F}^{n}$ in $(\mathbb{E}^{3},\ \cdot)$. In the case of the normal ferromagnetic model, the total energy of the $\mathbf{n}-$elastic magnetic curve is given by

$$\begin{aligned}\mathcal{E}^{n} &= \frac{m^{n}}{2}(\kappa+\tau)\sqrt{(\kappa^{2}+\tau^{2}-\frac{(\kappa\tau)_{s}}{\alpha})^{2}}\int_{0}^{s}\alpha ds\\ &\quad -\frac{m^{n}}{4}\int_{0}^{s}\frac{\alpha}{2}(\kappa+\tau)\sqrt{(\kappa^{2}+\tau^{2}-\frac{(\kappa\tau)_{s}}{\alpha})^{2}}ds,\end{aligned} \tag{76}$$

where q^{n} and m^{n} are given by Eq. (73).

Proof. It is obvious from Eqs. $(63-65,67-73)$.

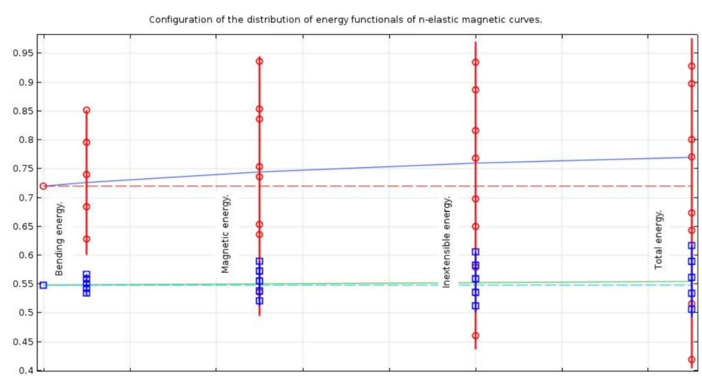

Figure 7. The energy of the $\mathbf{n}-$elastic magnetic curve.

Figure 7 shows the energy variation of the $\mathbf{n}-$elastic magnetic curve for appropriate values of the κ, τ, m^{n}, and α.

Teorem 4.4. Let $\gamma\in\phi^{e_{o}e_{1}}$ be an $\mathbf{n}-$elastic magnetic curve of the magnetic field $\mathcal{F}^{n}$ in $(\mathbb{E}^{3},\ \cdot)$. In the case of the normal superparamagnetic model, the total energy of the $\mathbf{n}-$elastic magnetic curve is given by

$$\mathcal{E}^{n}=c^{n}, \tag{77}$$

where $q^{n}(\kappa^{2}+\tau^{2})=(\kappa\tau)_{s}$ and c^{n} is a constant.

Proof. It is obvious from Eqs. $(63, 65-72, 74, 75)$.

Case 3 : In the third case, let $\gamma \in \phi^{e_o e_1}$ be a $\mathbf{b}-$elastic magnetic curve of the magnetic field $\mathcal{F}^b$ in $(\mathbb{E}^3, \cdot)$ such that $\mathcal{F}^b{=}\tau\mathbf{t}{+}\upsilon\mathbf{b}$ and

$$\sigma(\mathbf{t}) = \upsilon\mathbf{n},\ \sigma(\mathbf{n}) = -\upsilon\mathbf{t}{+}\tau\mathbf{b},\ \sigma(\mathbf{b}) = -\tau\mathbf{n},$$

where υ is the skew-symmetric Lorentz force operator. In the Frenet frame, the binormal bending elastic energy of the $\mathbf{b}-$elastic magnetic curve γ is computed by

$$\Phi^{\delta b}(\mathbf{b}) = \mathcal{E}_e^b = \frac{1}{2}\int_0^s \tau^2 ds, \tag{78}$$

and the binormal magnetic energy of the $\mathbf{b}-$elastic magnetic curve γ is computed by

$$\mathcal{E}_r^b = -\left\|\mathcal{M}^b\right\| \int_0^s \mathbf{b}\cdot\mathcal{F}^b ds. \tag{79}$$

