Meir-Keeler type fixed point theorem and its application on integral equations

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Abstract. The aim of this article to propose some generalization of Meir-Keeler fixed point theorem with the help of an α-admissible mapping. Further we prove the existence of solution of an infinite system of integral equations by using this generalized fixed point theorem involving measure of noncompactness in Banach space and illustrate the results with the help of an example. Finally, apply an iterative algorithm we find an approximate solution of an infinite system of integral equations.

Key Words: Measure of noncompactness; Meir-Keeler condensing operator; System of integral equations; Homotopy perturbation method.

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1. Introduction

Integral equations of different types lead as an important branch of applied functional analysis and obtain numerous applications to describing miscellaneous real life problems. Fixed point theory and measure of noncompactness are very useful tools to solve different types of integral equations which help us to come across in different real life situations. Schauder and Darbo's fixed point theorems have important contribution to study for existence of solution of different types of functional integral equations. Aghajani et al. [3] discussed the application of generalized Darbo fixed theorem for existence of solution of systems of integral equations. The existence of solutions of infinite systems of integral equations in two variables by utilizing measure of noncompactness and Darbo fixed point theorem studied by Arab et al. [7]. Banaś and Olszowy [9] introduced the class of measure of noncompactness in Banach algebra to investigate the existence of solution of nonlinear integral equations by using Darbo fixed point theorem (also see [17]) and references therein. In [18], Mursaleen and Rizvi employed Meir-Keeler fixed point

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theorem for solving infinite systems of second order differential equations in sequence spaces c_0 and ℓ_1 (see also [5, 10, 12, 23]).

Suppose $\mathcal{E}$ is a real Banach space with the norm $\|\ .\ \|$ and $\mathcal{B}(a,b)$ is a closed ball in $\mathcal{E}$ centered at a with radius b. If $\mathcal{X}$ is a nonempty subset of $\mathcal{E}$ then by $\bar{\mathcal{X}}$ and Conv$\mathcal{X}$ we denote the closure and convex closure of $\mathcal{X}$. Moreover, we denote $\mathcal{M}_{\mathcal{E}}$, the family of all nonempty and bounded subsets of $\mathcal{E}$ and $\mathcal{N}_{\mathcal{E}}$ its subfamily consisting of all relatively compact sets. Also we denote $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$.
We recall the following definition of measure of noncompactness which was defined by Banaś and Goebel [8].

Definition 1.1. *A function $\mu : \mathcal{M}_{\mathcal{E}} \to \mathbb{R}_+$ is called a measure of non-compactness in $\mathcal{E}$ if it satisfies the following conditions:*

(i) *for all $\mathcal{Y} \in \mathcal{M}_{\mathcal{E}}$, we have $\mu(\mathcal{Y}) = 0$ implies that $\mathcal{Y}$ is precompact.*
(ii) *the family ker $\mu = \{\mathcal{Y} \in \mathcal{M}_{\mathcal{E}} : \mu(\mathcal{Y}) = 0\}$ is nonempty and ker $\mu \subset \mathcal{N}_{\mathcal{E}}$.*
(iii) $\mathcal{Y} \subseteq \mathcal{Z} \implies \mu(\mathcal{Y}) \leq \mu(\mathcal{Z})$.
(iv) $\mu(\bar{\mathcal{Y}}) = \mu(\mathcal{Y})$.
(v) $\mu(\ Conv\mathcal{Y}) = \mu(\mathcal{Y})$.
(vi) $\mu(\lambda\mathcal{Y} + (1-\lambda)\mathcal{Z}) \leq \lambda\mu(\mathcal{Y}) + (1-\lambda)\mu(\mathcal{Z})$ *for* $\lambda \in [0,1]$.
(vii) *if* $\mathcal{Y}_n \in \mathcal{M}_{\mathcal{E}}, \mathcal{Y}_n = \bar{\mathcal{Y}}_n, \mathcal{Y}_{n+1} \subset \mathcal{Y}_n$ *for* $n = 1, 2, 3, \ldots$ *and* $\lim\limits_{n\to\infty} \mu(\mathcal{Y}_n) = 0$ *then* $\bigcap\limits_{n=1}^{\infty} \mathcal{Y}_n \neq \phi$.

The family $ker\mu$ is said to be the *kernel of measure* μ. Observe that the intersection set $\mathcal{Y}_\infty$ from (vii) is a member of the family $ker\mu$. In fact, since $\mu(\mathcal{Y}_\infty) \leq \mu(\mathcal{Y}_n)$ for any n, we infer that $\mu(\mathcal{Y}_\infty) = 0$. This gives $\mathcal{Y}_\infty \in ker\mu$.

