DIRECTIONAL MAGNETIC AND ELECTRIC VORTEX LINES AND THEIR GEOMETRIES

Talat KÖRPINAR[1]**, Rıdvan Cem DEMİRKOL**[2]**, Vedat ASİL**[3]
[1,2]Muş Alparslan University, Department of Mathematics,
49250, Muş, TURKEY.
[3]Fırat University, Department of Mathematics,
23100, Elazığ, TURKEY.

Abstract. In this study, we firstly introduce a different type of directional Fermi-Walker transportations along with vortex lines of a non-vanishing vector field in three-dimensional space. Thus we conclude that geometric quantities, which are used to characterize vortex lines, are also associated with the geometric phase and angular velocity vector (Darboux vector) of the system. Then we present directional magnetic vortex lines by computing the Lorentz force. Hence, we reach a remarkable relation between directional magnetic vortex lines and angular velocity vector of vortex lines with a non-rotating frame. We later determine the directional electric vortex lines by considering the electromagnetic force equation. We finally investigate the conditions of being uniform for magnetic fields of directional magnetic vortex lines and we improve a remarkable approach to find the electromagnetic curvature, which contains many geometrical features belonging to directional electric vortex line.

MSC 2010: 53Z05, 81Q70.
Keywords: Vortex lines; directional Fermi-Walker derivative; geometric phase; angular velocity vector; directional magnetic vortex lines.

1 Introduction

Recently, it has been intensively concentrated on the phenomena connected with topological and geometric features of some parameters evolving in time, which is given in the form of a space curve traced by the quantum or mechanical system. For instance, Berry proved that a quantum system can pick up a topological phase apart from the ordinary dynamical phase. Even though the complete phase of a system is not commonly estimated Berry argued a special case when the phase of a quantum system is observed due to the intervention with the phase of the alternative quantum system [1]. A traditional example for this case can be observed through the polarized light ray propagating along with optical fibers or waveguides. This observation provides detailed information regarding the related phases of the polarized light in both optical fibers and waveguides. Furthermore, phase dependence and geometric nature of the rotational polarized light ray in optical fibers or waveguides can be comprehended by the parallel transportation along with unit vectors of space curves described by optical fibers or waveguides [2, 3].

One of the well-known and highly important parallel transportation is given by the Fermi-Walker derivative. Fermi-Walker transportation is significant to understand various experiments and investigations. As is known an orthogonal frame undergoing rotational or linear acceleration can be determined by the Serret-Frenet frame. This frame together with the Fermi-Walker transportation is useful to characterize some essential physical events. They are effectively used in the investigation of the gravitational wave resonant detectors, in the search of the inertial effects on a Dirac particle, in the study of gyroscopic, Lense-Thirring and geodetic precession [4, 5, 6].

From the physical point of view, space curves are considered as the path or trajectory followed by the state of the system for a given specific parameter. In particular, they could symbolize vortex filaments, waveguides, polymer chains, optical fibers, elastic rods, etc. Many authors focus their attention on space curves to characterize the intrinsic properties of given systems. For example, Dandoloff and Zakrzewski found that two different phase-like quantities could help to represent a space curve and they concluded that these quantities are associated to the Berry phase, which emerges in the action of propagating of a light ray in a waveguide or optical fiber [7]. Dandoloff also defined other possible parallel transportation laws and their relations with the Berry phase [8] . Moreover, it has been investigated that the Berry phase can appear completely theoretical systems. For instance, in an integrable finite-dimensional Hamiltonian system, the straightforward analogue of the Berry phase is given by the Hannay angle [9]. As opposed to the former case if the integrability and existence of time-dependent parameters are not necessarily required then another type of phase is given as the straightforward anologue of the Aharanov-Anandan phase [10].

2 Geometric Background

In the pure geometric context, all phase-like quantities are evenly significant and closely linked. These quantities can be derived by considering the Serret-Frenet triad and Fermi-Walker parallel transportation for a given parameter of a vortex line.

According to this approach $s-$lines, which are vortex lines whose tangent vector is $\mathbf{t}$, is defined by $\mathbf{c} = \mathbf{c}(s)$, where s is treated as the arc-length parameter. Unit Serret-Frenet vectors along with $s-$lines are denoted by $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, which respectively stand for the unit tangent, principal normal and binormal vectors. Thus these vectors form a moving orthogonal frame for $s-$lines, which is known to satisfy the following Serret-Frenet equations.

$$\begin{aligned} \nabla_s \mathbf{t} &= \kappa \mathbf{n}, \\ \nabla_s \mathbf{n} &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \nabla_s \mathbf{b} &= -\tau \mathbf{n}, \end{aligned} \tag{1}$$

where $\kappa = \kappa(s)$, $\tau = \tau(s)$ are the curvature and torsion of the given $s-$lines. Here it is also assumed that $\nabla_s = d/ds$.

The angular velocity of the Serret-Frenet frame is given by the Darboux vector $\mathcal{D}$ and it has components along with unit vectors of $(\mathbf{t}, \mathbf{b})$ forming the rectifying plane and satisfying the following identities.

$$\begin{aligned} \nabla_s \mathbf{t} &= \mathcal{D} \times \mathbf{t}, \\ \nabla_s \mathbf{n} &= \mathcal{D} \times \mathbf{n}, \\ \nabla_s \mathbf{b} &= \mathcal{D} \times \mathbf{b}, \end{aligned} \tag{2}$$

where $\mathcal{D} = \tau \mathbf{t} + \kappa \mathbf{b}$ [11].

Now let us observe the normal plane spanned by unit vectors $(\mathbf{n}, \mathbf{b})$. The angular velocity of the Serret-Frenet triad around the tangent vector $\mathbf{t}$ is given by τ. Hence it can be found that Berry phase is $\mathcal{B}_1 = \int_0^{s_0} \tau(s)\, ds$ between unit vectors $(\mathbf{n}, \mathbf{b})$ and associated non-rotating frame, which is described by considering the ordinary Fermi-Walker derivative ∇° along with $s-$lines in the following way [7].

$$\nabla_s^\circ \mathcal{A} = \nabla_s \mathcal{A} - (\mathbf{t} \cdot \mathcal{A}) \nabla_s \mathbf{t} + (\nabla_s \mathbf{t} \cdot \mathcal{A}) \mathbf{t}, \tag{3}$$

which is induced to

$$\nabla_s \mathcal{A} = \kappa (\mathbf{b} \times \mathcal{A}) \tag{4}$$

if $\mathcal{A}$ is supposed to Fermi-Walker parallel transported.

If one observes the osculating plane spanned by unit vectors $(\mathbf{t}, \mathbf{n})$ then the angular velocity of the Serret-Frenet triad around the binormal vector $\mathbf{b}$ is given by κ. Hence it can be found that modified Berry phase is a $\mathcal{B}_2 = \int_0^{s_0} \kappa(s)\, ds$ between unit vectors $(\mathbf{t}, \mathbf{n})$ and associated non-rotating frame, which is described by considering the modified Fermi-Walker derivative $(\nabla^\circ)^M$ along with $s-$lines in the following way [7].

$$(\nabla^\circ)_s^M \mathcal{A} = \nabla_s \mathcal{A} - (\mathbf{b} \cdot \mathcal{A}) \nabla_s \mathbf{b} + (\nabla_s \mathbf{b} \cdot \mathcal{A}) \mathbf{b}, \tag{5}$$

which is induced to

$$\nabla_s \mathcal{A} = \tau (\mathbf{t} \times \mathcal{A}) \tag{6}$$

if $\mathcal{A}$ is supposed to modified Fermi-Walker parallel transported.

Finally, modified Fermi-Walker parallel transportation and ordinary Fermi-Walker parallel transportation yield the normal Fermi-Walker parallel transportation in the following way.

$$(\nabla^\circ)_s^N \mathcal{A} = \nabla_s \mathcal{A} - (\mathbf{n} \cdot \mathcal{A}) \nabla_s \mathbf{n} + (\nabla_s \mathbf{n} \cdot \mathcal{A}) \mathbf{n}, \tag{7}$$

which is induced to

$$\begin{aligned} \nabla_s \mathcal{A} &= \kappa (\mathbf{b} \times \mathcal{A}) + \tau (\mathbf{t} \times \mathcal{A}) \\ &= \mathcal{D} \times \mathcal{A} \end{aligned} \tag{8}$$

if $\mathcal{A}$ is supposed to normal Fermi-Walker parallel transported [7].