The above definitions of energy functionals are valid when γ is assumed to be inextensible. Therefore, the $\mathbf{b}-$elastic magnetic curve is assumed to satisfy the following functional in most applications

$$\mathcal{E}_\Sigma^b = \frac{1}{2}\int_0^s \Sigma^b(s) ds. \tag{80}$$

Moreover, in Eq. (79), binormal magnetic energy functional of the $\mathbf{b}-$elastic magnetic curve holds when γ is a binormal ferromagnetic. This definition can be extended to the following formula

$$\mathcal{E}_{rp}^b = -\int_0^s \pi^2\chi^2\varrho^2(1+2\pi\varrho)^{-1}(\mathbf{b}\cdot\mathcal{F}^b)^2 ds + c^b, \tag{81}$$

when γ is assumed to be a binormal superparamagnetic. Here, $\mathcal{M}^b$ is the binormal magnetic moment of γ, χ is the radius of the curvature of γ, ϱ is the magnetic susceptibility, and c^b is a constant term. The total energy of the $\mathbf{b}-$elastic magnetic curve is given by $\mathcal{E}^b{=}\ \mathcal{E}_e^b + \mathcal{E}_r^b + \mathcal{E}_\Sigma^b$. Now, we aim to obtain binormal magnetic force and torque acting on the $\mathbf{b}-$elastic magnetic curve by considering the energy functionals given by Eqs. $(78-81)$.

In the case of the binormal ferromagnetic model of the $\mathbf{b}-$elastic magnetic curve, the binormal torque is calculated by the following formula

$$\mathcal{T}^b = -\frac{m^b}{2}(\sigma(\mathbf{b}) \times \nabla_{\gamma_s}\sigma(\mathbf{b})) \times \mathcal{F}^b, \tag{82}$$

where m^b is any positive scalar. A straightforward calculation yields that

$$\mathcal{T}^b = \frac{m^b}{2}\tau^3(\kappa - \upsilon)\mathbf{n}. \tag{83}$$

The binormal torque acting on the local element of the $\mathbf{b}-$elastic magnetic curve is also obtained as follows

$$\mathcal{T}^b = \mathcal{M}^b \times \mathcal{F}^b. \tag{84}$$

Hence, we can obtain the binormal magnetic moment $\mathcal{M}^b$ along the $\mathbf{b}-$elastic magnetic curve by considering Eqs. $(83, 84)$ in the following manner

$$\mathcal{M}^b = -\frac{m^b}{2}\tau^3(\frac{\kappa}{\upsilon}\mathbf{t} + \frac{\upsilon}{\tau}\mathbf{b}), \tag{85}$$

both $\tau \neq 0$ and $\upsilon \neq 0$ along γ. Now, if we take into account the well-known formula of the binormal magnetic force acting on the $\mathbf{b}-$elastic magnetic curve then the force $\mathcal{G}^b$ is computed by

$$\mathcal{G}^b = q^b\tau\mathbf{n}, \tag{86}$$

where q^b is a non-zero scalar. Finally, let us consider the following identity

$$\mathcal{G}^b = -\nabla^2_{\gamma_s}\mathbf{b} + \Sigma^b\mathbf{b} + \mathcal{G}^b_r, \tag{87}$$

where $\mathcal{G}^b_r = -\left\|\mathcal{M}^b\right\|\mathcal{F}^b$. If we consider Eqs. $(82 - 86)$ and equate the respective coefficients of vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ in Eq. (87) then it is obtained that

$$\Sigma^b = -\tau^2 + \frac{m^b}{2}\tau^3\upsilon\sqrt{\frac{\kappa^2}{\upsilon^2} + \frac{\upsilon^2}{\tau^2}},\ q^b = \frac{\tau_s}{\tau},\ \frac{1}{m^b} = -\frac{\tau^3}{2\kappa}\sqrt{\frac{\kappa^2}{\upsilon^2} + \frac{\upsilon^2}{\tau^2}}. \tag{88}$$

Thus, we not only compute the inextensibility coefficient Σ^b but we also determine the scalars m^b and q^b in terms of the curvature and torsion functions of the $\mathbf{b}-$elastic magnetic curve.

In the case of the binormal superparamagnetic model of the $\mathbf{b}-$elastic magnetic curve, we have

$$\mathcal{G}^b_r = -\frac{2\pi^2\chi^2\varrho^2}{1 + 2\pi\varrho}(\mathbf{b} \cdot \mathcal{F}^b) \cdot \mathcal{F}^b. \tag{89}$$

From Eqs. $(82-87, 89)$ we have following equalities

$$\Sigma^b = -\tau^2 + \frac{2\pi^2\chi^2\varrho^2}{1+2\pi\varrho}\upsilon^2,\ q^b = \frac{\tau_s}{\tau},\ \frac{\kappa}{\upsilon} = -\frac{2\pi^2\chi^2\varrho^2}{1+2\pi\varrho}. \tag{90}$$

where $1+2\pi\varrho \neq 0,\ \tau \neq 0$, and $\upsilon \neq 0$.