For a bounded subset $\mathcal{S}$ of a metric space $\mathcal{X}$, the Kuratowski measure of noncompactness of $\mathcal{S}$ defined by Kuratowski [15] as follows:

$$\alpha(\mathcal{S}) = \inf\left\{\delta > 0 : \mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i,\ \operatorname{diam}(\mathcal{S}_i) \leq \delta\ for\ 1 \leq i \leq n, n \in \mathbb{N}\right\},$$

where $\operatorname{diam}(\mathcal{S}_i)$ denotes the diameter of the set $\mathcal{S}_i$, that is

$$\operatorname{diam}(\mathcal{S}_i) = \sup\{d(x,y) : x, y \in \mathcal{S}_i\}.$$

The Hausdorff measure of noncompactness for a bounded set $\mathcal{S}$ is defined as

$$\chi(\mathcal{S}) = \inf\{\epsilon > 0 : \mathcal{S} \text{ has finite } \epsilon - \text{net in } \mathcal{X}\}.$$

Definition 1.2. [8] *Let $\mathcal{X}$ be a nonempty subset of a Banach space $\mathcal{E}$ and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous operator transforming bounded subset of $\mathcal{X}$ to bounded ones. We say that $\mathcal{T}$ satisfies the Darbo condition with a constant k with respect to measure μ provided $\mu(\mathcal{T}\mathcal{Y}) \leq k\mu(\mathcal{Y})$ for each $\mathcal{Y} \in \mathcal{M}_{\mathcal{E}}$ such that $\mathcal{Y} \subset \mathcal{X}$.*

Now we recall the Shauder and Darbo fixed point theorems:

Theorem 1.3. [2, Shauder] *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a Banach space $\mathcal{E}$. Then every compact, continuous map $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ has at least one fixed point.*

Theorem 1.4. [11, Darbo] *Let $\mathcal{Z}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$. Let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a continuous mapping. Assume that there is a constant $k \in [0, 1)$ such that*

$$\mu(\mathcal{T}\mathcal{M}_1) \leq k\mu(\mathcal{M}_1),\ \mathcal{M}_1 \subseteq \mathcal{Z}.$$

Then $\mathcal{T}$ has a fixed point.

Definition 1.5. [6] *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Banach spaces and let μ_1 and μ_2 be arbitrary measure of noncompactness on $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. An operator f from $\mathcal{E}_1$ to $\mathcal{E}_2$ is called a (μ_1, μ_2)-condensing operator if it is continuous and $\mu_2(f(\mathcal{D})) < \mu_1(\mathcal{D})$ for every set $\mathcal{D} \subset \mathcal{E}_1$ with compact closure.*

Remark 1.6. *If $\mathcal{E}_1 = \mathcal{E}_2$ and $\mu_1 = \mu_2 = \mu$, then f is called a $\mu-$condensing operator.*

The contractive maps and the compact maps are condensing if we take as measure of noncompactness of the diameter of a set and the indicator function of a family of non-relatively compact sets, respectively (see [6]). In 1969, Meir and Keeler [16] proved the following very interesting fixed point theorem, which is a generalization of the Banach contraction principle.

Definition 1.7. [16] *Let $(\mathcal{X}, d)$ be a metric space. Then a mapping $\mathcal{T}$ on $\mathcal{X}$ is said to be a Meir-Keeler contraction if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta \implies d(\mathcal{T}x, \mathcal{T}y) < \epsilon,\ \forall x, y \in \mathcal{X}.$$

Theorem 1.8. [16] *Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a Meir-Keeler contraction, then $\mathcal{T}$ has a unique fixed point.*

Definition 1.9. [4] *Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{E}$ and let μ be an arbitrary measure of noncompactness on $\mathcal{E}$. We say that an operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a Meir-Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq \mu(\mathcal{X}) < \epsilon + \delta \implies \mu(\mathcal{T}(\mathcal{X})) < \epsilon$$

for any bounded subset $\mathcal{X}$ of $\mathcal{C}$.

Aghajani et al. [4], discussed the following result, which is very useful in our study (also see [10]).

Theorem 1.10. [4] *Let $\mathcal{C}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and let μ be an arbitrary measure of noncompactness on $\mathcal{E}$. If $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a continuous and Meir-Keeler condensing operator, then $\mathcal{T}$ has at least one fixed point and the set of all fixed points of $\mathcal{T}$ in $\mathcal{C}$ is compact.*

Hazarika et al. [13] proved a generalized version of Theorem 1.10 as follows:

Definition 1.11. [13] *Let Θ be class of all functions $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:*

(i) $\max\{u, v\} \leq \theta(u, v)$ *for* $u, v \geq 0$,
(ii) θ *is continuous and nondecreasing,*
(iii) $\theta(u + l, v + m) \leq \theta(u, v) + \theta(l, m)$ *for* $u, v, l, m \geq 0$.

Definition 1.12. [13] *Let $\mathcal{Z}$ be a nonempty subset of a Banach space $\mathcal{E}$ and μ be a measure of noncompactness on $\mathcal{E}$. We say that an operator $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ is a generalized Meir-Keeler type operator if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any subset $\mathcal{X}$ of $\mathcal{Z}$,*

$$\epsilon \leq \theta(\mu(\mathcal{X}), \psi(\mu(\mathcal{X}))) \leq \epsilon + \delta \implies \theta(\mu(\mathcal{T}\mathcal{X}), \psi(\mu(\mathcal{T}\mathcal{X}))) < \epsilon$$

where $\psi \in \Psi = \{\psi | \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *continuous and nondecreasing*$\}$ *and* $\theta \in \Theta$.

