BINORMAL SCHRODINGER EVOLUTION OF WAVE POLARIZATION VECTOR OF LIGHT IN THE NORMAL DIRECTION

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Abstract. In this paper, we mainly focus on the theory of evolution of wave polarization in the normal direction of the curved path, which is assumed to be the trajectory of the propagated light beam. The polarization state of the wave is described by the unit complex transverse field component by eliminating the longitudinal field component, which reduces the dimension of the problem. A Coriolis term is also effectively used to describe the relationship between the geometric phase and the parallel transport law of the wave polarization vector of the evolving light beam in the normal direction of the curved path. We further present a unified geometric interpretation of the binormal evolution of the wave polarization vector in the normal direction of the curved path via the nonlinear Schrodinger equation of repulsive type. Finally, we conclude these discussions by investigating the analytic solutions of the nonlinear Schrodinger equation of repulsive type, which represents binormal evolution of the polarization vector in the normal direction of the curved path trajectory, for some special cases by using the traveling wave hypothesis approach.

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1 Introduction

The exploration of the geometric phase in the late of 1970s raised attention in global geometrical and physical structures such as semiclassical equations of motion and topological monopoles with an emphasis on the evolution of quantum particles. The semiclassical equations in quantum mechanics is a counterpart of the geometrical optics approximation for the propagated light wave in an inhomogeneous medium. Electromagnetic waves naturally possess spin or polarization that causing the angular momentum transported by light. The wave polarization and its geometric phase are also crucial components to describe the optical Magnus effect or spin Hall effect of the light beam $[1 - 10]$. To measure the wave polarization and compute its geometric phase it is important to choose an appropriate coordinate frame having unit orthonormal vectors accompanying the light wave. This choice also affects the gauge of the geometric potential and the Stokes parameters at each point of the trajectory. Bliokh et al. managed to observe directly the precession of the Stock vector and the spin Hall effect in the evolution of light, which are perfectly agree with the theoretical predictions [11].

The flow of vortex filaments or the motion of curves in the ordinary space was firstly defined by Da Rios while he was examining the evolution of one dimensional filament of vortices in an incompressible fluid. This study leads to great interest and has been investigated from various perspectives by many scientists. For instance, Hasimoto [12] calculated a direct correlation between the nonlinear Schrodinger (NLS) equation and the vortex flow. Lamb [13] improved a beautiful formulation that expanded this correlation by including some other integrable equations. Later, Lakshmanan [14] described a Hasimoto transformation to show the equivalence relation between the Heisenberg ferromagnet model and the NLS equation in the ordinary space. The link between the sine-Gordon equation and the binormal motion of curves was discovered by Mukherjee and Balakrishnan [15]. We also showed that new kinds of binormal motions and related transformations can be used to characterize the evolution of the electric and magnetic field vectors on the monochromatic light wave coupling into an optical fiber and their solutions of evolution equations are associated with the solutions of the NLS type equations $[16 - 20]$.

The paper is organized as follows. Section 2 gives a short introduction to the geometric characterization of three-dimensional vector fields. It also introduces the binormal evolution of the coordinate frame's unit orthonormal vectors accompanying the light wave. Section 3 deals with the various aspect of the evolution of wave polarization vector in the normal direction of the curved path including the geometric phase, Coriolis term, and parallel transportation law. It also analyses a connection between the binormal evolution of the polarization vector in the normal direction of the curved path trajectory and the nonlinear Schrodinger equation of repulsive type.

Here, we finally present a unique approach to solve the nonlinear differential equation systems belonging to the binormal evolution of the wave polarization vector. In the final Section 4, we discuss fundamental geometric and physical interpretations of the binormal Schrodinger evolution of the wave polarization vector and mention the possible other applications.

2 Binormal motion of the coordinate frame's vector along the curved path in the normal direction

In this introductory section, we recall some of the formulae which are used to characterize a three-dimensional vector field and the geometry of curvature and torsion of vector lines in terms of anholonomic coordinates. Assuming that the light beam follows the curved path $\theta = \theta(s, n, b)$ whose trajectory corresponds to a three-dimensional space curve in the ordinary space, where s is the distance along the $s-lines$ of the curve in the tangential direction so that unit tangent vector of $s-lines$ is defined by $\overrightarrow{\mathbf{t}} = \overrightarrow{\mathbf{t}}(s, n, b) = \frac{\partial \theta}{\partial s}$, n is the distance along the $n-lines$ of the curve in the normal direction so that unit tangent vector of $n-lines$ is defined by $\overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{n}}(s, n, b) = \frac{\partial \theta}{\partial n}$, b is the distance along the $b-lines$ of the curve in the binormal direction so that unit tangent vector of $b-lines$ is defined by $\overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{b}}(s, n, b) = \frac{\partial \theta}{\partial b}$, the moving trihedron of orthonormal unit vectors $(\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$ provides a platform for investigating the intrinsic features of the light beam. Here, $\overrightarrow{\mathbf{t}}$ denotes the tangential vector, $\overrightarrow{\mathbf{n}}$ denotes the normal vector, and $\overrightarrow{\mathbf{b}}$ denotes the binormal vector of the curved path θ. Directional derivatives of the moving trihedron of the orthonormal unit vectors $(\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$ can be given by the extended