Teorem 4.5. Let $\gamma \in \phi^{e_o e_1}$ be a $\mathbf{b}-$elastic magnetic curve of the magnetic field $\mathcal{F}^b$ in $(\mathbb{E}^3,\ \cdot)$. In the case of the binormal ferromagnetic model, the total energy of the $\mathbf{b}-$elastic magnetic curve is given by

$$\mathcal{E}^b = -\frac{m^b}{2}\tau\sqrt{\frac{\kappa^2}{\upsilon^2}+\frac{\upsilon^2}{\tau^2}}\int_0^s \upsilon ds + \frac{m^b}{4}\int_0^s \tau^3\upsilon\sqrt{\frac{\kappa^2}{\upsilon^2}+\frac{\upsilon^2}{\tau^2}}ds,$$

where $q^b = \frac{\tau_s}{\tau},\ \frac{1}{m^b} = -\frac{\tau^3}{2\kappa}\sqrt{\frac{\kappa^2}{\upsilon^2}+\frac{\upsilon^2}{\tau^2}},\ \tau \neq 0,\ \kappa \neq 0$ and $\upsilon \neq 0$.

Proof. It is obvious from Eqs. $(78-80, 82-88)$.

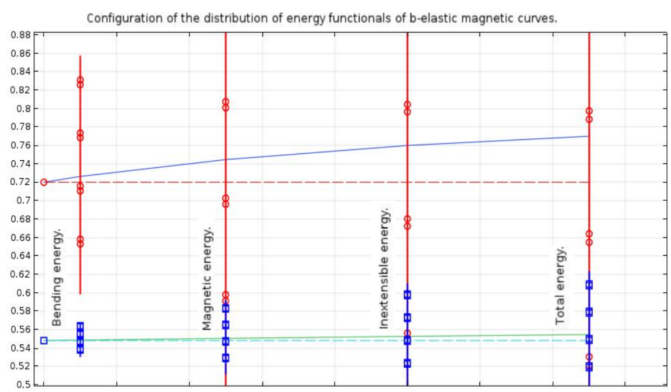

Figure 8. The energy of $\mathbf{b}-$elastic magnetic curve.

Figure 8 shows the energy variation of the $\mathbf{b}-$elastic magnetic curve for appropriate values of the $\kappa,\ \tau$, and υ.

Teorem 4.6. Let $\gamma \in \phi^{e_o e_1}$ be a $\mathbf{b}-$elastic magnetic curve of the magnetic field $\mathcal{F}^b$ in $(\mathbb{E}^3,\ \cdot)$. In the case of the binormal superparamagnetic model, the total energy of the $\mathbf{b}-$elastic magnetic curve is given by

$$\mathcal{E}^b = c^b, \tag{91}$$

where $q^b = \frac{\tau_s}{\tau},\ \frac{\kappa}{\upsilon} = -\frac{2\pi^2\chi^2\varrho^2}{1+2\pi\varrho},\ \tau \neq 0,\ \upsilon \neq 0$, and c^b is a constant.

Proof. It is obvious from Eqs. $(78, 80-87, 89, 90)$.

5 Analytical solutions and numerical simulations of elastic and elastic magnetic curves

In this section, we aim to obtain the analytical solutions and associated numerical simulations of Euler-Lagrange equations satisfied by elastic or elastic magnetic curves. We mainly consider a combined form of the Adomian decomposition method and Laplacian transform method to solve these nonlinear equations [21, 12]. The combined technique provides to compute the approximate solutions of the curvature functions belonging to elastic and elastic magnetic curves. Thus, it gives a great opportunity for numerical simulating of the analytical solutions of the Euler-Lagrange equations calculated in the previous sections.

Here, we only solve the Euler-Lagrange equations satisfied by $\mathbf{t}-$elastic curve and $\mathbf{t}-$elastic magnetic curve given by Def. 2.1 and Th. 3.4, respecctively. Solutions of the other cases can be obtained by using the similar argument and they are left to the reader.