Theorem 1.13. [13] *Let $\mathcal{Z}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and μ be an arbitrary measure of noncompactness on $\mathcal{E}$. Let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a continuous and generalized Meir-Keeler type condensing operator, then $\mathcal{T}$ has a fixed point in $\mathcal{Z}$.*

In order to establish our fixed point theorem, we used the following concept, which was introduced by Samet et al [22].

Definition 1.14. [22] *Let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ and $\alpha : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$. We say that $\mathcal{T}$ is α-admissible if for every $x, y \in \mathcal{Z}$*

$$\alpha(x, y) \geq 1 \implies \alpha(\mathcal{T}x, \mathcal{T}y) \geq 1.$$

Example 1.15. [22] *Let $\mathcal{Z} = [0, \infty)$. Define $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ by $\mathcal{T}z = \ln z$ for all $z \in \mathcal{Z}$, and $\alpha : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ defined by*

$$\alpha(x, y) = \begin{cases} 2, & x \geq y \\ 0, & x < y \end{cases}$$

Then $\mathcal{T}$ is α-admissible.

2. Generalized Meir-Keeler fixed point theorem

In this section we introduce a generalize version of Meir-Keeler fixed point theorem applying α-admissible mapping.

Definition 2.1. *$\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be an α-admissible mapping. We say that $\mathcal{T}$ is a generalized α-Meir-Keeler type condensing operator for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $z \in \mathcal{Z}$,*

$$\epsilon \leq \theta(\mu(\mathcal{Z}), \psi(\mu(\mathcal{Z}))) \leq \epsilon + \delta \implies \alpha(\mathcal{T}z, \mathcal{T}z)\theta(\mu(\mathcal{T}\mathcal{Z}), \psi(\mu(\mathcal{T}\mathcal{Z}))) < \epsilon$$

where $\psi \in \Psi = \{\psi | \psi : \mathbb{R}_+ \to \mathbb{R}_+$*continuous and nondecreasing*$\}$ *and* $\theta \in \Theta$.

Theorem 2.2. *Let $\mathcal{Z}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and μ be an arbitrary measure of noncompactness on $\mathcal{E}$. Let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a continuous and generalized α-Meir-Keeler type condensing operator, then $\mathcal{T}$ has a fixed point in $\mathcal{Z}$.*

Proof. Let $\mathcal{Z}_0 = \mathcal{Z}$. Construct the sequences of sets $\{\mathcal{Z}_n\}$ and elements $\{z_n\}$ as follows: $\mathcal{Z}_{n+1} = \text{Conv}(\mathcal{T}\mathcal{Z}_n)$ and $z_{n+1} = \mathcal{T}z_n$ with $\alpha(z_0, z_0) \geq 1$ and $z_n \in \mathcal{Z}_n$ for all $n \geq 0$. Now,

$$\begin{aligned}
&\mathcal{T}Z_0 = \mathcal{T}Z \subseteq \mathcal{Z} = \mathcal{Z}_0, \\
&\mathcal{Z}_1 = \text{Conv}(\mathcal{T}\mathcal{Z}_0) \subseteq \mathcal{Z} = \mathcal{Z}_0, \\
&\mathcal{Z}_2 = \text{Conv}(\mathcal{T}\mathcal{Z}_1) \subseteq \text{Conv}(\mathcal{T}\mathcal{Z}_0) = \mathcal{Z}_1
\end{aligned}$$

and so on.

Therefore we obtain $\mathcal{Z}_0 \supseteq \mathcal{Z}_1 \supseteq \mathcal{Z}_2 \supseteq \ldots \supseteq \mathcal{Z}_n \supseteq \mathcal{Z}_{n+1} \supseteq \ldots$ and $\mathcal{T}\mathcal{Z}_{n+1} \subseteq \mathcal{T}\mathcal{Z}_n \subseteq \text{Conv}(\mathcal{T}\mathcal{Z}_n) = \mathcal{Z}_{n+1}$. Thus $\mathcal{T}\mathcal{Z}_n \subseteq \mathcal{Z}_n$ for all $n \geq 0$.

If there exists a natural number N such that $\mu(\mathcal{Z}_N) = 0$ then $\mathcal{Z}_N$ is compact. By Schauder's fixed point theorem we conclude that $\mathcal{T}$ has a fixed point.

So we assume that $\mu(\mathcal{Z}_n) > 0$ for some $n \geq 0$.

Define $\epsilon_n = \theta(\mu(\mathcal{Z}_n), \psi(\mu(\mathcal{Z}_n)))$.

For $\alpha(z_0, z_0) \geq 1 \implies \alpha(\mathcal{T}z_0, \mathcal{T}z_0) \geq 1 \implies \alpha(z_1, z_1) \geq 1$.

Proceeding in a similar manner we obtain, $\alpha(z_n, z_n) \geq 1$ for all $n \geq 0$ and

$$\epsilon_n = \theta(\mu(\mathcal{Z}_n), \psi(\mu(\mathcal{Z}_n))) \geq \theta(\mu(\mathcal{Z}_{n+1}), \psi(\mu(\mathcal{Z}_{n+1}))) = \epsilon_{n+1}.$$

Therefore ϵ_n is a positive non increasing sequence and there exists $b \geq 0$ such that $\epsilon_n \to b$ as $n \to \infty$.