Serret-Frenet relations in the following forms [21, 22].

$$\frac{\partial}{\partial s}\begin{pmatrix}\overrightarrow{\mathbf{t}}\\ \overrightarrow{\mathbf{n}}\\ \overrightarrow{\mathbf{b}}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{\mathbf{t}}\\ \overrightarrow{\mathbf{n}}\\ \overrightarrow{\mathbf{b}}\end{pmatrix}, \tag{1}$$

$$\frac{\partial}{\partial n}\begin{pmatrix}\overrightarrow{\mathbf{t}}\\ \overrightarrow{\mathbf{n}}\\ \overrightarrow{\mathbf{b}}\end{pmatrix} = \begin{pmatrix}0 & \delta_{ns} & \pi_b+\tau\\ -\delta_{ns} & 0 & -\overrightarrow{\mathbf{b}}\\ -(\pi_b+\tau) & div\,\overrightarrow{\mathbf{b}} & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{\mathbf{t}}\\ \overrightarrow{\mathbf{n}}\\ \overrightarrow{\mathbf{b}}\end{pmatrix}, \tag{2}$$

$$\frac{\partial}{\partial b}\begin{pmatrix}\overrightarrow{\mathbf{t}}\\ \overrightarrow{\mathbf{n}}\\ \overrightarrow{\mathbf{b}}\end{pmatrix} = \begin{pmatrix}0 & -(\pi_n+\tau) & \delta_{bs}\\ (\pi_n+\tau) & 0 & \kappa+div\,\overrightarrow{\mathbf{n}}\\ -\delta_{bs} & -\left(\kappa+div\,\overrightarrow{\mathbf{n}}\right) & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{\mathbf{t}}\\ \overrightarrow{\mathbf{n}}\\ \overrightarrow{\mathbf{b}}\end{pmatrix}, \tag{3}$$

where $\delta_{ns} = \overrightarrow{\mathbf{n}}\frac{\partial}{\partial n}\overrightarrow{\mathbf{t}}$ and $\delta_{bs} = \overrightarrow{\mathbf{b}}\frac{\partial}{\partial b}\overrightarrow{\mathbf{t}}$. The gradient operator ∇ is expressed by

$$\nabla = \overrightarrow{\mathbf{t}}\frac{\partial}{\partial s} + \overrightarrow{\mathbf{n}}\frac{\partial}{\partial n} + \overrightarrow{\mathbf{b}}\frac{\partial}{\partial b}. \tag{4}$$

Thus, other geometric quantities are computed by the vector analysis formulae in the following manner.

$$div\,\overrightarrow{\mathbf{t}} = \nabla\cdot\overrightarrow{\mathbf{t}} = \delta_{ns}+\delta_{bs}, \tag{5}$$

$$div\,\overrightarrow{\mathbf{n}} = \nabla\cdot\overrightarrow{\mathbf{n}} = -\kappa + \overrightarrow{\mathbf{b}}\cdot\frac{\partial}{\partial b}\overrightarrow{\mathbf{n}}, \tag{6}$$

$$div\,\overrightarrow{\mathbf{b}} = \nabla\cdot\overrightarrow{\mathbf{b}} = -\overrightarrow{\mathbf{b}}\cdot\frac{\partial}{\partial n}\overrightarrow{\mathbf{n}}, \tag{7}$$

$$curl\,\overrightarrow{\mathbf{t}} = \nabla\times\overrightarrow{\mathbf{t}} = \pi_s\overrightarrow{\mathbf{t}} + \kappa\overrightarrow{\mathbf{b}}, \tag{8}$$

$$curl\,\overrightarrow{\mathbf{n}} = \nabla\times\overrightarrow{\mathbf{n}} = -(div\,\overrightarrow{\mathbf{b}})\overrightarrow{\mathbf{t}} + \pi_n\overrightarrow{\mathbf{n}} + \delta_{ns}\overrightarrow{\mathbf{b}}, \tag{9}$$