In three-dimensions, the intrinsic characterization of vortex lines is given by the curvature and binormal functions together with the arc-length parameter defined along the vortex lines. However, investigating the intrinsic characterization of a vector field is significantly sophisticated as the vector field may be defined by non-holonomic coordinates which includes much more partial differential equations and parameters. It can be introduced the orthonormal basis at points along with the tangent vector $(\mathbf{t})$, normal vector $(\mathbf{n})$ and binormal vector $(\mathbf{b})$ on a given vortex line of the non-vanishing vector field [12, 13]. Directional derivative in the tangential direction is given by the Eq. (1) and it is known as the Serret-Frenet triad. Directional derivatives in the normal and binormal directions were introduced by Marris and Passman in the following manner [14, 15].

$$\begin{aligned} \nabla_n \mathbf{t} &= \delta_{ns} \mathbf{n} + (\pi_b + \tau) \mathbf{b}, \\ \nabla_n \mathbf{n} &= -\delta_{ns} \mathbf{t} - (\operatorname{div} \mathbf{b}) \mathbf{b}, \\ \nabla_n \mathbf{b} &= -(\pi_b + \tau) \mathbf{t} + (\operatorname{div} \mathbf{b}) \mathbf{n}, \end{aligned} \tag{9}$$

where $\nabla_n = d/dn$ is the directional derivative in the principal normal direction.

$$\begin{aligned}
\nabla_b \mathbf{t} &= -(\pi_n + \tau)\,\mathbf{n} + \delta_{bs}\mathbf{b}, \\
\nabla_b \mathbf{n} &= (\pi_n + \tau)\,\mathbf{t} + (\kappa + \operatorname{div}\mathbf{n})\mathbf{b}, \\
\nabla_b \mathbf{b} &= -\delta_{bs}\mathbf{t} - (\kappa + \operatorname{div}\mathbf{n})\mathbf{n},
\end{aligned} \tag{10}$$

where $\nabla_b = d/db$ is the directional derivative in the binormal direction. Here, ∇ is considered as gradient operator and it can be expressed by

$$\nabla = \mathbf{b}\nabla_b + \mathbf{n}\nabla_n + \mathbf{t}\nabla_s. \tag{11}$$

Geometric constants δ_{ns} and δ_{bs} are defined by Bjorgum [13] as follows.

$$\delta_{ns} = \mathbf{n}\nabla_n\mathbf{t}, \quad \delta_{bs} = \mathbf{b}\nabla_b\mathbf{t}, \tag{12}$$

and

$$\begin{aligned}
(\nabla \cdot \mathbf{t}) &= \operatorname{div}\mathbf{t} = \delta_{ns} + \delta_{bs}, \\
(\nabla \cdot \mathbf{n}) &= \operatorname{div}\mathbf{n} = -\kappa + \mathbf{b}\cdot\nabla_b\mathbf{n}, \\
(\nabla \cdot \mathbf{b}) &= \operatorname{div}\mathbf{b} = -\mathbf{b}\nabla_n\mathbf{n}.
\end{aligned} \tag{13}$$

Moreover,

$$\begin{aligned}
\operatorname{curl}\mathbf{t} &= \pi_s\mathbf{t} + \kappa\mathbf{b}, \\
\operatorname{curl}\mathbf{n} &= -(\operatorname{div}\mathbf{b})\mathbf{t} + \pi_n\mathbf{n} + \delta_{ns}\mathbf{b}, \\
\operatorname{curl}\mathbf{b} &= (\kappa + \operatorname{div}\mathbf{n})\mathbf{t} - \delta_{bs}\mathbf{n} + \pi_b\mathbf{b},
\end{aligned} \tag{14}$$

where

$$\operatorname{curl} = \mathbf{b}\times\nabla_b + \mathbf{n}\times\nabla_b + \mathbf{t}\times\nabla_s, \tag{15}$$

and

$$\begin{aligned}
\operatorname{curl}\mathbf{t}\cdot\mathbf{t} &= \pi_s = \mathbf{b}\cdot\nabla_n\mathbf{t} - \mathbf{n}\cdot\nabla_b\mathbf{t}, \\
\operatorname{curl}\mathbf{n}\cdot\mathbf{n} &= \pi_n = \mathbf{t}\cdot\nabla_b\mathbf{n} - \tau, \\
\operatorname{curl}\mathbf{b}\cdot\mathbf{b} &= \pi_b = -\tau - \mathbf{t}\cdot\nabla_n\mathbf{b}.
\end{aligned} \tag{16}$$

Here $\pi_s,\ \pi_n,\ \pi_b$ are called abnormality functions of the unit Serret-Frenet vectors $\mathbf{t},\ \mathbf{n},\ \mathbf{b}$. These abnormalities represent total moments of the unit tangent, principal normal and binormal vectors, respectively [$13-15$].

3 Directional Fermi-Walker Parallel Transportations and Geometric Phases of Vortex Lines

In this section, we focus our attention to define a new type of directional Fermi-Walker derivatives in three dimensions to reach a complete understanding with the parallelism of vortex lines of a non-vanishing vector field. Different kind of Fermi-Walker parallel transportations for $s-$lines, which are curves whose tangent vector is $\mathbf{t}$, has been given by Eqs. $(1,3,5,7)$. These parallel transportations can be considered as the directional Fermi-Walker derivative in the tangential direction. Now we will define directional Fermi-Walker derivative in the principal normal direction for $n-$lines and directional Fermi-Walker derivative in the binormal direction for $b-$lines, respectively.

3.1 Fermi-Walker Derivative and Geometric Phases in the Principal Normal Direction

In this subsection, we consider a three dimensional non-vanishing vector field defined along with $n-$lines, which are vortex lines whose tangent vector is $\mathbf{n}$, to investigate its Fermi-Walker parallelism in the principal normal direction.

Definition 1. $\mathcal{A}$ be any three dimensional non-vanishing vector field defined along with $n-$lines.

i. Ordinary Fermi-Walker derivative in the principal normal direction is defined by

$$\nabla_n^{\circ}\mathcal{A} =\nabla_n\mathcal{A}-(\mathbf{t}\cdot\mathcal{A})\nabla_n\mathbf{t}+(\nabla_n\mathbf{t}\cdot\mathcal{A})\mathbf{t}. \tag{17}$$

By using Eqs. $(9,17)$ one can obtain the following formula which is interchangeable with the above expression.

$$\nabla_n^{\circ}\mathcal{A} =\nabla_n\mathcal{A}-\delta_{ns}\left(\mathbf{b}\times\mathcal{A}\right)+\left(\pi_b+\tau\right)\left(\mathbf{n}\times\mathcal{A}\right). \tag{18}$$

ii. Normal Fermi-Walker derivative in the principal normal direction is defined by

$$\left(\nabla^{\circ}\right)_n^N\mathcal{A} =\nabla_n\mathcal{A}-(\mathbf{n}\cdot\mathcal{A})\nabla_n\mathbf{n}+(\nabla_n\mathbf{n}\cdot\mathcal{A})\mathbf{n}. \tag{19}$$

By using Eqs. $(9,19)$ one can obtain the following formula which is interchangeable with the above expression.

$$\left(\nabla^{\circ}\right)_n^N\mathcal{A} =\nabla_n\mathcal{A}-\delta_{ns}\left(\mathbf{b}\times\mathcal{A}\right)+\operatorname{div}\mathbf{b}\left(\mathbf{t}\times\mathcal{A}\right). \tag{20}$$

iii. Modified Fermi-Walker derivative in the principal normal direction is defined by

$$(\nabla^{\circ})_{n}^{M}\mathcal{A}=\nabla_{n}\mathcal{A}-(\mathbf{b}\cdot\mathcal{A})\nabla_{n}\mathbf{b}+(\nabla_{n}\mathbf{b}\cdot\mathcal{A})\mathbf{b}. \tag{21}$$

By using Eqs. $(9,21)$ one can obtain the following formula which is interchangeable with the above expression.

$$(\nabla^{\circ})_{n}^{M}\mathcal{A}=\nabla_{n}\mathcal{A}+(\pi_{b}+\tau)(\mathbf{n}\times\mathcal{A})+\operatorname{div}\mathbf{b}\,(\mathbf{t}\times\mathcal{A}). \tag{22}$$

Theorem 2. **i**. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines. The vector field $\mathcal{A}=\mathcal{A}_{1}\mathbf{t}+\mathcal{A}_{2}\mathbf{n}+\mathcal{A}_{3}\mathbf{b}$ is an ordinary Fermi-Walker parallel transported in the principal normal direction if and only if

$$\frac{d\mathcal{A}_{1}}{dn}=0,\ \frac{d}{dn}\begin{bmatrix}\mathcal{A}_{2}\\ \mathcal{A}_{3}\end{bmatrix}=\begin{bmatrix}0 & -\operatorname{div}\mathbf{b}\\ \operatorname{div}\mathbf{b} & 0\end{bmatrix}\begin{bmatrix}\mathcal{A}_{2}\\ \mathcal{A}_{3}\end{bmatrix}, \tag{23}$$

where $\nabla_{n}=d/dn$.

ii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines. The vector field $\mathcal{A}=\mathcal{A}_{1}\mathbf{t}+\mathcal{A}_{2}\mathbf{n}+\mathcal{A}_{3}\mathbf{b}$ is a normal Fermi-Walker parallel transported in the principal normal direction if and only if

$$\frac{d\mathcal{A}_{2}}{dn}=0,\ \frac{d}{dn}\begin{bmatrix}\mathcal{A}_{1}\\ \mathcal{A}_{3}\end{bmatrix}=\begin{bmatrix}0 & (\pi_{b}+\tau)\\ -(\pi_{b}+\tau) & 0\end{bmatrix}\begin{bmatrix}\mathcal{A}_{1}\\ \mathcal{A}_{3}\end{bmatrix}, \tag{24}$$

where $\nabla_{n}=d/dn$.

iii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines. The vector field $\mathcal{A}=\mathcal{A}_{1}\mathbf{t}+\mathcal{A}_{2}\mathbf{n}+\mathcal{A}_{3}\mathbf{b}$ is a modified Fermi-Walker parallel transported in the principal normal direction if and only if

$$\frac{d\mathcal{A}_{3}}{dn}=0,\ \frac{d}{dn}\begin{bmatrix}\mathcal{A}_{1}\\ \mathcal{A}_{2}\end{bmatrix}=\begin{bmatrix}0 & \delta_{ns}\\ -\delta_{ns} & 0\end{bmatrix}\begin{bmatrix}\mathcal{A}_{1}\\ \mathcal{A}_{2}\end{bmatrix}, \tag{25}$$

where $\nabla_{n}=d/dn$.

Proof. **i**. $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines such that it has a form of $\mathcal{A}=\mathcal{A}_{1}\mathbf{t}+\mathcal{A}_{2}\mathbf{n}+\mathcal{A}_{3}\mathbf{b}$. Let assume that $\mathcal{A}$ is an ordinary Fermi-Walker parallel transported in the principal normal direction. Thus one has from the Eq. (18)

$$\nabla_{n}^{\circ}\mathcal{A}=0,$$

which implies that

$$\nabla_n \mathcal{A} = \delta_{ns} (\mathbf{b} \times \mathcal{A}) - (\pi_b + \tau) (\mathbf{n} \times \mathcal{A}) .$$

Now if one reconsiders the Eq. (9) and solves the above equality then it is obtained that

$$\frac{d\mathcal{A}_1}{dn} = 0, \ \frac{d\mathcal{A}_2}{dn} = -\operatorname{div} \mathbf{b} \mathcal{A}_3, \ \frac{d\mathcal{A}_3}{dn} = \operatorname{div} \mathbf{b} \mathcal{A}_2.$$

The converse part of the proof is trivial.