If $b > 0$ then there exists $m \in \mathbb{N}$ such that $n > m$ gives $b \leq \epsilon_n \leq b + \delta(b)$, where $\delta(b) > 0$. Therefore by Definition 2.1 we get

$$\alpha(\mathcal{T}z_n, \mathcal{T}z_n)\epsilon_{n+1} < b \implies \alpha(z_{n+1}, z_{n+1})\epsilon_{n+1} < b.$$

Since $\alpha(z_{n+1}, z_{n+1}) \geq 1$ therefore $\epsilon_{n+1} < b$ which is a contradiction. Thus we conclude $b = 0$. Since the sequence $(\mathcal{Z}_n)$ is nested in view of axiom (vii), we conclude that $\mathcal{Z}_\infty = \bigcap_{n=1}^{\infty} \mathcal{Z}_n$ is nonempty, closed and convex subset of $\mathcal{Z}$. Moreover, $\mathcal{Z}_\infty \in ker\mu$. So $\mathcal{Z}_\infty$ is compact $\mathcal{Z}_\infty$ and invariant under $\mathcal{T}$. Thus Schauder's theorem implies that $\mathcal{T}$ has a fixed point in $\mathcal{Z}_\infty \subseteq \mathcal{Z}$. This completes the proof. □

Corollary 2.3. *If we take $\alpha(x, y) = 1$ then generalized α-Meir-Keeler condensing operator becomes generalized Meir-Keeler condensing operator.*

Theorem 2.4. *Let $\mathcal{Z}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and μ be an arbitrary measure of noncompactness on $\mathcal{E}$. Let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a continuous and generalized Meir-Keeler type condensing operator, then $\mathcal{T}$ has a fixed point in $\mathcal{Z}$.*

Proof. The result follows by taking $\alpha(x, y) = 1$ for every $x, y \in \mathcal{Z}$ in Theorem 2.2. □

Theorem 2.5. *Let $\mathcal{Z}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and μ be an arbitrary measure of noncompactness on $\mathcal{E}$. Let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a continuous operator satisfying*

(i) *$\mathcal{T}$ is α-admissible mapping on $\mathcal{Z}$,*

(ii) *for each* $\epsilon > 0$ *such that* $\delta > 0$ *such that for all* $z \in \mathcal{Z}$ *we have*

$$\epsilon \leq \mu(\mathcal{T}\mathcal{Z}) + \psi\left(\mu(\mathcal{T}\mathcal{Z})\right) < \epsilon + \delta \implies \alpha(\mathcal{T}z, \mathcal{T}z)\left(\mu(\mathcal{Z}) + \psi\left(\mu(\mathcal{Z})\right)\right) < \epsilon,$$

(iii) $\psi \in \Psi$,

then $\mathcal{T}$ *has at least one fixed point in* $\mathcal{Z}$.

Proof. The result follows by taking $\theta(l, m) = l + m$ in Theorem 2.2. □

Theorem 2.6. *Let* $\mathcal{Z}$ *be a nonempty, bounded, closed and convex subset of a Banach space* $\mathcal{E}$ *and* μ *be an arbitrary measure of noncompactness on* $\mathcal{E}$*.Let* $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ *be a continuous operator satisfying*

(i) $\mathcal{T}$ *is* α*- admissible mapping on* $\mathcal{Z}$.

(ii) *for each* $\epsilon > 0$ *such that* $\delta > 0$ *such that for all* $z \in \mathcal{Z}$ *we have*

$$\epsilon \leq \mu(\mathcal{Z}) < \epsilon + \delta \implies \alpha(\mathcal{T}z, \mathcal{T}z)\mu(\mathcal{T}\mathcal{Z}) < \epsilon,$$

then $\mathcal{T}$ *has at least one fixed point in* $\mathcal{Z}$.

Proof. The result follows by taking $\psi \equiv 0$ in Theorem 2.5. □

3. Application of generalized Meir-Keeler fixed point theorem

The Hausdorff measure of noncompactness χ in the Banach space $(c_0, \| \,.\, \|_{c_0})$ defined by Banaś and Goebel [8] as follows:

$$\chi_{c_0}\left(\hat{D}\right) = \lim_{n\to\infty}\left[\sup_{z\in\hat{D}}\left(\max_{k\geq n} \mid z_k \mid\right)\right], \tag{3.1}$$

where $z = (z_i)_{i=1}^{\infty} \in c_0$ and $\hat{D} \in \mathcal{M}_{c_0}$.

Let us denote by $C(I, c_0)$ the space of all continuous functions on $I = [0, T]$ with values in c_0. Then $C(I, c_0)$ is also a Banach space with norm $\| z(t) \|_{C(I,c_0)} = \sup\left\{\| z(t) \|_{c_0} : t \in I\right\}$, where $z(t) = (z_i(t))_{i=1}^{\infty} \in C(I, c_0)$.