$$curl\,\overrightarrow{\mathbf{b}} = \nabla\times\overrightarrow{\mathbf{b}} = (\kappa+div\,\overrightarrow{\mathbf{n}})\overrightarrow{\mathbf{t}} - \delta_{bs}\overrightarrow{\mathbf{n}} + \pi_b\overrightarrow{\mathbf{b}}, \tag{10}$$

where

$$\pi_s = curl\,\overrightarrow{\mathbf{t}}\cdot\overrightarrow{\mathbf{t}} = \overrightarrow{\mathbf{b}}\cdot\frac{\partial}{\partial n}\overrightarrow{\mathbf{t}} - \overrightarrow{\mathbf{n}}\cdot\frac{\partial}{\partial b}\overrightarrow{\mathbf{t}}, \tag{11}$$

$$\pi_n = curl\,\overrightarrow{\mathbf{n}}\cdot\overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{t}}\cdot\frac{\partial}{\partial b}\overrightarrow{\mathbf{n}} - \tau, \tag{12}$$

$$\pi_b = curl\,\overrightarrow{\mathbf{b}}\cdot\overrightarrow{\mathbf{b}} = -\tau - \overrightarrow{\mathbf{t}}\cdot\frac{\partial}{\partial n}\overrightarrow{\mathbf{b}}, \tag{13}$$

which are called abnormalities of the $\overrightarrow{\mathbf{t}} - field$, $\overrightarrow{\mathbf{n}} - field$, $\overrightarrow{\mathbf{b}} - field$, respectively.

Now, we can define the binormal evolution of curved path $\theta(s,n,b;t)$, which characterizes the trajectory of the light beam, in the normal direction. For this purpose, we assume that the tangent vector of the curved path $\theta(s,n,b;t)$ satisfies the following identity.

$$\overrightarrow{\mathbf{t}}_t(s,n,b;t) = \overrightarrow{\mathbf{t}}(s,n,b;t) \times \frac{\partial^2}{\partial n^2}\overrightarrow{\mathbf{t}}(s,n,b;t). \tag{14}$$

Thus the binormal evolution of the tangent vector of the curved path $\theta(s,n,b;t)$ is induced to the following form by using Eqs. $(2,14)$.

$$\begin{aligned}\frac{\partial}{\partial t}\overrightarrow{\mathbf{t}}(s,n,b;t) &= (\delta_{ns} div\overrightarrow{\mathbf{b}}(s,n,b;t) - \frac{\partial}{\partial n}(\pi_b+\tau))\overrightarrow{\mathbf{n}}(s,n,b;t) \\ &+(\frac{\partial}{\partial n}\delta_{ns} + (\pi_b+\tau)div\overrightarrow{\mathbf{b}}(s,n,b;t))\overrightarrow{\mathbf{b}}(s,n,b;t).\end{aligned} \tag{15}$$

Here, we can further compute the evolution of the normal and binormal vectors of the curved path $\theta(s,n,b;t)$. From the nature of the orthonormality of Frenet-Serret vectors, it is canonically true that

$$\overrightarrow{\mathbf{t}}(s,n,b;t)\cdot\overrightarrow{\mathbf{n}}(s,n,b;t) = 0, \tag{16}$$
$$\overrightarrow{\mathbf{t}}(s,n,b;t)\cdot\overrightarrow{\mathbf{b}}(s,n,b;t) = 0, \tag{17}$$

and

$$\overrightarrow{\mathbf{n}}(s,n,b;t)\cdot\overrightarrow{\mathbf{n}}(s,n,b;t) = 1, \tag{18}$$
$$\overrightarrow{\mathbf{b}}(s,n,b;t)\cdot\overrightarrow{\mathbf{b}}(s,n,b;t) = 1. \tag{19}$$

Thus, if we take the derivative of both identities in Eqs. $(16,17)$ and $(18,19)$ with respect to t then it is obtained that

$$\frac{\partial}{\partial t}\overrightarrow{\mathbf{t}}(s,n,b;t)\cdot\overrightarrow{\mathbf{n}}(s,n,b;t) = -\overrightarrow{\mathbf{t}}(s,n,b;t)\cdot\frac{\partial}{\partial t}\overrightarrow{\mathbf{n}}(s,n,b;t), \tag{20}$$
$$\frac{\partial}{\partial t}\overrightarrow{\mathbf{t}}(s,n,b;t)\cdot\overrightarrow{\mathbf{b}}(s,n,b;t) = -\overrightarrow{\mathbf{t}}(s,n,b;t)\cdot\frac{\partial}{\partial t}\overrightarrow{\mathbf{b}}(s,n,b;t), \tag{21}$$

and

$$\frac{\partial}{\partial t}\overrightarrow{\mathbf{n}}(s,n,b;t)\cdot\overrightarrow{\mathbf{n}}(s,n,b;t) = 0, \tag{22}$$
$$\frac{\partial}{\partial t}\overrightarrow{\mathbf{b}}(s,n,b;t)\cdot\overrightarrow{\mathbf{b}}(s,n,b;t) = 0. \tag{23}$$