The rest of the proof can be completed by using the similar argument as in the first case.

Lemma 3. **i**. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines. The derivative of the vector field $\mathcal{A} = \mathcal{A}_1 \mathbf{t} + \mathcal{A}_2 \mathbf{n} + \mathcal{A}_3 \mathbf{b}$ in the principal normal direction coincides with the ordinary Fermi-Walker derivative in the principal normal direction if and only if

$$\mathcal{A}_1 = 0, \ \mathcal{A}_2 = -(\pi_b + \tau), \ \mathcal{A}_3 = \delta_{ns}. \tag{26}$$

ii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines. The derivative of the vector field $\mathcal{A} = \mathcal{A}_1 \mathbf{t} + \mathcal{A}_2 \mathbf{n} + \mathcal{A}_3 \mathbf{b}$ in the principal normal direction coincides with the normal Fermi-Walker derivative in the principal normal direction if and only if

$$\mathcal{A}_2 = 0, \ \mathcal{A}_1 = -\operatorname{div} \mathbf{b}, \ \mathcal{A}_3 = \delta_{ns}. \tag{27}$$

iii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines. The derivative of the vector field $\mathcal{A} = \mathcal{A}_1 \mathbf{t} + \mathcal{A}_2 \mathbf{n} + \mathcal{A}_3 \mathbf{b}$ in the principal normal direction coincides with the modified Fermi-Walker derivative in the principal normal direction if and only if

$$\mathcal{A}_3 = 0, \ \mathcal{A}_1 = \operatorname{div} \mathbf{b}, \ \mathcal{A}_2 = (\pi_b + \tau). \tag{28}$$

Proof. **i**. $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $n-$lines such that it has a form of $\mathcal{A} = \mathcal{A}_1 \mathbf{t} + \mathcal{A}_2 \mathbf{n} + \mathcal{A}_3 \mathbf{b}$. The derivative of the vector field $\mathcal{A}$ in the principal normal direction coincides with the ordinary Fermi-Walker derivative in the principal normal direction if and only if $\nabla_n^{\circ} \mathcal{A} = \nabla_n \mathcal{A}$, which is induced by Eqs. $(17, 18)$. If one considers the Eq. (18) then one should have

$$\delta_{ns} (\mathbf{b} \times \mathcal{A}) = (\pi_b + \tau) (\mathbf{n} \times \mathcal{A}) .$$

Thus it is obtained that this equality holds when $\mathcal{A}_1 = 0$, $\mathcal{A}_2 = -(\pi_b + \tau)$, $\mathcal{A}_3 = \delta_{ns}$. The rest of the proof can be completed by using the similar argument as in the first case.

Lemma 4. **i**. Non-vanishing Serret-Frenet vectors defined along with $n-$lines are ordinary Fermi-Walker parallel transported in the principal normal direction if and only if

$$\operatorname{div} \mathbf{b} = 0. \tag{29}$$

ii. Non-vanishing Serret-Frenet vectors defined along with $n-$lines are normal Fermi-Walker parallel transported in the principal normal direction if and only if

$$(\pi_b + \tau) = 0. \tag{30}$$

iii. Non-vanishing Serret-Frenet vectors defined along with $n-$lines are modified Fermi-Walker parallel transported in the principal normal direction if and only if

$$\delta_{ns} = 0. \tag{31}$$

Proof. Non-vanishing Serret-Frenet vectors defined along with $n-$lines are ordinary Fermi-Walker parallel transported in the principal normal direction if and only if

$$\nabla_n^\circ \mathbf{t} = 0, \;\; \nabla_n^\circ \mathbf{n} = 0, \;\; \nabla_n^\circ \mathbf{b} = 0.$$

Hence the proof is evident if one uses Eqs. $(9, 18)$. The rest of the proof can be completed by using the similar argument as in the first case.

Main Results 1. So far it has been given a mathematical insight of the Fermi-Walker parallel transportation in the principal normal direction based on the basic definitions and some elementary computations. Now, we will present the geometric and physical interpretation of the obtained data.

i. The ordinary Fermi-Walker derivative in the principal normal direction is defined by Eqs. $(17, 18)$ in Definition 1. The Eq. (23) in Theorem 2 explains an ordinary Fermi-Walker parallel transportation for any three-dimensional non-vanishing vector field defined along with $n-$lines. According to the Eq. (23) it is obtained that the angular velocity of the normal plane, which is spanned by $(\mathbf{n}, \mathbf{b})$, while it rotates around the tangent vector $\mathbf{t}$ is given by $\operatorname{div} \mathbf{b}$ along with $n-$lines. Thus the ordinary geometric phase, which is improved by the system between the principal normal and binormal vectors, is computed by $(\mathcal{B}_1)_n = \int_0^{n_0} \operatorname{div} \mathbf{b} dn$ when n has increased from $n = 0$ to $n = n_0$ in the principal normal direction.

One can also obtain an ordinary Darboux vector of the non-rotating frame, which is defined along with n-lines if one considers Eqs. $(26, 29)$. Thus the ordinary Darboux vector in the principal normal direction is written by

$$\nabla_n \mathbf{t} = \mathcal{D}_n \times \mathbf{t},\ \nabla_n \mathbf{n} = \mathcal{D}_n \times \mathbf{n},\ \nabla_n \mathbf{b} = \mathcal{D}_n \times \mathbf{b},$$

where $\mathcal{D}_n = -(\pi_b + \tau)\mathbf{n} + \delta_{ns}\mathbf{b}$ and $\nabla_n = d/dn$.

ii. The normal Fermi-Walker derivative in the principal normal direction is defined by Eqs. $(19, 20)$ in Definition 1. The Eq. (24) in Theorem 2 explains a normal Fermi-Walker parallel transportation for any three-dimensional non-vanishing vector field defined along with $n-$lines. According to the Eq. (24) it is obtained that the angular velocity of the rectifying plane, which is spanned by $(\mathbf{t}, \mathbf{b})$, while it rotates around the principal normal vector $\mathbf{n}$ is given by $(\pi_b + \tau)$ along with $n-$lines. Thus the normal geometric phase, which is improved by the system between the tangent and binormal vectors, is computed by $(\mathcal{B}_1)_n^N = \int_0^{n_0} (\pi_b + \tau)\, dn$ when n has increased from $n = 0$ to $n = n_0$ in the principal normal direction.

One can also obtain a normal Darboux vector of the non-rotating frame, which is defined along with n-lines if one considers Eqs. $(27, 30)$. Thus the normal Darboux vector in the principal normal direction is written by

$$\nabla_n \mathbf{t} = \mathcal{D}_n^N \times \mathbf{t},\ \nabla_n \mathbf{n} = \mathcal{D}_n^N \times \mathbf{n},\ \nabla_n \mathbf{b} = \mathcal{D}_n^N \times \mathbf{b},$$

where $\mathcal{D}_n^N = -(\operatorname{div} \mathbf{b})\, \mathbf{t} + \delta_{ns}\mathbf{b}$ and $\nabla_n = d/dn$.

iii. The modified Fermi-Walker derivative in the principal normal direction is defined by Eqs. $(21, 22)$ in Definition 1. The Eq. (25) in Theorem 2 explains a modified Fermi-Walker parallel transportation for any three-dimensional non-vanishing vector field defined along with $n-$lines. According to the Eq. (25) it is obtained that the angular velocity of the osculating plane, which is spanned by $(\mathbf{t}, \mathbf{n})$, while it rotates around the tangent vector $\mathbf{t}$ is given by $\operatorname{div} \mathbf{b}$ along with $n-$lines. Thus the modified geometric phase, which is improved by the system between the tangent and principal normal vectors, is computed by $(\mathcal{B}_1)_n^M = \int_0^{n_0} \delta_{ns} dn$ when n has increased from $n = 0$ to $n = n_0$ in the principal normal direction.

One can also obtain a modified Darboux vector of the non-rotating frame, which is defined along with n-lines if one considers Eqs. $(28, 31)$. Thus the modified Darboux vector in the principal normal direction is written by

$$\nabla_n \mathbf{t} = \mathcal{D}_n^M \times \mathbf{t},\ \nabla_n \mathbf{n} = \mathcal{D}_n^M \times \mathbf{n},\ \nabla_n \mathbf{b} = \mathcal{D}_n^M \times \mathbf{b},$$

where $\mathcal{D}_n^M = -(\operatorname{div}\mathbf{b})\,\mathbf{t} - (\pi_b + \tau)\,\mathbf{n}$ and $\nabla_n = d/dn$.

3.2 Fermi-Walker Derivative and Geometric Phases in the Binormal Direction

In this subsection, we consider a three-dimensional non-vanishing vector field defined along with $b-$lines, which are vortex lines whose tangent vector is $\mathbf{b}$, to investigate its Fermi-Walker parallelism in the binormal direction.