For any non-empty bounded subset $\mathcal{Z}$ of $C(I, c_0)$ and $t \in I$, let $\mathcal{Z}(t) = \{z(t) : z \in \mathcal{Z}\}$. Now, using (3.1), we conclude that the Housdorff measure of noncompactness for $\mathcal{Z} \subset C(I, c_0)$ can be defined by

$$\chi_{C(I,c_0)}(\mathcal{Z}) = \sup\left\{\chi_{c_0}(\mathcal{Z}(t)) : t \in I\right\}.$$

In this part we study the solvability of the following infinite system of integral equations

$$z_n(t) = P_n\left(t, z(t), \int_0^t Q_n(t, s, z(s))ds\right), t \in [0, 1] = I. \tag{3.2}$$

where $z(t) = (z_n(t))_{n=1}^{\infty}$ and $z_n(t) \in C(I, c_0)$ for all $n \in \mathbb{N}$.

We consider the following assumptions

(1) The functions $P_n : I\times C(I,c_0)\times\mathbb{R}\to\mathbb{R}$ is continuous with $K_n = \sup\left\{\left|P_n(t,z^0(t),0)\right| : t\in I\right\}$, where $z^0(t) = \left(z_n^0(t)\right)_{n=1}^{\infty}$ and $z_n^0(t) = 0$ for all $t\in I,\ n\in\mathbb{N}$ such that $(K_n)_{n=1}^{\infty}$ converges to zero. Also,

$$|P_n(t,z(t),l) - P_n(t,\bar{z}(t),m)| \le C_n(t)\,|z_n(t)-\bar{z}_n(t)| + D_n(t)\,|l-m|\,,$$

where $C_n, D_n : I\to\mathbb{R}_+$ are continuous functions for all $n\in\mathbb{N}$ and $\bar{z}(t) = (\bar{z}_n(t))_{n=1}^{\infty}\in C(I,c_0)$.

(2) $Q_n : I\times I\times C(I,c_0)\to\mathbb{R}$ is continuous functions for all $n\in\mathbb{N}$ and there exists G_n such that

$$G_n = \sup\left\{D_n(t)\left|\int_0^t Q_n(t,s,z(s))ds\right| : t\in I\right\}$$

and (G_n) converges to zero.

(3) Define an operator $\mathcal{T}$ on $I\times C(I,c_0)$ to $C(I,c_0)$ as follows $(t,z(t))\to(\mathcal{T}z)(t)$, where $(\mathcal{T}z)(t) = ((\mathcal{T}_n z)(t))_{n=1}^{\infty} = \left(P_n\left(t,z(t),\int_0^t Q_n(t,s,z(s))ds\right)\right)_{n=1}^{\infty}$.

(4) Let $\sup_n G_n = G,\ \sup_n K_n = K,\ \sup\{C_n(t) : n\in\mathbb{N},\ t\in I\} = C,\ \sup\{D_n(t) : n\in\mathbb{N},\ t\in I\} = D$ and $0 < C < 1$.

Theorem 3.1. *Under the hypothesis (1)-(4), the infinite system of equations (3.2) has at least one solution in $C(I,c_0)$.*

Proof. By using (3.2) and assumption $(1)-(4)$, for arbitrary fixed $t\in I$, we get

$$\begin{aligned}
&\| z(t) \|_{c_0}\\
&= \sup_{n\ge1}\left|P_n\left(t,z(t),\int_0^t Q_n(t,s,z(s))ds\right)\right|\\
&\le \sup_{n\ge1}\left|P_n\left(t,z(t),\int_0^t Q_n(t,s,z(s))ds\right) - P_n\left(t,z^0(t),0\right)\right| + \sup_{n\ge1}\left|P_n\left(t,z^0(t),0\right)\right|\\
&\le \sup_{n\ge1}\left\{C_n(t)\,|z_n(t)| + D_n(t)\left|\int_0^t Q_n(t,s,z(s))ds\right|\right\} + K\\
&\le C \,\| z(t) \|_{c_0} + G + K
\end{aligned}$$

i.e.

$$(1-C)\,\| z(t) \|_{c_0}\le G+K$$
$$\| z(t) \|_{c_0}\le \frac{G+K}{1-C} = r \text{ (say)}.$$

Therefore $\| z(t) \|_{C(I,c_0)}\le r$. Let $B = \left\{z\in C(I,c_0) : \| z \|_{C(I,c_0)}\le r\right\}$. By assumption (3), the operator $\mathcal{T}$ satisfies the condition that $(\mathcal{T}z)(t)\in C(I,c_0)$.

Therefore for any arbitrary $t\in I$,

$$\begin{aligned}
&\| (\mathcal{T}z)(t) \|_{c_0}\le r\\
\Longrightarrow &\sup_{t\in I}\| (\mathcal{T}z)(t) \|_{c_0}\le r\\
\Longrightarrow &\| (\mathcal{T}z) \|_{C(I,c_0)}\le r
\end{aligned}$$

i.e. $\mathcal{T}$ is a self mapping on B.