Finally, we can construct the binormal evolution of Frenet-Serret vectors in the normal direction as in the following form.

$$\frac{\partial}{\partial t}\overrightarrow{\mathbf{n}}(s,n,b;t) = (-\delta_{ns} div\overrightarrow{\mathbf{b}}(s,n,b;t) + \frac{\partial}{\partial n}(\pi_b+\tau))\overrightarrow{\mathbf{t}}(s,n,b;t) + \Lambda\overrightarrow{\mathbf{b}}(s,n,b;t)\,, \tag{24}$$

$$\frac{\partial}{\partial t}\overrightarrow{\mathbf{b}}(s,n,b;t) = (-\frac{\partial}{\partial n}\delta_{ns} - (\pi_b+\tau) div\overrightarrow{\mathbf{b}}(s,n,b;t))\overrightarrow{\mathbf{t}}(s,n,b;t) - \Lambda\overrightarrow{\mathbf{n}}(s,n,b;t)\,, \tag{25}$$

where Λ is a sufficiently smooth well-defined arbitrary function defined along the curved path $\theta(s,n,b;t)$. From now on, we will use following notations for the t-parameter included vectors for the purpose of brevity and clarity.

$$\overrightarrow{\mathbf{t}} = \overrightarrow{\mathbf{t}}(s,n,b;t)\,,\ \overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{n}}(s,n,b;t)\,,\ \overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{b}}(s,n,b;t)\,. \tag{26}$$

3 Evolution of the wave polarization along the curved path in the normal direction

The evolution of electromagnetic vector waves requires advanced techniques since it includes both external and internal degrees of freedom of the wave. Geometrodynamical evolution equations of monochromatic electric field $\overrightarrow{\mathbf{E}}$ in a smooth inhomogeneous medium satisfies the following type of Maxwell equations.

$$(\lambda_0^2\nabla^2 + n^2)\overrightarrow{\mathbf{E}} - \lambda_0^2\nabla(\nabla\overrightarrow{\mathbf{E}}) = 0, \tag{27}$$

where $n^2 = \epsilon$ is the dielectric constant of the medium. The Eq. (27) has almost similar interpretations with the Helmholz equation. However, Eq. (27) involves wave polarization term different from the Helmholz equation, which also makes the Eq. (27) non-diagonal. Thus, it is guaranteed that the electric field of the polarized wave remains almost transverse with respect to the momentum:

$$\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E}}_\perp + \mathbf{E}_\parallel\overrightarrow{\mathbf{t}},\ \overrightarrow{\mathbf{E}}_\perp \perp \overrightarrow{\mathbf{t}},\ \left|\mathbf{E}_\parallel\right| \ll \left|\mathbf{E}_\perp\right|. \tag{28}$$

Here, $\overrightarrow{\mathbf{E}}_\perp$ is the projection of the electric field on the normal plane, $\mathbf{E}_\parallel = \overrightarrow{\mathbf{E}}\overrightarrow{\mathbf{t}}$ is the longitudinal component of the field, and $\overrightarrow{\mathbf{t}}$ is the unit tangent vector of the ray trajectory. For the rest of the paper, we eliminate the longitudinal field component $\mathbf{E}_{\parallel}$ since the wave polarization is fundamentally determined by the transverse field components $\overrightarrow{\mathbf{E}}_{\perp}$. After a short introductory, we are ready to concentrate on computing the evolution of the wave polarization vector in the normal direction.

In order to observe the evolution of the wave polarization the first immediate action should be introduced a coordinate frame to describe the evolution of the transverse field $\overrightarrow{\mathbf{E}}_{\perp}$. We will consider the Frenet-Serret coordinate frame whose orthonormal unit vectors are denoted by $(\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$. Then it can be derived that $(\overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$ vectors provide an inherent basis for linear polarizations:

$$\overrightarrow{\mathbf{E}}_{\perp} = E^{+}\overrightarrow{\varpi} + E^{-}\overrightarrow{\varpi}^{*},\ \varpi = \frac{\overrightarrow{\mathbf{n}} + i\overrightarrow{\mathbf{b}}}{\sqrt{2}},\ E^{\pm} = \frac{E^{n} \mp iE^{b}}{\sqrt{2}}, \tag{29}$$

where $*$ denotes the complex conjugate. As the unit tangent vector $\overrightarrow{\mathbf{t}}$ varies in the normal direction of the curved path in an inhomogeneous medium the coordinate frame $(\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$ experiences a rotation with some angular velocity ${}_{n}\overrightarrow{\mathbf{\Omega}}$, which satisfies following identities.