Definition 5. $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines.

i. Ordinary Fermi-Walker derivative in the binormal direction is defined by

$$\nabla_b^\circ \mathcal{A} = \nabla_b \mathcal{A} - (\mathbf{t}\cdot\mathcal{A})\nabla_b \mathbf{t} + (\nabla_b \mathbf{t}\cdot\mathcal{A})\mathbf{t}. \tag{32}$$

By using Eqs. $(10, 32)$ one can obtain the following formula which is interchangeable with the above expression.

$$\nabla_b^\circ \mathcal{A} = \nabla_b \mathcal{A} + \delta_{bs}\,(\mathbf{n}\times\mathcal{A}) + (\pi_n + \tau)\,(\mathbf{b}\times\mathcal{A}). \tag{33}$$

ii. Normal Fermi-Walker derivative in the binormal direction is defined by

$$(\nabla^\circ)_b^N \mathcal{A} = \nabla_b \mathcal{A} - (\mathbf{n}\cdot\mathcal{A})\nabla_b \mathbf{n} + (\nabla_b \mathbf{n}\cdot\mathcal{A})\mathbf{n}. \tag{34}$$

By using Eqs. $(10, 34)$ one can obtain the following formula which is interchangeable with the above expression.

$$(\nabla^\circ)_b^N \mathcal{A} = \nabla_b \mathcal{A} - (\kappa + \operatorname{div}\mathbf{n})\,(\mathbf{t}\times\mathcal{A}) + (\pi_n + \tau)\,(\mathbf{b}\times\mathcal{A}). \tag{35}$$

iii. Modified Fermi-Walker derivative in the binormal direction is defined by

$$(\nabla^\circ)_b^M \mathcal{A} = \nabla_b \mathcal{A} - (\mathbf{b}\cdot\mathcal{A})\nabla_b \mathbf{b} + (\nabla_b \mathbf{b}\cdot\mathcal{A})\mathbf{b}. \tag{36}$$

By using Eqs. $(10, 36)$ one can obtain the following formula which is interchangeable with the above expression.

$$(\nabla^\circ)_b^M \mathcal{A} = \nabla_b \mathcal{A} - (\kappa + \operatorname{div}\mathbf{n})\,(\mathbf{t}\times\mathcal{A}) + \delta_{bs}\,(\mathbf{n}\times\mathcal{A}). \tag{37}$$

Theorem 6. **i**. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines. The vector field $\mathcal{A} = \mathcal{A}_1\mathbf{t} + \mathcal{A}_2\mathbf{n} + \mathcal{A}_3\mathbf{b}$ is an ordinary Fermi-Walker parallel transported in the binormal direction if and only if

$$\frac{d\mathcal{A}_1}{db}=0,\ \frac{d}{db}\begin{bmatrix}\mathcal{A}_2\\ \mathcal{A}_3\end{bmatrix}=\begin{bmatrix}0 & (\kappa+\operatorname{div}\mathbf{n})\\ -(\kappa+\operatorname{div}\mathbf{n}) & 0\end{bmatrix}\begin{bmatrix}\mathcal{A}_2\\ \mathcal{A}_3\end{bmatrix}, \tag{38}$$

where $\nabla_b = d/db$.

ii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines. The vector field $\mathcal{A}=\mathcal{A}_1\mathbf{t}+\mathcal{A}_2\mathbf{n}+\mathcal{A}_3\mathbf{b}$ is a normal Fermi-Walker parallel transported in the binormal direction if and only if

$$\frac{d\mathcal{A}_2}{db}=0,\ \frac{d}{db}\begin{bmatrix}\mathcal{A}_1\\ \mathcal{A}_3\end{bmatrix}=\begin{bmatrix}0 & \delta_{bs}\\ -\delta_{bs} & 0\end{bmatrix}\begin{bmatrix}\mathcal{A}_1\\ \mathcal{A}_3\end{bmatrix}, \tag{39}$$

where $\nabla_b = d/db$.

iii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines. The vector field $\mathcal{A}=\mathcal{A}_1\mathbf{t}+\mathcal{A}_2\mathbf{n}+\mathcal{A}_3\mathbf{b}$ is a modified Fermi-Walker parallel transported in the binormal direction if and only if

$$\frac{d\mathcal{A}_3}{db}=0,\ \frac{d}{db}\begin{bmatrix}\mathcal{A}_1\\ \mathcal{A}_2\end{bmatrix}=\begin{bmatrix}0 & -(\pi_n+\tau)\\ (\pi_n+\tau) & 0\end{bmatrix}\begin{bmatrix}\mathcal{A}_1\\ \mathcal{A}_2\end{bmatrix}, \tag{40}$$

where $\nabla_b = d/db$.

Proof. **i**. $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines such that it has a form of $\mathcal{A}=\mathcal{A}_1\mathbf{t}+\mathcal{A}_2\mathbf{n}+\mathcal{A}_3\mathbf{b}$. Let assume that $\mathcal{A}$ is an ordinary Fermi-Walker parallel transported in the binormal direction. Thus one has from the Eq. (33)

$$\nabla_b^{\circ}\mathcal{A}=0,$$

which implies that

$$\nabla_b\mathcal{A}=-\delta_{bs}\left(\mathbf{n}\times\mathcal{A}\right)-\left(\pi_n+\tau\right)\left(\mathbf{b}\times\mathcal{A}\right).$$

Now if one reconsiders the Eq. (10) and solves the above equality then it is obtained that

$$\frac{d\mathcal{A}_1}{db}=0,\ \frac{d\mathcal{A}_2}{db}=\left(\kappa+\operatorname{div}\mathbf{n}\right)\mathcal{A}_3,\ \frac{d\mathcal{A}_3}{db}=-\left(\kappa+\operatorname{div}\mathbf{n}\right)\mathcal{A}_2.$$

The converse part of the proof is trivial.

The rest of the proof can be completed by using the similar argument as in the first case.

Lemma 7. **i**. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines. The derivative of the vector field $\mathcal{A} = \mathcal{A}_1\mathbf{t} + \mathcal{A}_2\mathbf{n} + \mathcal{A}_3\mathbf{b}$ in the binormal direction coincides with the ordinary Fermi-Walker derivative in the binormal direction if and only if

$$\mathcal{A}_1 = 0,\ \mathcal{A}_2 = \delta_{bs},\ \mathcal{A}_3 = (\pi_n + \tau)\,. \tag{41}$$

ii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines. The derivative of the vector field $\mathcal{A} = \mathcal{A}_1\mathbf{t} + \mathcal{A}_2\mathbf{n} + \mathcal{A}_3\mathbf{b}$ in the binormal direction coincides with the normal Fermi-Walker derivative in the binormal direction if and only if

$$\mathcal{A}_2 = 0,\ \mathcal{A}_1 = (\kappa + \operatorname{div}\mathbf{n})\,,\ \mathcal{A}_3 = -\,(\pi_n + \tau)\,. \tag{42}$$

iii. Let $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines. The derivative of the vector field $\mathcal{A} = \mathcal{A}_1\mathbf{t} + \mathcal{A}_2\mathbf{n} + \mathcal{A}_3\mathbf{b}$ in the binormal direction coincides with the modified Fermi-Walker derivative in the binormal direction if and only if

$$\mathcal{A}_3 = 0,\ \mathcal{A}_1 = (\kappa + \operatorname{div}\mathbf{n})\,,\ \mathcal{A}_2 = -\delta_{bs}. \tag{43}$$

Proof. **i**. $\mathcal{A}$ be any three-dimensional non-vanishing vector field defined along with $b-$lines such that it has a form of $\mathcal{A} = \mathcal{A}_1\mathbf{t} + \mathcal{A}_2\mathbf{n} + \mathcal{A}_3\mathbf{b}$. The derivative of the vector field $\mathcal{A}$ in the binormal direction coincides with the ordinary Fermi-Walker derivative in the binormal direction if and only if $\nabla_b^{\circ}\mathcal{A} = \nabla_b\mathcal{A}$, which is induced by Eqs. $(32, 33)$. If one considers the Eq. (33) then one should have

$$-\delta_{bs}\,(\mathbf{n} \times \mathcal{A}) = (\pi_n + \tau)\,(\mathbf{b} \times \mathcal{A})\,.$$

Thus it is obtained that this equality holds when $\mathcal{A}_1 = 0$, $\mathcal{A}_2 = \delta_{ns}$, $\mathcal{A}_3 = -\,(\pi_n + \tau)$. The converse part of the proof is trivial.

The rest of the proof can be completed by using the similar argument as in the first case.

Lemma 8. **i**. Non-vanishing Serret-Frenet vectors defined along with $b-$lines are ordinary Fermi-Walker parallel transported in the binormal direction if and only if

$$(\kappa + \operatorname{div}\mathbf{n}) = 0. \tag{44}$$

ii. Non-vanishing Serret-Frenet vectors defined along with $b-$lines are normal Fermi-Walker parallel transported in the binormal direction if and only if

$$\delta_{bs}=0. \tag{45}$$

iii. Non-vanishing Serret-Frenet vectors defined along with $b-$lines are modified Fermi-Walker parallel transported in the binormal direction if and only if

$$(\pi_n + \tau) = 0. \tag{46}$$

Proof. **i**. Non-vanishing Serret-Frenet vectors defined along with $b-$lines are ordinary Fermi-Walker parallel transported in the binormal direction if and only if

$$\nabla_b^{\circ}\mathbf{t} = 0,\ \nabla_b^{\circ}\mathbf{n} = 0,\ \nabla_b^{\circ}\mathbf{b} = 0.$$

Hence the proof is evident if one uses Eqs. $(10, 33)$.

The rest of the proof can be completed by using the similar argument as in the first case.

Main Results 2. So far it has been given a mathematical insight of the Fermi-Walker parallel transportation in the binormal direction based on the basic definitions and some elementary computations. Now, we will present the geometric and physical interpretation of the obtained data.

i. The ordinary Fermi-Walker derivative in the binormal direction is defined by Eqs. $(32, 33)$ in Definition 5. The Eq. (38) in Theorem 6 explains an ordinary Fermi-Walker parallel transportation for any three-dimensional non-vanishing vector field defined along with $b-$lines. According to the Eq. (38) it is obtained that the angular velocity of the normal plane, which is spanned by $(\mathbf{n}, \mathbf{b})$, while it rotates around the tangent vector $\mathbf{t}$ is given by $(\kappa + \operatorname{div}\mathbf{n})$ along with $b-$lines. Thus the ordinary geometric phase, which is improved by the system between the principal normal and binormal vectors, is computed by $(\mathcal{B}_1)_b = \int_0^{b_0} (\kappa + \operatorname{div}\mathbf{n})\, db$ when b has increased from $b = 0$ to $b = b_0$ in the binormal direction.