Now, we prove that $\mathcal{T}$ is continuous on B. Let $\epsilon > 0$ be arbitrary and $\| z - \bar{z} \|_{C(I,c_0)} < \frac{\epsilon}{2C}$ for $z, \bar{z} \in B$. Thus for arbitrary $t \in I$ we have

$$\begin{aligned}
&\| (\mathcal{T}z)(t) - (\mathcal{T}\bar{z})(t) \|_{c_0} \\
&= \sup_{n\geq 1} \left| P_n\left(t, z(t), \int_0^t Q_n(t,s,z(s))ds\right) - P_n\left(t, \bar{z}(t), \int_0^t Q_n(t,s,\bar{z}(s))ds\right) \right| \\
&\leq \sup_{n\geq 1} \left\{ C_n(t)\left|z_n(t) - \bar{z}_n(t)\right| + D_n(t) \left| \int_0^t Q_n(t,s,z(s))ds - \int_0^t Q_n(t,s,\bar{z}(s))ds \right| \right\} \\
&\leq C \| z(t) - \bar{z}(t) \|_{c_0} + \sup_{n\geq 1} \left\{ D_n(t) \int_0^t \left|Q_n(t,s,z(s)) - Q_n(t,s,\bar{z}(s))\right| ds \right\} \\
&< \frac{\epsilon}{2} + D \sup_{n\geq 1} \left\{ \int_0^t \left|Q_n(t,s,z(s)) - Q_n(t,s,\bar{z}(s))\right| ds \right\}.
\end{aligned}$$

As Q_n is continuous for all $n \in \mathbb{N}$ and $I \times I \times B$ is compact, therefore Q_n is uniformly convergent. Therefore

$$|Q_n(t,s,z(s)) - Q_n(t,s,\bar{z}(s))| < \frac{\epsilon}{2(TD+1)}, \text{ for } \| z - \bar{z} \|_{C(I,c_0)} < \frac{\epsilon}{2C}.$$

Hence we have

$$\begin{aligned}
&\| (\mathcal{T}z)(t) - (\mathcal{T}\bar{z})(t) \|_{c_0} < \frac{\epsilon}{2} + D\int_0^t \frac{\epsilon}{2(TD+1)} < \epsilon \\
&\text{i.e. } \| (\mathcal{T}z)(t) - (\mathcal{T}\bar{z})(t) \|_{C(I,c_0)} < \epsilon.
\end{aligned}$$

Thus, $\mathcal{T}$ is continuous on B.

We have for any $t \in I$,

$$\begin{aligned}
&\chi_{c_0}(\mathcal{T}(B)) \\
&= \lim_{n\to\infty} \sup_{z(t)\in B} \sup_{k\geq n} \left| P_n\left(t, z(t), \int_0^t Q_n(t,s,z(s))ds\right) \right| \\
&\leq \lim_{n\to\infty} \sup_{z(t)\in B} \sup_{k\geq n} \{C_n(t)\left|z_n(t)\right| + G_n + K_n\} \\
&\leq C\chi_{c_0}(B).
\end{aligned}$$

Therefore

$$\sup_{t\in I} \chi_{c_0}(\mathcal{T}(B)) \leq C \sup_{t\in I} \chi_{c_0}(B)$$

gives

$$\chi_{C(I,c_0)}(\mathcal{T}(B)) \leq C\chi_{C(I,c_0)}(B).$$

Observe that $\chi_{C(I,c_0)}(\mathcal{T}(B)) \leq C\chi_{C(I,c_0)}(B) < \epsilon$ gives $\chi_{C(I,c_0)}(B) < \frac{\epsilon}{C}$.
Considering $\delta = \frac{\epsilon(1-C)}{C}$, we get $\epsilon \leq \chi_{C(I,c_0)}(B) < \epsilon + \delta$. Therefore $\mathcal{T}$ satisfies all the conditions of Theorem 2.6 for $\alpha(x,y) = 1$ for all x, y which implies that $\mathcal{T}$ has at least one fixed point on B. Therefore the system (3.2) has at least a solution in $B \subset C(I, c_0)$. $\square$

Example 3.2. Consider the following system of equations

$$z_n(t) = \frac{tz_n(t)}{(1+t)n^4} + \frac{1}{n^7}\int_0^t \frac{\cos(z_n(s))}{5+\sin\left(\sum\limits_{j=1}^{n} z_j(s)\right)} ds \tag{3.3}$$

for $t \in [0,1] = I,\ n \in \mathbb{N}$.

Here we have

$$P_n(t, z(t), y_n(z(t))) = \frac{tz_n(t)}{(1+t)n^4} + \frac{y_n(z(t))}{n^7},$$
$$y_n(z(t)) = \int_0^t Q_n\left(t, s, z(s)\right) ds,$$
$$Q_n\left(t, s, z(s)\right) = \frac{\cos(z_n(s))}{5+\sin\left(\sum\limits_{j=1}^{n} z_j(s)\right)}.$$