$$\frac{\partial}{\partial n}\overrightarrow{\mathbf{t}} = {}_{n}\overrightarrow{\mathbf{\Omega}} \times \overrightarrow{\mathbf{t}},\ \frac{\partial}{\partial n}\overrightarrow{\mathbf{n}} = {}_{n}\overrightarrow{\mathbf{\Omega}} \times \overrightarrow{\mathbf{n}},\ \frac{\partial}{\partial n}\overrightarrow{\mathbf{b}} = {}_{n}\overrightarrow{\mathbf{\Omega}} \times \overrightarrow{\mathbf{b}}, \tag{30}$$

$${}_{n}\overrightarrow{\mathbf{\Omega}} = (\frac{\partial}{\partial n}\overrightarrow{\mathbf{t}} \cdot \overrightarrow{\mathbf{n}})\overrightarrow{\mathbf{b}} + (\frac{\partial}{\partial n}\overrightarrow{\mathbf{n}} \cdot \overrightarrow{\mathbf{b}})\overrightarrow{\mathbf{t}} + (\frac{\partial}{\partial n}\overrightarrow{\mathbf{b}} \cdot \overrightarrow{\mathbf{t}})\overrightarrow{\mathbf{n}}. \tag{31}$$

If we consider Eqs. $(2, 30, 31)$ then the angular velocity vector is computed by

$${}_{n}\overrightarrow{\mathbf{\Omega}} = (-div\,\overrightarrow{\mathbf{b}})\overrightarrow{\mathbf{t}} - (\pi_b + \tau)\overrightarrow{\mathbf{n}} + \delta_{ns}\overrightarrow{\mathbf{b}}. \tag{32}$$

Thus the longitudinal component of the angular velocity vector in the normal direction is also computed by

$${}_{n}\overrightarrow{\mathbf{\Omega}} = {}_{n}\overrightarrow{\mathbf{\Omega}}_{\parallel} \cdot \overrightarrow{\mathbf{t}} + \overrightarrow{\mathbf{t}} \times \frac{\partial}{\partial n}\overrightarrow{\mathbf{t}}, \tag{33}$$

${}_{n}\overrightarrow{\mathbf{\Omega}}_{\parallel} = -div\,\overrightarrow{\mathbf{b}}$. Coriolis effect is known to be caused by the rotation of the coordinate frame defined along the curved path. The Coriolis term is determined by ${}_{n}\overrightarrow{\mathbf{\Omega}} \times \overrightarrow{\mathbf{e}}$ and it can be considered to find differentiation of the wave polarization field vector $(\overrightarrow{\mathbf{e}})$ in the normal direction of the curved path that the light beam supposed to follow.

$$\frac{\partial}{\partial n}\overrightarrow{\mathbf{e}} = ({}_{n}\overrightarrow{\mathbf{\Omega}} - {}_{n})\overrightarrow{\mathbf{\Omega}}_{\parallel} \times \overrightarrow{\mathbf{e}}, \tag{34}$$

where $\overrightarrow{\mathbf{e}} = \frac{\overrightarrow{\mathbf{E}}_{\perp}}{\mathbf{E}_{\perp}}$. We will assume $\mathbf{E}_{\perp} = 1$ for the rest of the paper due to the simplicity reason. If we consider Eqs. $(2, 28, 29, 32, 34)$ then it is obtained that

$$\frac{\partial}{\partial n}\overrightarrow{\mathbf{e}} = (k_0\varphi_0 + k_1\varphi_1)\overrightarrow{\mathbf{t}}, \tag{35}$$

where

$$\begin{aligned} \varphi_0 &= -\frac{1}{\sqrt{2}}e^{-i\int_0^n div\overrightarrow{\mathbf{b}}dn}(\delta_{ns} + i(\pi_b + \tau)), && (36) \\ \varphi_1 &= \frac{1}{\sqrt{2}}e^{i\int_0^n div\overrightarrow{\mathbf{b}}dn}(-\delta_{ns} + i(\pi_b + \tau)), && (37) \end{aligned}$$

Here we also calculate that

$$\frac{\partial}{\partial n}E^{+} = -iE^{+}div\overrightarrow{\mathbf{b}}k_0, \ \frac{\partial}{\partial n}E^{-} = iE^{-}div\overrightarrow{\mathbf{b}}k_1, \tag{38}$$

which implies that

$$E^{+} = e^{-i\int_0^n div\overrightarrow{\mathbf{b}}dn}k_0, \ E^{-} = e^{i\int_0^n div\overrightarrow{\mathbf{b}}dn}k_1,$$

where k_0, k_1 are chosen as arbitrary constants and

$$\frac{\partial}{\partial n}E^{n} = -E^{b}div\overrightarrow{\mathbf{b}}, \ \frac{\partial}{\partial n}E^{b} = E^{n}div\overrightarrow{\mathbf{b}}. \tag{39}$$