Now, let us first consider the Euler-Lagrange equation satisfied by the $\mathbf{t}-$elastic curve as follows

$$\kappa_{ss} + \frac{\kappa^3}{2} - \kappa\tau^2 - \frac{\delta\kappa}{2} = 0, \tag{92}$$

$$\kappa\tau_s + 2\kappa_s\tau = 0, \tag{93}$$

where δ is a real valued constant. We can solve the Eq. (93) in the following way

$$\frac{d\tau}{\tau} = -2\frac{d\kappa}{\kappa},$$

which implies that

$$\tau = p_1\kappa^{-2}.$$

If we replace $\tau = p_1\kappa^{-2}$ in the Eq. (92), we get the following identities

$$\kappa_{ss} + \frac{\kappa^3}{2} - p_1 - \frac{\delta\kappa}{2} = 0. \tag{94}$$

Thus, we can consider the nonlinear problem given below

$$\kappa_{ss} + \frac{\kappa^3}{2} - \delta\frac{\kappa}{2} - p_1 = 0, \tag{95}$$

$$\kappa(0) = 1, \qquad \kappa'(0) = 1. \tag{96}$$

If we take the Laplace transform method of both sides of the Eq. (95) then we get that

$$z^2\mathcal{L}[\kappa]-z\kappa(0)-\kappa'(0)=-\mathcal{L}[\frac{\kappa^3}{2}]+\frac{\delta}{2}\mathcal{L}[\kappa]+p_1. \tag{97}$$

The initial conditions of the Eq. (96) imply that

$$\mathcal{L}[\kappa]=\frac{1}{z}+\frac{1}{z^2}-\frac{1}{2z^2}\mathcal{L}[\kappa^3]+\frac{\delta}{2z^2}\mathcal{L}[\kappa]+\frac{p_1}{z^2}. \tag{98}$$

Following the algorithm, if we assume an infinite series solution is of the form $\kappa=\sum_{n=0}^{\infty}\kappa_n$, then we obtain that

$$\mathcal{L}[\sum_{n=0}^{\infty}\kappa_n]=\frac{1}{z}+\frac{1}{z^2}-\frac{1}{2z^2}\mathcal{L}[\sum_{n=0}^{\infty}Q_n]+\frac{\delta}{2z^2}\mathcal{L}[\sum_{n=0}^{\infty}\kappa_n]+\frac{p_1}{z^2}, \tag{99}$$

where the nonlinear operator $f(\kappa)=\kappa^3$ is decomposed as in terms of the Adomian polynomials. In general, we have

$$\begin{aligned}\mathcal{L}[\kappa_0] &= \frac{1}{z}+\frac{1}{z^2},\\ \mathcal{L}[\kappa_{n+1}] &= -\frac{1}{2z^2}\mathcal{L}[\sum_{n=0}^{\infty}Q_n]+\frac{\delta}{2z^2}\mathcal{L}[\sum_{n=0}^{\infty}\kappa_n]+\frac{p_1}{z^2},\ n\geq 0.\end{aligned} \tag{100}$$

The first four Adomian polynomials for $f(\kappa)=\kappa^3$ are given by

$$\begin{aligned}Q_0 &= \kappa_0^3,\\ Q_1 &= 3\kappa_0^2\kappa_1,\\ Q_2 &= 3\kappa_0^2\kappa_2+3\kappa_0\kappa_1^2,\\ Q_3 &= 3\kappa_0^2\kappa_3+6\kappa_0\kappa_1\kappa_2+\kappa_1^3,\\ &\ \ \ldots\end{aligned} \tag{101}$$

By operating with the Laplace inverse on both sides of the Eq. (100) , we compute that

$$
\begin{aligned}
\mathcal{L}[\kappa_0] &= \frac{1}{z}+\frac{1}{z^2}, \\
\kappa_0 &= 1+s, \\
\mathcal{L}[\kappa_1] &= -\frac{1}{2z^2}\mathcal{L}[Q_0]+\frac{\delta}{2z^2}\mathcal{L}[\kappa_0]+\frac{p_1}{z^2} \\
&= -\frac{1}{2z^2}[\frac{1}{z}+\frac{3}{z^2}+\frac{6}{z^3}+\frac{6}{z^4}]+\frac{\delta}{2z^2}[\frac{1}{z}+\frac{1}{z^2}]+\frac{p_1}{z^2}, \\
\kappa_1 &= -\frac{1}{2}[\frac{s^2}{2}+\frac{s^3}{2}+\frac{s^4}{4}+\frac{s^5}{20}]+\frac{\delta}{2}[\frac{s^2}{2}+\frac{s^3}{6}]+sp_1, \\
\mathcal{L}[\kappa_2] &= -\frac{1}{2z^2}\mathcal{L}[Q_1]+\frac{\delta}{2z^2}\mathcal{L}[\kappa_1]+\frac{p_1}{z^2}, \\
& ...
\end{aligned} \tag{102}
$$