One can also obtain an ordinary Darboux vector of the non-rotating frame, which is defined along with b-lines if one considers Eqs. $(41, 44)$. Thus the ordinary Darboux vector in the binormal direction is written by

$$\nabla_b\mathbf{t} = \mathcal{D}_b \times \mathbf{t},\ \nabla_b\mathbf{n} = \mathcal{D}_b \times \mathbf{n},\ \nabla_b\mathbf{b} = \mathcal{D}_b \times \mathbf{b},$$

where $\mathcal{D}_b = -\delta_{bs}\mathbf{n} - (\pi_n + \tau)\,\mathbf{b}$ and $\nabla_b = d/db$.

ii. The normal Fermi-Walker derivative in the binormal direction is defined by Eqs. $(34, 35)$ in Definition 5. The Eq. (39) in Theorem 6 explains a normal Fermi-Walker parallel transportation for any three-dimensional non-vanishing vector field defined along with $b-$lines. According to the Eq. (39) it is obtained that the angular velocity of the rectifiying plane, which is spanned by $(\mathbf{t}, \mathbf{b})$, while it rotates around the principal normal vector $\mathbf{n}$ is given by δ_{bs} along with $b-$lines. Thus the normal geometric phase, which is improved by the system between the tangent and binormal vectors, is computed by $(\mathcal{B}_1)_b^N = \int_0^{b_0} \delta_{bs} db$ when b has increased from $b = 0$ to $b = b_0$ in the binormal direction.

One can also obtain a normal Darboux vector of the non-rotating frame, which is defined along with b-lines if one considers Eqs. $(42, 45)$. Thus the normal Darboux vector in the binormal direction is written by

$$\nabla_b \mathbf{t} = \mathcal{D}_b^N \times \mathbf{t},\ \nabla_b \mathbf{n} = \mathcal{D}_b^N \times \mathbf{n},\ \ \nabla_b \mathbf{b} = \mathcal{D}_b^N \times \mathbf{b},$$

where $\mathcal{D}_b^N = (\kappa + \operatorname{div} \mathbf{n})\, \mathbf{t} - (\pi_n + \tau)\, \mathbf{b}$ and $\nabla_b = d/db$.

iii. The modified Fermi-Walker derivative in the binormal direction is defined by Eqs. $(36, 37)$ in Definition 5. The Eq. (40) in Theorem 6 explains a modified Fermi-Walker parallel transportation for any any three-dimensional non-vanishing vector field defined along with $b-$lines. According to the Eq. (40) it is obtained that the angular velocity of the osculating plane, which is spanned by $(\mathbf{t}, \mathbf{n})$, while it rotates around the tangent vector $\mathbf{t}$ is given by $(\pi_n + \tau)$ along with $b-$lines. Thus the modified geometric phase, which is improved by the system between the tangent and principal normal vectors, is computed by $(\mathcal{B}_1)_b^M = \int_0^{b_0} (\pi_n + \tau)\, db$ when b has increased from $b = 0$ to $b = b_0$ in the binormal direction.

One can also obtain a modified Darboux vector of the non-rotating frame, which is defined along with b-lines if one considers Eqs. $(43, 46)$. Thus the modified Darboux vector in the principal normal direction is written by

$$\nabla_b \mathbf{t} = \mathcal{D}_b^M \times \mathbf{t},\ \nabla_b \mathbf{n} = \mathcal{D}_b^M \times \mathbf{n},\ \ \nabla_b \mathbf{b} = \mathcal{D}_b^M \times \mathbf{b},$$

where $\mathcal{D}_b^M = (\kappa + \operatorname{div} \mathbf{n})\, \mathbf{t} - \delta_{bs} \mathbf{n}$ and $\nabla_b = d/db$.

4 Directional Magnetic Vortex Lines

In the literature magnetic curves, magnetic fields and magnetic flows of vortex filaments or charged particles have been intensively studied. For instance, Barros et al. used a variational method and Lorentz force equation to investigate the relation between Killing magnetic fields and magnetic flows. They have obtained a remarkable connection between the Hall effect and elastic theory [16]. Some other solutions of the Lorentz force equation and some special magnetic flows and magnetic curves have been characterized by Bozkurt et al. [17]. A similar approach is used in various studies to understand the behavior of magnetic curves and their flows in different geometric structure. For example, Druta-Romaniuc and Munteanu [18, 19] studied on magnetic curves associated with the Killing magnetic fields in both Euclidean and Minkowski space separately. Munteanu and Nistor [20] generalized local description of magnetic trajectories corresponding to Killing vector fields in $\mathbb{S}^2 \times \mathbb{R}$. Classification of the trajectories of the charged particle corresponding to Kahler magnetic fields in Kahler manifolds; Killing vector fields in Walker manifold; contact magnetic fields in Sasakian, quasi-Sasakian, quasi-para-Sasakian, and cosymplectic manifolds were given by $[21-26]$. In $[27, 28]$, we defined frictional and gravitational magnetic curves together with their energy functionals and uniformity conditions on the 3D Riemannian surface. Even though all these studies have distinct consequences and physical interpretations magnetic flows obtained through the solution of the Lorentz force equation belong to $s-$lines, which are vortex lines whose tangent vector is $\mathbf{t}$. Following is the summary of the common approach that has been mainly considered so far.

A magnetic field can be described on an n-dimensional Riemannian manifold $(\mathcal{K}^n, \cdot)$ as a closed two-form $\mathbf{V} \in \mathbf{\Lambda}(\mathcal{K}^n, \cdot)$ such that anti-symmetric Lorentz force operator Π satisfies

$$\Pi(\mathcal{S}) \cdot \mathcal{Z} = \mathbf{V}(\mathcal{S}, \mathcal{Z}), \tag{47}$$

where $\mathcal{S}, \mathcal{Z} \in (\mathcal{K}^n, \cdot)$. Then magnetic trajectories associated with the magnetic field $\mathbf{V}$ are magnetic curves in $(\mathcal{K}^n, \cdot)$ provided that their tangent vectors $\mathbf{t}$ satisfy the non-linear second-order Lorentz force equation

$$\nabla_s \mathbf{t} = \Pi(\mathbf{t}). \tag{48}$$

In three-dimensional space, magnetic fields possess very elegant features that make this case special. In three dimensions, 2-forms and vector fields, magnetic fields and divergence-free vector fields, uniform magnetic fields, and parallel vector fields are said to equivalent to each other and their definitions allow one to interchange each concept with other. Finally, these facts imply that the Lorentz force equation (48) can be written in terms of vector product by

$$\nabla_s \mathbf{t} = \Pi(\mathbf{t}) = \mathbf{V} \times \mathbf{t}. \tag{49}$$

In the following subsections, the solution of the Lorentz force equation on $n-$lines, which are vortex lines whose tangent vector is $\mathbf{n}$ and the Lorentz force equation on $b-$lines, which are vortex lines whose tangent vector is $\mathbf{b}$ are computed. Thus we define directional magnetic vortex lines of $n-$lines in the principal normal direction and directional magnetic vortex lines for $b-$lines in the binormal direction, respectively. Consequently, it is aimed to establish a relation between directional magnetic vector fields of directional magnetic vortex lines and angular velocity vectors of vortex lines with non-rotating frame.

4.1 Magnetic Vector Fields of Directional Magnetic Vortex Lines in the Principal Normal Direction

In this subsection, it is firstly defined the adapted Lorentz force equation for $n-$lines, which are vortex lines whose tangent vector is $\mathbf{n}$. Later it is defined directional magnetic vortex lines and associated magnetic fields in the principal normal direction.

Definition 9. **i**. Tangent magnetic vortex lines along with $n-$lines ($nm_{\mathbf{t}}$) are defined by

$$\Pi_n(\mathbf{t}) = \nabla_n \mathbf{t} = \mathbf{V}_1 \times \mathbf{t}, \tag{50}$$

where Π_n is the ordinary Lorentz force equation in the principal normal direction and $\mathbf{V}_1$ is a vector field with $\operatorname{div}(\mathbf{V}_1) = 0$.

ii. Principal normal magnetic vortex lines along with $n-$lines ($nm_{\mathbf{n}}$) are defined by

$$\Pi_n^N(\mathbf{n}) = \nabla_n \mathbf{n} = \mathbf{V}_2 \times \mathbf{n}, \tag{51}$$

where Π_n^N is the normal Lorentz force equation in the principal normal direction and $\mathbf{V}_2$ is a vector field with $\operatorname{div}(\mathbf{V}_2) = 0$.

iii. Binormal magnetic vortex lines along with $n-$lines ($nm_{\mathbf{b}}$) are defined by

$$\Pi_n^M(\mathbf{b}) = \nabla_n \mathbf{b} = \mathbf{V}_3 \times \mathbf{b}, \tag{52}$$

where Π_n^M is the modified Lorentz force equation in the principal normal direction and $\mathbf{V}_3$ is a vector field with $\operatorname{div}(\mathbf{V}_3) = 0$.