Eqs. $(35 - 40)$ has significant implications on estimating the polarization measured in the ray-accompanying coordinate frame $(\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$ and describing the polarization dynamics. According to these equations, we firstly arrive that $\overrightarrow{\mathbf{e}}$ does not rotate locally around $\overrightarrow{\mathbf{t}}$ in the normal direction and it satisfies a new kind of parallel transportation law given below.

$$\frac{\partial}{\partial n}\overrightarrow{\mathbf{e}} = -(\overrightarrow{\mathbf{e}} \cdot \frac{\partial}{\partial n}\overrightarrow{\mathbf{t}})\overrightarrow{\mathbf{t}}. \tag{40}$$

We also verify that polarization measured in the ray-accompanying $(\overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$ plane rotates by an angle $\pm div\overrightarrow{\mathbf{b}}$, which is equal to the longitudinal component of the angular velocity vector in the normal direction as we expected. Thus, the divergence of binormal vector $\overrightarrow{\mathbf{b}}$ characterizes local rotation of polarization vector about the tangent vector $\overrightarrow{\mathbf{t}}$, and the geometric phase of the polarization vector in the normal direction is

$${}_{n}\Phi = \sigma\int_0^n div\overrightarrow{\mathbf{b}}dn, \tag{41}$$

where $\sigma = \pm 1$. Now, we will check the consequences of the binormal evolution of the polarization vector in the normal direction along the curved path. If we take into account Eqs. $(2, 15, 24, 25, 36, 37)$ then the evolution of the polarization vector in the normal direction with respect to t parameter is computed by the following:

$$\frac{\partial}{\partial t}\overrightarrow{\mathbf{e}} = i(k_0 \frac{\partial}{\partial n}\varphi_0 - k_1 \frac{\partial}{\partial n}\varphi_1)\overrightarrow{\mathbf{t}} + \Lambda\overrightarrow{\mathbf{\Psi}}_1 + \Lambda\overrightarrow{\mathbf{\Psi}}_2, \tag{42}$$

where Λ is a sufficiently smooth well-defined arbitrary function along the curved path and

$$\overrightarrow{\mathbf{\Psi}}_1 = -i\frac{k_0}{\sqrt{2}} e^{-i\int_0^n div\overrightarrow{\mathbf{b}}dn}(\overrightarrow{\mathbf{n}} + i\overrightarrow{\mathbf{b}}), \tag{43}$$

$$\overrightarrow{\mathbf{\Psi}}_2 = i\frac{k_1}{\sqrt{2}} e^{i\int_0^n div\overrightarrow{\mathbf{b}}dn}(\overrightarrow{\mathbf{n}} - i\overrightarrow{\mathbf{b}}). \tag{44}$$

Here, $\overrightarrow{\mathbf{\Psi}}_1$ and $\overrightarrow{\mathbf{\Psi}}_2$ are called binormal Hasimoto transformations in the normal direction of the curved path. If we further consider the compatibility and integrability conditions of the polarization vector $(\frac{\partial}{\partial t}\frac{\partial}{\partial n}\overrightarrow{\mathbf{e}} = \frac{\partial}{\partial n}\frac{\partial}{\partial t}\overrightarrow{\mathbf{e}})$ together with Eqs. $(2, 35-37, 42-44)$ then we obtain that

$$\frac{\partial}{\partial t}\varphi_0 = i\frac{\partial^2}{\partial n^2}\varphi_0 - i\Lambda\varphi_0, \tag{45}$$

$$\frac{\partial}{\partial t}\varphi_1 = -i\frac{\partial^2}{\partial n^2}\varphi_1 + i\Lambda\varphi_1. \tag{46}$$

Thus, it can be observed that Eqs. $(45, 46)$ are exactly the nonlinear Schrodinger equations of repulsive type. Thus, we can conclude that the binormal evolution of the polarization vector in the normal direction of the curved path trajectory is equivalent to the nonlinear Schrodinger equation of repulsive type.

Finally, we can sum up these discussions by investigating the analytic solutions of the nonlinear Schrodinger equation of repulsive type, which represents binormal evolution of the polarization vector in the normal direction of the curved path trajectory, for some special cases by using the traveling wave hypothesis approach.