Finally, the approximate solution is given by

$$
\begin{aligned}
\kappa &= \kappa_0+\kappa_1+\kappa_2+... \\
&= 1+s-\frac{1}{2}[\frac{s^2}{2}+\frac{s^3}{2}+\frac{s^4}{4}+\frac{s^5}{20}] \\
&\quad +\frac{\delta}{2}[\frac{s^2}{2}+\frac{s^3}{6}]+sc_1+....
\end{aligned} \tag{103}
$$

In the case of the $\mathbf{t}-$elastic magnetic curve, for the purpose of brevity, we prefer to induce the Euler-Lagrange equation given by the Th. 3.4 to the linear equation. Hence, we have the following special linear equations

$$\varsigma_s+\kappa_s=0, \tag{104}$$

$$\varsigma_{ss}\kappa-\kappa_{ss}\varsigma=0. \tag{105}$$

By using the Eq. (104), we have

$$\kappa = y_2+sy_3, \tag{106}$$

$$\varsigma = -sy_3-y_2+y_1, \tag{107}$$

where $y_1,\ y_2,\ y_3$ are constants of integration. For $y_1=y_2=y_3=1,$ we have the following demonstraions of the functions κ and ς.

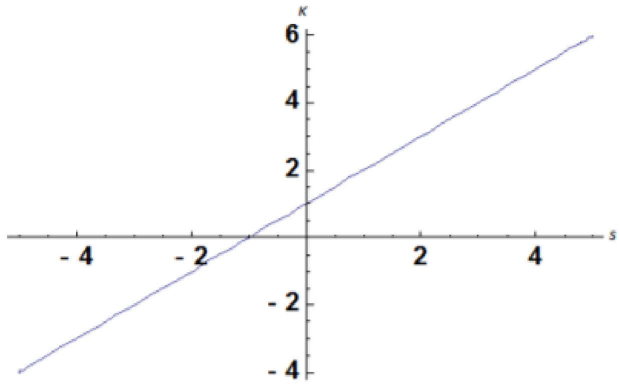

Figure 9. The 2D graphic for the κ of Eqs. $(104, 105)$.

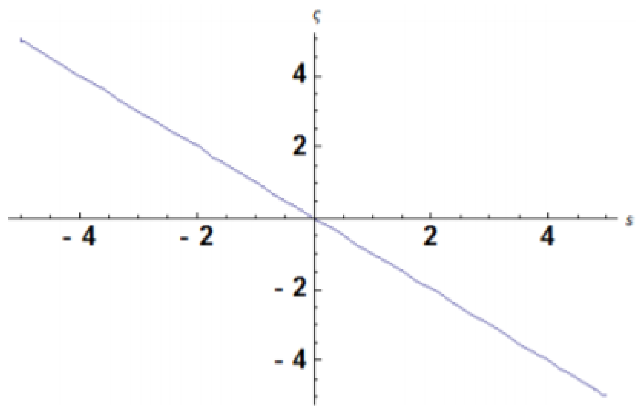

Figure 10. The 2D graphic for the ς of Eqs. $(104, 105)$.

Figures 9-10 show the 2D graphic of the numerical solutions of κ and ς with respect to obtained data.

6 Conclusion

The benign process of computing the critical curves of bending energy functionals led to the research of an enormous family of elastic curves. This significant process has been addressed by many researchers in a series of seminar papers, conferences, research articles, and experimental studies. So far, the representation and analysis of critical curves of space curves have been exploited by considering the classical variational approach. However, in this manuscript, we mainly concentrate on finding the critical curves of magnetic curves of a given magnetic field by improving an alternative variational approach. Our approach deviates from traditional techniques in the fact that the choice of the variational vector is not arbitrary and it is also used the definition of the Lorentz force during the computation process. This distinction leads to investigate surprising physically relevant subjects known as ferromagnetic and superparamagnetic models of magnetically driven elastic curves. This paper is a remarkable first step toward our eventual aim of comprehending the complete generalization of the energy functionals. In this manuscript, an evident omission is the large deformation of elastic curves including the twisting and stretching energy functionals. We plan to concentrate on this issue for future research. This study also plays an outstanding role to describe the elasticity of different types of magnetic curves such as frictional elastic magnetic curves, gravitational elastic magnetic curves, elastic electromagnetic curves, etc. We aim to investigate these curves in different structures to understand the distinct effect of the deformations thoroughly including Minkowski spacetime, De-Sitter spacetime, and anti-De-Sitter spacetime, surfaces, manifolds, etc. We eventually hope to give the geometric characterization of the elastic magnetic surface and membranes formed by the evolution of the elastic magnetic curves and strips.