Theorem 10. **i.** $nm_{\mathbf{t}}$ is a magnetic trajectory of the magnetic field $\mathbf{V}_1$ if and only if $\mathbf{V}_1 = \varepsilon_1 \mathbf{t} - (\pi_b + \tau)\,\mathbf{n} + \delta_{ns}\mathbf{b}$ along with $n-$lines.

ii. $nm_{\mathbf{n}}$ is a magnetic trajectory of the magnetic field $\mathbf{V}_2$ if and only if $\mathbf{V}_2 = -(\operatorname{div}\mathbf{b})\,\mathbf{t} - \varepsilon_2\mathbf{n} + \delta_{ns}\mathbf{b}$ along with $n-$lines.

iii. $nm_{\mathbf{b}}$ is a magnetic trajectory of the magnetic field $\mathbf{V}_3$ if and only if $\mathbf{V}_3 = -(\operatorname{div}\mathbf{b})\,\mathbf{t} - (\pi_b + \tau)\,\mathbf{n} + \varepsilon_3\mathbf{b}$ along with $n-$lines.

Proof. i. Let assume first that $nm_{\mathbf{t}}$ is a magnetic trajectory of the magnetic field $\mathbf{V}_1$ along with $n-$lines. Then the ordinary Lorentz force equation of the orthonormal frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ in the principal normal direction can be computed if one considers the Eq. (50) and following identities

$$\Pi_n(\mathbf{t})\cdot\mathbf{n} = -\Pi_n(\mathbf{n})\cdot\mathbf{t},\ \Pi_n(\mathbf{t})\cdot\mathbf{b} = -\Pi_n(\mathbf{b})\cdot\mathbf{t},\ \Pi_n(\mathbf{n})\cdot\mathbf{b} = -\Pi_n(\mathbf{b})\cdot\mathbf{n}. \tag{53}$$

Thus we have

$$\begin{gathered}\Pi_n(\mathbf{t}) = \delta_{ns}\mathbf{n} + (\pi_b + \tau)\,\mathbf{b},\ \Pi_n(\mathbf{n}) = -\delta_{ns}\mathbf{t} + \varepsilon_1\mathbf{b}, \\ \Pi_n(\mathbf{b}) = -(\pi_b + \tau)\,\mathbf{t} - \varepsilon_1\mathbf{n},\end{gathered} \tag{54}$$

where ε_1 is an arbitrarily choosen smooth function along with the $nm_{\mathbf{t}}$. Now, one can observe that $\mathbf{V}_1$ can be spanned by $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ that is

$$\mathbf{V}_1 = v_1\mathbf{t} + v_2\mathbf{n} + v_3\mathbf{b}, \tag{55}$$

where $v_j,\ 1 \leq j \leq 3$ are sufficiently smooth fucntions. If one considers Eqs. $(50, 54)$ then we have following two facts:

$$\begin{gathered}\mathbf{V}_1\times\mathbf{t} = \delta_{ns}\mathbf{n} + (\pi_b + \tau)\,\mathbf{b}, \\ 0 = \Pi_n(\mathbf{V}_1) = v_1\Pi_n(\mathbf{t}) + v_2\Pi_n(\mathbf{n}) + v_3\Pi_n(\mathbf{b}).\end{gathered} \tag{56}$$

Finally, it is computed by the Eq. (56) that $\mathbf{V}_1 = \varepsilon_1\mathbf{t} - (\pi_b + \tau)\,\mathbf{n} + \delta_{ns}\mathbf{b}$.

The rest of the proof can be completed by using the similar argument as in the first case.

Main Results 3. i. The ordinary Darboux vector field of the non-rotating frame, which is defined along with n-lines, is said to coincide with the magnetic vector field of tangent magnetic vortex lines in the principal normal direction when the arbitrarily chosen smooth function ε_1 vanishes.

ii. The normal Darboux vector field of the non-rotating frame, which is defined along with n-lines, is said to coincide with the magnetic vector field of principal normal magnetic vortex lines in the principal normal direction when the arbitrarily chosen smooth function ε_2 vanishes.

iii. The modified Darboux vector field of the non-rotating frame, which is defined along with n-lines, is said to coincide with the magnetic vector field of binormal magnetic vortex lines in the principal normal direction when the arbitrarily chosen smooth function ε_3 vanishes.

4.2 Magnetic Vector Fields of Directional Magnetic Vortex Lines in the Binormal Direction

In this subsection, it is firstly defined the adapted Lorentz force equation for $b-$lines, which are vortex lines whose tangent vector is $\mathbf{b}$. Later it is defined directional magnetic vortex lines and associated magnetic fields in the binormal direction.

Definition 11. **i**. Tangent magnetic vortex lines along with $b-$lines ($bm_{\mathbf{t}}$) are defined by

$$\Pi_b(\mathbf{t}) = \nabla_b \mathbf{t} = \mathbf{W}_1 \times \mathbf{t}, \tag{57}$$

where Π_b is the ordinary Lorentz force equation in the binormal direction and $\mathbf{W}_1$ is a vector field with $\operatorname{div}(\mathbf{W}_1) = 0$.

ii. Principal normal magnetic vortex lines along with $b-$lines ($bm_{\mathbf{n}}$) are defined by

$$\Pi_b^N(\mathbf{n}) = \nabla_b \mathbf{n} = \mathbf{W}_2 \times \mathbf{n}, \tag{58}$$

where Π_b^N is the normal Lorentz force equation in the binormal direction and $\mathbf{W}_2$ is a vector field with $\operatorname{div}(\mathbf{W}_2) = 0$.

iii. Binormal magnetic vortex lines along with $b-$lines ($bm_{\mathbf{b}}$) are defined by

$$\Pi_b^M(\mathbf{b}) = \nabla_b \mathbf{b} = \mathbf{W}_3 \times \mathbf{b}, \tag{59}$$

where Π_b^M is the modified Lorentz force equation in the binormal direction and $\mathbf{W}_3$ is a vector field with $\operatorname{div}(\mathbf{W}_3) = 0$.

Theorem 12. **i.** $bm_{\mathbf{t}}$ is a magnetic trajectory of the magnetic field $\mathbf{W}_1$ if and only if $\mathbf{W}_1 = \eta_1 \mathbf{t} - \delta_{bs}\mathbf{n} - (\pi_n + \tau)\mathbf{b}$ along with $b-$lines.

ii. $bm_{\mathbf{n}}$ is a magnetic trajectory of the magnetic field $\mathbf{W}_2$ if and only if $\mathbf{W}_2 = (\kappa + \operatorname{div}\mathbf{n})\mathbf{t} - \eta_2\mathbf{n} - (\pi_n + \tau)\mathbf{b}$ along with $b-$lines.

iii. $bm_{\mathbf{b}}$ is a magnetic trajectory of the magnetic field $\mathbf{W}_3$ if and only if $\mathbf{W}_3 = (\kappa + \operatorname{div}\mathbf{n})\mathbf{t} - \delta_{bs}\mathbf{n} + \eta_3\mathbf{b}$ along with $b-$lines.

Proof. **i**. Let assume first that $bm_{\mathbf{t}}$ is a magnetic trajectory of the magnetic field $\mathbf{W}_1$ along with $b-$lines. Then the ordinary Lorentz force equation on the orthonormal frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ in the binormal direction can be computed if one considers the Eq. (57) and following identities

$$\Pi_b(\mathbf{t}) \cdot \mathbf{n} = -\Pi_b(\mathbf{n}) \cdot \mathbf{t},\ \Pi_b(\mathbf{t}) \cdot \mathbf{b} = -\Pi_b(\mathbf{b}) \cdot \mathbf{t},\ \Pi_b(\mathbf{n}) \cdot \mathbf{b} = -\Pi_b(\mathbf{b}) \cdot \mathbf{n}. \tag{60}$$

Thus we have

$$\Pi_b(\mathbf{t}) = -(\pi_n+\tau)\,\mathbf{n}+\delta_{bs}\mathbf{b},\ \Pi_b(\mathbf{n}) = (\pi_n+\tau)\,\mathbf{t}+\eta_1\mathbf{b},\qquad (61)$$
$$\Pi_b(\mathbf{b}) = -\delta_{bs}\mathbf{t}-\eta_1\mathbf{n},$$

where η_1 is an arbitrarily choosen smooth function along with the $bm_{\mathbf{t}}$. Now, one can observe that $\mathbf{W}_1$ can be spanned by $(\mathbf{t},\mathbf{n},\mathbf{b})$ that is

$$\mathbf{W}_1 = w_1\mathbf{t}+w_2\mathbf{n}+w_3\mathbf{b}, \qquad (62)$$

where $w_j,\ 1\leq j\leq 3$ are sufficiently smooth fucntions. If one considers Eqs. $(57,61)$ then we have following two facts:

$$\mathbf{W}_1{\times}\mathbf{t} = -(\pi_n+\tau)\,\mathbf{n}+\delta_{bs}\mathbf{b},$$
$$0 = \Pi_b(\mathbf{W}_1) = w_1\Pi_b(\mathbf{t})+w_2\Pi_b(\mathbf{n})+w_3\Pi_b(\mathbf{b}). \qquad (63)$$

Finally, it is computed by the Eq. (63) that $\mathbf{W}_1 = \eta_1\mathbf{t}-\delta_{bs}\mathbf{n}-(\pi_n+\tau)\,\mathbf{b}$.

The rest of the proof can be completed by using the similar argument as in the first case.

Main Results 4. **i**. The ordinary Darboux vector field of the non-rotating frame, which is defined along with b-lines, is said to coincide with the magnetic vector field of tangent magnetic vortex lines in the binormal direction when the arbitrarily chosen smooth function η_1 vanishes.

ii. The normal Darboux vector field of the non-rotating frame, which is defined along with b-lines, is said to coincide with the magnetic vector field of principal normal magnetic vortex lines in the binormal direction when the arbitrarily chosen smooth function η_2 vanishes.

iii. The modified Darboux vector field of the non-rotating frame, which is defined along with b-lines, is said to coincide with the magnetic vector field of binormal magnetic vortex lines in the binormal direction when the arbitrarily chosen smooth function η_3.

5 Directional Electric Vortex Lines

The idea of electric lines was firstly presented by M. Faraday in his famous research on electromagnetism. He discussed that the forces of gravity, magnetism and electricity are all well-defined by fields, charactrized with field lines.