If $z(t) \in C(I, c_0)$ then $(P_n(t, z(t), y_n(z(t))))_{n=1}^{\infty} \in C(I, c_0)$. Again, if $\bar{z}(t) = (\bar{z}_i(t))_{n=1}^{\infty} \in C(I, c_0)$ then we have

$$\begin{aligned} &|P_n(t, z(t), l) - P_n(t, \bar{z}(t), m)| \\ &= \left| \frac{tz_n(t)}{(1+t)\, n^4} + \frac{l}{n^7} - \frac{t\bar{z}_n(t)}{(1+t)\, n^4} - \frac{m}{n^7} \right| \\ &\le \frac{t}{(1+t)\, n^4} \left| z_n(t) - \bar{z}_n(t) \right| + \frac{1}{n^7} \left| l - m \right|. \end{aligned}$$

Here $C_n(t) = \frac{t}{(1+t)n^4}$ and $D_n(t) = \frac{1}{n^7}$ are both continuous functions for all $n \in \mathbb{N}$. Also, $K_n = 0$, therefore (K_n) converges to zero and $0 < C < 1$.
Again,

$$G_n = \sup_{t\in I} \left\{ \frac{1}{n^7} \left| \int_0^t \frac{\cos(z_n(s))}{5+\sin\left(\sum\limits_{j=1}^{n} z_j(s)\right)} ds \right| \right\} = \frac{1}{n^7}.$$

Therefore (G_n) converges to zero and $D = 1,\ G = 1$. It is obvious that P_n and Q_n are continuous functions. So all assumptions from $(1) - (4)$ are satisfied. Hence by Theorem 3.1 we conclude that equation (3.3) has a solution in $C(I, c_0)$.

3.1. **Homotopy perturbation and adomain decomposition method to solve (3.3).** In [1, 21] authors solved nonlinear problems by using Adomian decomposition method. We use homotopy perturbation method to transform a nonlinear problem to a simple problem and apply Adomian polynomials to avoid nonlinearity. We also construct an iteration algorithm to find the solution of infinite system of nonlinear integral equations. In general case we consider the following nonlinear problem with boundary conditions

$$\mathcal{A}(x_n) - h(t) = 0,\ t \in \Omega \tag{3.4}$$

with

$$\mathcal{B}\left(x_n, \frac{\partial x_n}{\partial r}\right) = 0,\ r \in \Gamma, \tag{3.5}$$

where $\mathcal{A}$ is a general differential operator, $\mathcal{B}$ is a boundary operator and h is a known analytic function. As in [14, 19, 20], we define the following homotopy perturbation operator by q embedding parameter

$$\mathcal{H}(w_n, q) = (1-q)(\mathcal{L}(w_n) - \mathcal{L}(w_0)) + q(\mathcal{A}(w_n) - h(t)) = 0,\ n \in \mathbb{N},\ q \in [0,1], \tag{3.6}$$

where $\mathcal{L}$ is a linear operator and

$$z_n(t) \simeq w_n(t) = w_{0,n}(t) + qw_{1,n}(t) + p^2 w_{2,n}(t) + p^3 w_{3,n}(t) + \dots \tag{3.7}$$

also $w_0(t)$ is an initial approximation of solution that is defined by the initial condition of (3.3). By variation of q from 0 to 1, we obtain $w_n(t) = w_0(t)$ to $\mathcal{A}(w_n) - h(t) = 0$. So we obtain the solution of (3.4) for $q = 1$ and $z_n(t) \simeq \lim\limits_{q\to 1} w_n(t)$.

Consider the following infinite system of integral equations,

$$z_n(t) - f(t,n)\int_0^t \frac{\cos(z_n(s))}{5 + \sin\left(\sum_{j=1}^n z_j(s)\right)} ds = 0, \tag{3.8}$$

where $f(t,n) = \frac{t+1}{(1+t)n^4 - t} \cdot \frac{1}{n^3}$ and $n \in \mathbb{N}$. We take $\mathcal{L}$ and $\mathcal{A}$ operators for (3.8) as follows

$$\mathcal{L}(z_n) = z_n(t),\ \mathcal{A}(z_n) = z_n(t) - f(t,n)\int_0^t \frac{\cos(z_n(s))}{5 + \sin\left(\sum_{j=1}^n z_j(s)\right)} ds. \tag{3.9}$$

Applying (3.7) and (3.9) in (3.6) we get

$$(1-q)(w_n(t) - w_0(t)) + q\left(w_n(t) - f(t,n)\int_0^t \frac{\cos(w_n(s))}{5 + \sin\left(\sum\limits_{j=1}^n w_j(s)\right)} ds\right) = 0$$

and

$$\left(\sum_{i=0}^{\infty} q^i w_{i,n}(t) - w_0(t)\right) + q\left(w_0(t) - f(t,n)\int_0^t \frac{\cos\left(\sum\limits_{i=0}^{\infty} q^i w_{i,n}(s)\right)}{5 + \sin\left(\sum\limits_{j=1}^n \sum\limits_{i=0}^{\infty} q^i w_{i,j}(s)\right)} ds\right) = 0.$$