REFERENCES

[1] L. Euler, Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattisimo sensu accepti, Bousquet, Lausannae et Genevea E 65A., O.O. Ser. I **24** (1744).

[2] J. Radon, Zum Problem von Lagrange. 4 Vorträge, gehalten im Mathematischen Seminar der Hamburgischen Universität (7-24 Juli 1928) (German) JFM 54.0530.01. Abhandlungen Hamburg **6** (1928) 273-299.

[3] J. Langer, D.A. Singer, Lagrangian aspect of the Kirchhoff elastic rod, SIAM Rev. **38** (1986) 605-618.

[4] J. Langer, D.A. Singer, The total squared curvature of closed curves, Journal of Differential Geometry **20**(1) (1984) 1-22.

[5] D.A. Singer, Lectures on elastic curves and rods, AIP Conference Proceedings: American Institute of Physics **1002**(1) (2008) 3-32.

[6] O. Ameline, S. Haliyo, X. Huang, J.A. Cognet, Classifications of ideal 3D elastica shapes at equilibrium, Journal of Mathematical Physics **58**(6)

(2017) 062902.

[7] R. Huang, A note on the p-elastica in a constant sectional curvature manifold, Journal of Geometry and Physics **49** (2004) 343-349.

[8] O.J. Garay, S. Montaldo, Elasticae in Killing Submersions, Mediterranean Journal of Mathematics **13**(5) (2016) 3281-3302.

[9] B. Palmer, Á. Pámpano, Anisotropic bending energies of curves, Annals of Global Analysis and Geometry **57**(2) (2020) 257-287.

[10] K.S. Thorne, R.D. Blandford, Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics. Princeton University Press (2017).

[11] Ó.J. Garay, Extremals of the generalized Euler-Bernoulli energy and applications, Proceedings of the Tenth International Conference on Geometry, Integrability and Quantization. Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences (2009) 56-87.

[12] M. Barros, J.L. Cabrerizo, M. Fernández, A. Romero, Magnetic vortex filament flows, Journal of Mathematical Physics **48**(8) (2007) 082904.

[13] M. Barros, A. Ferrández, Ó.J. Garay, Dynamics of charges and solitons, Journal of Geometry and Physics **125** (2018) 12-22.

[14] T. Körpınar, R.C. Demirkol, Z. Körpınar, Soliton propagation of electromagnetic field vectors of polarized light ray traveling in a coiled optical fiber in Minkowski space with Bishop equations, The European Physical Journal D **73**(9) (2019) 203.

[15] T. Körpınar, R.C. Demirkol, Z. Körpınar, Soliton propagation of electromagnetic field vectors of polarized light ray traveling in a coiled optical fiber in the ordinary space, International Journal of Geometric Methods in Modern Physics **16**(08) (2019) 1950117.

[16] T. Körpınar, R.C. Demirkol, Z. Körpınar, Soliton propagation of electromagnetic field vectors of polarized light ray traveling along with coiled optical fiber on the unit 2-sphere $\mathbb{S}^2$, Revista Mexicana de Física **65**(6) (2019) 626-633.

[17] T. Körpınar, R.C. Demirkol, On the new approach for the energy of elastica, Acta Scientiarum. Technology **40** (2018) e35493-e35493.

[18] T. Körpınar, R.C. Demirkol, A New Approach on the Energy of Elastica and Non-Elastica in Minkowski Space $\mathbb{E}_2^4$, Bulletin of the Brazilian Mathematical Society, New Series **49**(1) (2018) 159-177.

[19] Z. Bozkurt, I. Gök, Y. Yaylı, F.N. Ekmekci, A new approach for magnetic curves in 3D Riemannian manifolds, Journal of Mathematical Physics **55**(5) (2014) 053501.

[20] A. Cebers, K. Erglis, Flexible magnetic filaments and their applications, Advanced Functional Materials **26**(22) (2016) 3783-3795.

[21] A. M. Wazwaz, The combined Laplace transform–Adomian decomposition method for handling nonlinear Volterra integro–differential equations, Applied Mathematics and Computation **216**(4) (2010) 1304-1309.

[21] M. Khan, M. Hussain, Application of Laplace decomposition method on semi-infinite domain, Numer Algorithms **56** (2011) 211–218.