The fundamental of electrodynamics together with the electromagnetic field, energy, force, and momentum, which are closely connected with each other via the Lorentz force law and the theory of the Poynting vector, have been constructed upon the theory of Maxwell. This theory also governs the electromagnetic energy flow and its exchange between magnetic field ($\mathbf{V}$) and electric field ($\mathbf{E}$).

The well-known electromagnetic force on the moving particle whose trajectory is defined to be a curve (Υ) in three dimensional space is given by

$$\mathbf{F} = m\nabla_s(\nabla_s\Upsilon) = q(\mathbf{E}+\nabla_s\Upsilon \times \mathbf{V}), \tag{64}$$

where q is the charge of the particle. For the sake of clarity, it is assumed that no other forces acts on the given system. It is also considered a non-relativistic case for the simplicity purpose.

In the following subsections, the solution of the electromagnetic force equation of the positively charged particle moving under the action of electric and magnetic fields along with $n-$lines, which are vortex lines whose tangent vector is $\mathbf{n}$ and the solution of the electromagnetic force equation of the positively charged particle moving under the action of electric and magnetic fields along with $b-$lines, which are vortex lines whose tangent vector is $\mathbf{b}$ are computed. Thus we define directional electric vortex lines of $n-$lines in the principal normal direction and directional electric vortex lines for $b-$lines in the binormal direction, respectively. Consequently, it is aimed to investigate the physical and geometrical dynamics of the electric field lines in the principal normal and binormal directions.

5.1 Electric Vector Fields of Directional Electric Vortex Lines in the Principal Normal Direction

In this subsection, it is firstly defined the adapted electromagnetic force equations of the positively charged particle moving under the action of electric and magnetic fields along with $n-$lines, which are vortex lines whose tangent vector is $\mathbf{n}$. Later it is obtained both electric fields in the principal normal direction and some further investigations on the dynamics of the charged particle.

Definition 13. **i**. Tangent electric vortex lines along with $n-$lines ($ne_{\mathbf{t}}$) are defined by

$$\mathbf{F} = m\nabla_n(\nabla_n\Upsilon) = q(\mathbf{E}_1+\nabla_n\Upsilon \times \mathbf{V}_1), \tag{65}$$

where $\mathbf{V}_1$ is the magnetic vector field of tangent magnetic vortex lines in the principal normal direction.

Here if one considers Eqs. $(9, 65)$ and the Theorem 10 (i) then electric vector field of tangent electric vortex lines $(\mathbf{E}_1)$ in the principal normal direction is written by

$$\mathbf{E}_1 = -\delta_{ns}(1 + \frac{m}{q})\mathbf{t} + (\varepsilon_1 - \frac{m}{q}\operatorname{div}\mathbf{b})\mathbf{b}. \tag{66}$$

ii. Principal normal electric vortex lines along with $n-$lines $(ne_{\mathbf{n}})$ are defined by

$$\mathbf{F} = m\nabla_n(\nabla_n\Upsilon) = q(\mathbf{E}_2 + \nabla_n\Upsilon \times \mathbf{V}_2), \tag{67}$$

where $\mathbf{V}_2$ is the magnetic vector field of principal normal magnetic vortex lines in the principal normal direction.

Here if one considers Eqs. $(9, 67)$ and the Theorem 10 (ii) then electric vector field of principal normal electric vortex lines $(\mathbf{E}_2)$ in the principal normal direction is written by

$$\mathbf{E}_2 = -\delta_{ns}(1 + \frac{m}{q})\mathbf{t} - \operatorname{div}\mathbf{b}(1 + \frac{m}{q})\mathbf{b}. \tag{68}$$

iii. Binormal electric vortex lines along with $n-$lines $(ne_{\mathbf{b}})$ are defined by

$$\mathbf{F} = m\nabla_n(\nabla_n\Upsilon) = q(\mathbf{E}_3 + \nabla_n\Upsilon \times \mathbf{V}_3), \tag{69}$$

where $\mathbf{V}_3$ is the magnetic vector field of binormal magnetic vortex lines in the principal normal direction.

Here if one considers Eqs. $(9, 69)$ and the Theorem 10 (iii) then electric vector field of binormal electric vortex lines $(\mathbf{E}_3)$ in the principal normal direction is written by

$$\mathbf{E}_3 = -(\varepsilon_3 + \frac{m}{q}\delta_{ns})\mathbf{t} - \operatorname{div}\mathbf{b}(1 + \frac{m}{q})\mathbf{b}. \tag{70}$$

5.2 Electric Vector Fields of Directional Electric Vortex Lines in the Binormal Direction

In this subsection, it is firstly defined the adapted electromagnetic force equations of the positively charged particle moving under the action of electric and magnetic fields along with $b-$lines, which are vortex lines whose tangent vector is $\mathbf{b}$. Later it is obtained both electric fields in the principal normal direction and some further investigations on the dynamics of the charged particle.

Definition 14. **i**. Tangent electric vortex lines along with $b-$lines ($be_{\mathbf{t}}$) are defined by

$$\mathbf{F} = m\nabla_b(\nabla_b\Upsilon) = q(\mathbf{G}_1 + \nabla_b\Upsilon \times \mathbf{W}_1), \tag{71}$$

where $\mathbf{W}_1$ is the magnetic vector field of tangent magnetic vortex lines in the binormal direction.

Here if one considers Eqs. $(10, 71)$ and the Theorem 12 (i) then electric vector field of tangent electric vortex lines ($\mathbf{G}_1$) in the binormal direction is written by

$$\mathbf{G}_1 = -\delta_{bs}(1 + \frac{m}{q})\mathbf{t} - (\eta_1 + \frac{m}{q}(\kappa + \operatorname{div}\mathbf{n}))\mathbf{n}. \tag{72}$$

ii. Principal normal electric vortex lines along with $b-$lines ($be_{\mathbf{n}}$) are defined by

$$\mathbf{F} = m\nabla_b(\nabla_b\Upsilon) = q(\mathbf{G}_2 + \nabla_b\Upsilon \times \mathbf{W}_2), \tag{73}$$

where $\mathbf{W}_2$ is the magnetic vector field of principal normal magnetic vortex lines in the binormal direction.

Here if one considers Eqs. $(10, 73)$ and the Theorem 12 (ii) then electric vector field of principal normal electric vortex lines ($\mathbf{G}_2$) in the binormal direction is written by

$$\mathbf{G}_2 = (-\eta_2 - \frac{m}{q}\delta_{bs})\mathbf{t} - (\kappa + \operatorname{div}\mathbf{n})(1 + \frac{m}{q})\mathbf{b}. \tag{74}$$

iii. Binormal electric vortex lines along with $b-$lines ($be_{\mathbf{b}}$) are defined by

$$\mathbf{F} = m\nabla_b(\nabla_b\Upsilon) = q(\mathbf{G}_3 + \nabla_b\Upsilon \times \mathbf{W}_3), \tag{75}$$

where $\mathbf{W}_3$ is the magnetic vector field of binormal magnetic vortex lines in the binormal direction.

Here if one considers Eqs. $(10, 75)$ and the Theorem 12 (iii) then electric vector field of binormal electric vortex lines ($\mathbf{E}_3$) in the binormal direction is written by

$$\mathbf{G}_3 = -\delta_{bs}(1 + \frac{m}{q})\mathbf{t} - (\kappa + \operatorname{div}\mathbf{n})(1 + \frac{m}{q})\mathbf{n}. \tag{76}$$

6 Conclusion

We have used anholonomic coordinates of a three-dimensional vector field, which are supposed to govern the intrinsic features of vortex lines, to deduce particular geometric consequences of physical importance including the Fermi-Walker transportations of vortex lines, the angular velocity vectors of vortex lines, the magnetic fields of vortex lines. We also give relations between the obtained consequences. As one of the most important conclusions of the study, we finalize the paper by presenting the uniformness of the magnetic vector fields of directional magnetic vortex lines.

Uniformness of a magnetic field in a surface is the significant part of Landau-Hall problem, which deals with obtaining constant curvature curves in the given surface. Based on this study one can deduce the conditions of being uniform for each magnetic vector field of directional magnetic vortex lines in the principal normal and binormal direction by considering the fact that uniform magnetic fields are made up of parallel two forms. That is a magnetic field $\mathbf{V}$ is said to uniform if and only if

$$\nabla \cdot \mathbf{V} = 0. \tag{77}$$

For instance, magnetic vector field of tangent magnetic vortex lines ($\mathbf{V}_1$) in the principal normal direction is said to uniform if and only if

$$\frac{d}{ds}\varepsilon_1 - \frac{d}{dn}\left(\pi_b + \tau\right) + \frac{d}{db}\delta_{ns} = -\varepsilon_1\left(\delta_{ns} + \delta_{bs}\right) + \left(\pi_b + \tau\right)\operatorname{div}\mathbf{n} + \delta_{ns}\operatorname{div}\mathbf{b}. \tag{78}$$

One should consider the Theorem 10 (i) and the Eq. (77) for the validity of the above expression. It is a known fact that uniform magnetic fields are Killing magnetic fields. Then one can conclude that the magnetic field of tangent magnetic vortex lines ($\mathbf{V}_1$) in the principal normal direction is also a Killing magnetic field if the Eq. (78) is satisfied.