In the first case, we assume that the arbitrary function Λ given in the Eqs. $(45, 46)$ is selected in the following way.

$$\Lambda = -\varphi_0\varphi_1. \tag{47}$$

Hence, by considering the Eqs. $(45, 46, 47)$, we have the following identity.

$$\frac{\partial}{\partial t}\varphi_0 = i\frac{\partial^2}{\partial n^2}\varphi_0 + i\varphi_0^2\varphi_1, \tag{48}$$

$$\frac{\partial}{\partial t}\varphi_1 = -i\frac{\partial^2}{\partial n^2}\varphi_1 - i\varphi_0\varphi_1^2. \tag{49}$$

Here we consider the traveling wave transformation for Eqs. $(48, 49)$ by using expressions given below.

$$\varphi_0 = u(\phi), \tag{50}$$

$$\varphi_1 = w(\phi), \qquad \phi = n - Qt, \tag{51}$$

where Q describe the speed of the wave. If we plug the Eqs. $(50, 51)$ into the Eqs. $(48, 49)$ and consider imaginary section then it is obtained that

$$u''(\phi) + u^2(\phi)w(\phi) + Qu(\phi) - u(\phi) = 0, \tag{52}$$

$$w''(\phi) + u(\phi)w^2(\phi) - Qw(\phi) - w(\phi) = 0. \tag{53}$$

Moreover, solutions of the Eqs. $(52, 53)$ can be written as the series expansion solutions as the following way.

$$u(\phi) = \alpha_0 + \alpha_1 G(\phi) + \alpha_2 G^{-1}(\phi), \tag{54}$$

$$w(\phi) = \beta_0 + \beta_1 G(\phi) + \beta_2 G^{-1}(\phi), \tag{55}$$

where $\alpha_0,\ \alpha_1,\ \alpha_2,\ \beta_0,\ \beta_1,\ \beta_2$ are functions to be determined later and $G(\phi)$ satisfies the fractional Riccati equation given as follows.

$$G'(\phi) = \sigma + G^2(\phi), \tag{56}$$

where σ is an arbitrary constant. N is obtained with the aid of balance between the highest order derivatives and the nonlinear terms in the Eqs. $(52, 53)$. A few special solutions of the Eq. (56) are given by following cases.
1) When $\sigma < 0$, we have

$$G_1(\phi) = -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\phi), \tag{57}$$

$$G_2(\phi) = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\phi), \tag{58}$$

2) When $\sigma > 0$, we have

$$G_3(\phi) = \sqrt{\sigma}\tan(\sqrt{\sigma}\phi), \tag{59}$$

$$G_4(\phi) = \sqrt{\sigma}\cot(\sqrt{\sigma}\phi), \tag{60}$$

3) When $\sigma = 0,\ \rho = const.$, we have

$$G_5(\phi) = -\frac{1}{\phi+\rho}, \tag{61}$$

Now, if we replace the Eqs. (54, 55) and (56) into the Eqs. (52, 53) and equate the all coefficients of $G(\phi)$ then we can solve these equations and obtain the following functions.

$$\alpha_0 = 0,\ \alpha_1 = \frac{(-1+3Q+2\sigma)\alpha_2}{6\sigma^2}, \tag{62}$$

$$\beta_0 = 0,\ \beta_1 = \frac{1+3Q-2\sigma}{3\alpha_2},\ \beta_2 = -\frac{2\sigma^2}{\alpha_2}. \tag{63}$$

If we take $\sigma = 1,\ G(\phi) = \sqrt{\sigma}\tan(\sqrt{\sigma}\phi)$ then we obtain that

$$\begin{aligned} u(\phi) &= \frac{(3Q+1)\alpha_2}{6}\tan(\phi) + \alpha_2(\tan(\phi))^{-1}, \\ w(\phi) &= \frac{3Q-1}{3\alpha_2}\tan(\phi) + -\frac{2}{\alpha_2}(\tan(\phi))^{-1}. \end{aligned}$$

Thus, we get the following set of solution systems.

$$\varphi_0 = \frac{(3Q+1)\alpha_2}{6}\tan(n-Qt) + \alpha_2(\tan(n-Qt))^{-1}, \tag{64}$$

$$\varphi_1 = \frac{3Q-1}{3\alpha_2}\tan(n-Qt) + -\frac{2}{\alpha_2}(\tan(n-Qt))^{-1}. \tag{65}$$

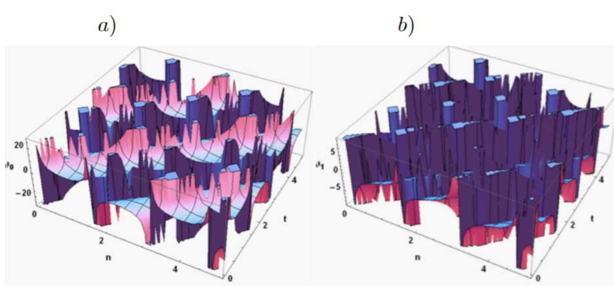

Figure 1: The 3D graphic for analytical solutions of the Eqs. (48, 49) for $Q = 1.2,\ \alpha_2 = 2$. $a)$ for φ_0 in solutions (64, 65), $b)$ for φ_1 in solutions (64, 65).