As stated before uniform magnetic fields correspond to parallel vector fields. Then a question which arises is whether a uniform magnetic field corresponds to a Fermi-Walker parallel vector field or not. Even though we have not reached a decisive answer for that question we can investigate every single case that has been investigated so far and determine when the uniform magnetic field is Fermi-Walker transported in these special cases. For example, the magnetic vector field of tangent magnetic vortex lines ($\mathbf{V}_1$) corresponds to an ordinary Fermi-Walker parallel transported in the principal normal direction if and only if

$$\frac{d}{dn}\varepsilon_1 = 0,\ \frac{d}{dn}\left(\pi_b + \tau\right) = \delta_{ns}\operatorname{div}\mathbf{b},\ \frac{d}{dn}\delta_{ns} = -\left(\pi_b + \tau\right)\operatorname{div}\mathbf{b};$$

it corresponds to a normal Fermi-Walker transported in the principal normal direction if and only if

$$\frac{d}{dn}\left(\pi_b+\tau\right)=0,\ \frac{d}{dn}\varepsilon_1=\delta_{ns}\left(\pi_b+\tau\right),\ \frac{d}{dn}\delta_{ns}=-\varepsilon_1\left(\pi_b+\tau\right);$$

it corresponds to a modified Fermi-Walker transported in the principal normal direction if and only if

$$\frac{d}{dn}\delta_{ns}=0,\ \frac{d}{dn}\varepsilon_1=-\delta_{ns}\left(\pi_b+\tau\right),\ \frac{d}{dn}\left(\pi_b+\tau\right)=-\delta_{ns}\varepsilon_1.$$

One should consider Theorem 10 (i) and Eqs. $(17,18,23)$ for the validity of the above expressions. We consider the magnetic vector field of tangent magnetic vortex lines ($\mathbf{V}_1$) in the principal normal direction when we determine the relationship between the uniformness condition and Fermi-Walker transportation as a sample case and other cases left to the reader.

In the three-dimensional space, an electric field line is supposed to be a curve whose direction is equal to the electric field's direction. Namely, the electric field line Ψ must meet the following equation:

$$\frac{d}{dx}\Psi=\mathbf{E},$$

where $\mathbf{E}$ is the electric field and $\frac{d}{dx}$ represents the derivative in the x direction [29]. Since we have investigated distinct electric vortex lines in the principal normal and binormal directions we can transform the above equation into the following forms:

$$\frac{d}{dn}\Psi = \mathbf{E}, \tag{79}$$

$$\frac{d}{db}\Psi = \mathbf{E}. \tag{80}$$

Thus we can define a curvature of the curve Ψ in the principal normal and binormal directions respectively in the following manner:

$$\kappa = \frac{\left|\frac{d}{dn}\Psi\times\frac{d}{dn}(\frac{d}{dn}\Psi)\right|}{\left|\frac{d}{dn}\Psi\right|^3}, \tag{81}$$

$$\kappa = \frac{\left|\frac{d}{db}\Psi\times\frac{d}{db}(\frac{d}{db}\Psi)\right|}{\left|\frac{d}{db}\Psi\right|^3}. \tag{82}$$

Thanks to these definitions we are able to determine the electromagnetic curvature of a tangent, principal normal, binormal electric vortex lines with

$n-$lines, which are vortex lines whose tangent vector is $\mathbf{n}$ and tangent, principal normal, binormal electric vortex lines with $b-$lines. For instance, the electromagnetic curvature of the tangent electric vortex line along with $n-$lines can be computed by

$$\kappa = \frac{\left|\mathbf{E}_1 \times \frac{d}{dn}(\mathbf{E}_1)\right|}{\left|\mathbf{E}_1\right|^3}$$

where $\frac{d}{dn} ne_{\mathbf{t}} = \mathbf{E}_1$ and $\mathbf{E}_1$ is given by the Eq. (66). Hence the electromagnetic curvature of the tangent electric vortex line along with $n-$lines is written by

$$\begin{aligned}
\kappa \;=\; & \frac{1}{(\delta_{ns}^2(1+\frac{m}{q})^2+(\varepsilon_1-\frac{m}{q}\operatorname{div}\mathbf{b})^2)^{\frac{3}{2}}}((-\delta_{ns}^2(1+\frac{m}{q}) \\
& +(\varepsilon_1-\frac{m}{q}\operatorname{div}\mathbf{b})\operatorname{div}\mathbf{b})^2(\delta_{ns}(1+\frac{m}{q})^2+(\varepsilon_1-\frac{m}{q}\operatorname{div}\mathbf{b})^2) \\
& +(-\delta_{ns}^2(1+\frac{m}{q})^2(\pi_b+\tau)+\delta_{ns}(1+\frac{m}{q})\frac{d}{dn}(\varepsilon_1-\frac{m}{q}\operatorname{div}\mathbf{b}) \\
& -(\varepsilon_1-\frac{m}{q}\operatorname{div}\mathbf{b})(1+\frac{m}{q})\frac{d}{dn}(\delta_{ns})-(\varepsilon_1-\frac{m}{q}\operatorname{div}\mathbf{b})^2(\pi_b+\tau))^2)^{\frac{1}{2}}.
\end{aligned}$$

This study will be also a fundamental source for anyone whose aim is to concentrate on the various type of flows (inextensible flows, Beltrami flows, Complex-Lamellar flows, etc.) of vortex lines and directional magnetic (or electric) vortex lines in principal and binormal direction. In the future, we will further investigate solitonic behavior of directional magnetic vortex lines and the directional electric vortex lines and their features of motions (uniformly accelerated motions, uniformly circular motion, unchanged direction motion).

REFERENCES

[1] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. Roy. Soc. London A **392** (1984) 45.

[2] A. Tomita, R.Y. Chiao, Observation of Berry's phase topological phase by use of an optical fiber, Physical Review Letters **57**(8) (1986) 937.

[3] F.D.M. Haldane, Path dependence of the geometric rotation of polarization in optical fibers, Optics Letters **11**(11) (1986) 730.

[4] I. Ciufolini, J.A. Wheeler, Gravitation and Inertia, Princeton University Press, 2018.

[5] F.W. Hehl, W.T. Ni, Inertial effects of a Dirac particle, Physical Review D **42**(6) (1990) 2045.

[6] B.R. Iyer, C.V. Vishveshwara, Frenet-Serret description of gyroscopic precession, Physical Review D **48**(12) (1993) 5706.

[7] R. Dandoloff, W.J. Zakrzewski, Parallel transport along a space curve and related phases, J. Phys. A: Math Gen. **22** (1989) L461.

[8] R. Dandoloff, Berry's Phase and Fermi-Walker Parallel Transport, Physics Letters A **139**(1,2) (1989) 19.

[9] J.H. Hannay, Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian, J. Phys. A: Math Gen. **18**(2) (1985) 221.

[10] Y. Aharonov, J. Anandan, Phase chance during a cyclic quantum evolution, Physical Review Letters **58**(16) (1987) 1593.

[11] M. P. Do Carmo, Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition, Courier Dover Publications, 2016.

[12] A. Masotti, Decomposione intrinseca del vortice e sue applicazioni, Istituto Lombardo di Scienze e Lettere, Rendiconti **60**(2) (1927) 869.

[13] O. Bjorgum, T. Godal, On Beltrami vector fields and flows, Universitet I Bergaen, 1951.

[14] A. W. Marris, S.L. Passman, Vector fields and flows on developable surfaces, Arch. Ration. Mech. Analysis **32** (1969) 29.

[15] W. K. Schief, C. Rogers, Binormal motion of curves constant curvature and torsion. Generation of soliton surfaces, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **455** (1999) 3163.

[16] M. Barros, J.L. Cabrerizo, M. Fernandez, A. Romero, Magnetic vortex filament flows. Journal of Mathematical Physics, **48**(8) (2007) 082904.

[17] Z. Bozkurt, I. Gök, Y. Yaylı, F. N. Ekmekçi, A new approach for magnetic curves in 3D Riemannian manifolds. Journal of Mathematical Physics, **55** (2014) 053501.

[19] S.L. Druta-Romaniuc, M.I. Munteanu, Magnetic Curves corresponding to Killing magnetic fields in E^3. Journal of Mathematical Physics, **52**(11) (2011) 1-11.

[19] S.L. Druta-Romaniuc, M.I. Munteanu, Killing magnetic curves in a Minkowski 3-space. Nonlinear Analysis:Real Word Application, **14**(1) (2013) 383-396.

[20] M.I. Munteanu, A.I. Nistor, The classification of Killing magnetic curves in $S^2 \times R$. Journal of Geometry and Physics, **62**(2) (2012) 170-182.

[21] D.A. Kalinin, Trajectory of charged particles in Kahler magnetic fields. Reports on Mathematical Physics, **39**(3) (1997) 299-309.

[22] S.L. Druta-Romaniuc, C.L. Bejan, Walker manifolds and Killing magnetic curves. Differential Geometry and its Applications, **35** (2014) 106-116.

[23] S.L. Druta-Romaniuc, J. Inoguchi, M.I. Munteanu, A.I. Nistor, Magnetic curves in Sasakian manifolds. Journal of Nonlinear Mathematical Physics, **22**(3) (2015) 428-447.

[24] J. Inoguchi, M.I. Munteanu, A.I. Nistor, Magnetic curves in quasi-Sasakian 3-manifolds. Analysis and Mathematical Physics, (2017) 1-19.

[25] M. Crasmareanu, C. Calin, Magnetic Curves in Three-Dimensional Quasi-Para-Sasakian Geometry. Mediterranean Journal of Mathematics, **13**(4) (2016) 2087-2097.

[26] S.L. Druta-Romaniuc, J. Inoguchi, M.I. Munteanu, A.I. Nistor, Magnetic curves in Cosymplectic Manifolds. Reports on Mathematical Physics, **78**(1) (2016) 33-48.

[27] T. Körpınar, R.C. Demirkol, Frictional magnetic curves in 3D Riemannian manifolds. International Journal of Geometric Methods in Modern Physics, **15**(2) (2018) 1850020.

[28] T. Körpınar, R.C. Demirkol, Gravitational magnetic curves on 3D Riemannian manifolds. International Journal of Geometric Methods in Modern Physics, 15(11) (2018) 1850184.

[29] Y. Hadad, E. Cohen, I. Kaminer, A.C. Elitzur, The hidden geometry of electromagnetism. Journal of Physics: Conference Series, **626** (2015) 012024.