In the second case, we assume that the arbitrary function Λ is selected by

$$\Lambda = -i\varphi_0\varphi_1. \tag{66}$$

Hence, by considering the Eqs. $(45, 46, 66)$, we have the following identity.

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_0 &= i\frac{\partial^2}{\partial n^2}\varphi_0 - i\varphi_0^2\varphi_1, && (67)\\
\frac{\partial}{\partial t}\varphi_1 &= -i\frac{\partial^2}{\partial n^2}\varphi_1 + i\varphi_0\varphi_1. && (68)
\end{aligned}$$

Here we consider the traveling wave transformation for Eqs. $(67, 68)$ by using expressions given below.

$$\begin{aligned}
\varphi_0 &= u(\phi), && (69)\\
\varphi_1 &= w(\phi), \qquad \phi = n - Qt, && (70)
\end{aligned}$$

where Q describe the speed of the wave. If we plug the Eqs. $(69, 70)$ into the Eqs. $(67, 68)$ and consider imaginary section then it is obtained that

$$\begin{aligned}
u''(\phi) + Qu(\phi) - u(\phi) &= 0, && (71)\\
w''(\phi) - Qw(\phi) - w(\phi) &= 0. && (72)
\end{aligned}$$

By solving the Eqs. $(71, 72)$ we obtain the followings.

$$\begin{aligned}
u(\phi) &= e^{\sqrt{1-Q}\phi}c_1 + e^{-\sqrt{1-Q}\phi}c_2,\\
w(\phi) &= e^{\sqrt{1+Q}\phi}c_3 + e^{-\sqrt{1+Q}\phi}c_4.
\end{aligned}$$

Thus, we get the following set of solution systems.

$$\begin{aligned}
\varphi_0 &= \frac{(3Q+1)\alpha_2}{6}\tan(n-Qt) + \alpha_2(\tan(n-Qt))^{-1}, && (73)\\
\varphi_1 &= \frac{3Q-1}{3\alpha_2}\tan(n-Qt) + -\frac{2}{\alpha_2}(\tan(n-Qt))^{-1}. && (74)
\end{aligned}$$

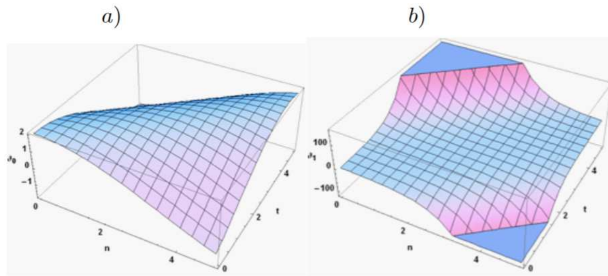

Figure 2: The 3D graphic for analytical solutions of the Eqs. $(66, 67)$ for $Q = 1.2, c_1 = c_2 = c_4 = 1, c_3 = -1$. $a)$ for φ_0 in solutions $(73, 74)$, $b)$ for φ_1 in solutions $(73, 74)$.

4 Conclusion

One of the direct consequences of Maxwell's equations is the geometric phases of the light beam, which has an also decent connection with the parallel transportation law of the electric field. In this study, we improve an alternative approach to derive the geometric phases and parallel transport law of wave polarization vector of light in the normal direction. In this case, the geometric phase is measured by the divergence of the binormal and it leads to a rotation in $(\overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}})$ plane. This paper serves as a basis for further research on investigating the spin Hall effect (SHE) of light in the normal direction since the SHE and the geometric phase are reciprocal concepts. SHE of light is an important phenomenon since it has potential, natural, and dynamical interpretations and applications besides the purely geometric meanings. This paper will also be a helpful source to connect the geometric phase of the wave polarization vector in the normal direction with different gauge fields in different nonadiabatic or adiabatic evolution of light beam. As the wave polarization vector travels along the curved path the geometry of the curve reveals an interesting model to determine its evolution equation in the normal direction via the binormal motion. Interestingly, this evolution equation is found to be associated with the very well-known formula of the Schrodinger equation. We will further investigate such evolution in the binormal direction and aim to complete our series of research on the evolution of wave polarization vector in the tangent, normal and binormal direction.

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