Applying Adomian polynomials to approximate the above integrand we obtain

$$\left(w_{0,n}(t) + qw_{1,n}(t) + q^2 w_{2,n}(t) + \dots - w_0(t)\right) + q\left(w_0(t) - f(t,n)\int_0^t \sum_{i=0}^{\infty} q^i A_{i,n}(s) ds\right) = 0, \tag{3.10}$$

where the Adomian polynomials are given by

$$A_{k,n}(s) = \frac{1}{k!}\left[\frac{d^k}{dq^k}\left(\frac{\cos\left(\sum\limits_{i=0}^{\infty} q^i w_{i,n}(s)\right)}{5+\sin\left(\sum\limits_{j=1}^{n}\sum\limits_{i=0}^{\infty} q^i w_{i,j}(s)\right)}\right)\right]_{q=0}. \tag{3.11}$$

Rearranging (3.10) in terms of powers of q we obtain

$$\begin{aligned}
&q^0 : (w_{0,n}(t) - w_0(t)),\\
&q^1 : (w_{1,n}(t) + w_0(t) - f(t,n)\int_0^t A_{0,n}(s)ds),\\
&q^j : (w_{j,n}(t) - f(t,n)\int_0^t A_{j-1,n}(s)ds)
\end{aligned}$$

where $j = 2, 3, \ldots$ and $n \in \mathbb{N}$. By (3.10) we observe that the coefficients of different powers of q are equal to zero. Consequently we obtain the following algorithm to obtain numerical solution of (3.3).

Algorithm:

$$\begin{aligned}
w_{0,n}(t) &= w_0(t),\\
w_{1,n}(t) &= -w_0(t) + f(t,n)\int_0^t A_{0,n}(s)ds,\\
w_{j,n}(t) &= f(t,n)\int_0^t A_{j-1,n}(s)ds.
\end{aligned}$$

where $j = 2, 3, \ldots$ and $n \in \mathbb{N}$. By (3.8), we obtain $z_n(0) = 0$ and by (3.7) we obtain $w_{0,n}(0) = w_{1,n}(0) = w_{2,n}(0) = \ldots = 0$. Therefore we choose $w_{0,n}(t) = w_0(t) = 0$ in algorithm and we have

$$\begin{aligned}
w_{0,n}(t) &= 0\\
w_{j,n}(t) &= \frac{t+1}{(1+t)n^4 - t}.\frac{1}{n^3}\int_0^t A_{j-1,n}(s)ds,
\end{aligned} \tag{3.12}$$

where $j = 1, 3, \ldots$ and $n \in \mathbb{N}$.
Since $w_{0,n}(t) = 0$ for all $n \in \mathbb{N}$, then $A_{0,n}(s) = \frac{1}{5}$. Therefore, by (3.12) the first three terms of the series (3.7) for $n = 1, 10, 100$ are given by,

$$\begin{aligned}
&n = 1,\ w_{0,1}(t) = 0,\ w_{1,1}(t) = (1+t)\int_0^t A_{0,n}(s)ds = \frac{t(1+t)}{5},\\
&n = 10,\ w_{0,10}(t) = 0,\ w_{1,10}(t) = \frac{1+t}{(1+t)10^4 - t}.\frac{1}{10^3}\int_0^t A_{0,10}(s)ds = \frac{(1+t)t}{5\times 10^3(10^4 + 9999t)},\\
&n = 100,\ w_{0,100}(t) = 0,\ w_{1,100}(t) = \frac{1+t}{(1+t)10^8 - t}.\frac{1}{10^6}\int_0^t A_{0,100}(s)ds = \frac{(1+t)t}{5\times 10^6(10^8 + 99999999t)}.
\end{aligned}$$

The approximate solution of equation (3.3)is given by

$$z_n(t) = w_{0,n}(t) + w_{1,n}(t) \text{ for all } n \in \mathbb{N}.$$

To show the convergence of the sequence $(z_n(t))_{n=1}^{\infty}$ some terms of this sequence $n = 1, 10, 100$ are drawn in the following figures:

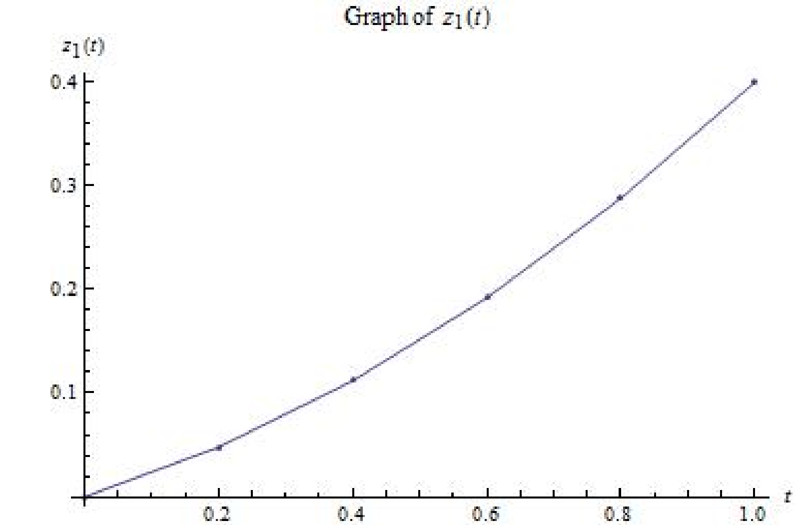

Figure 1

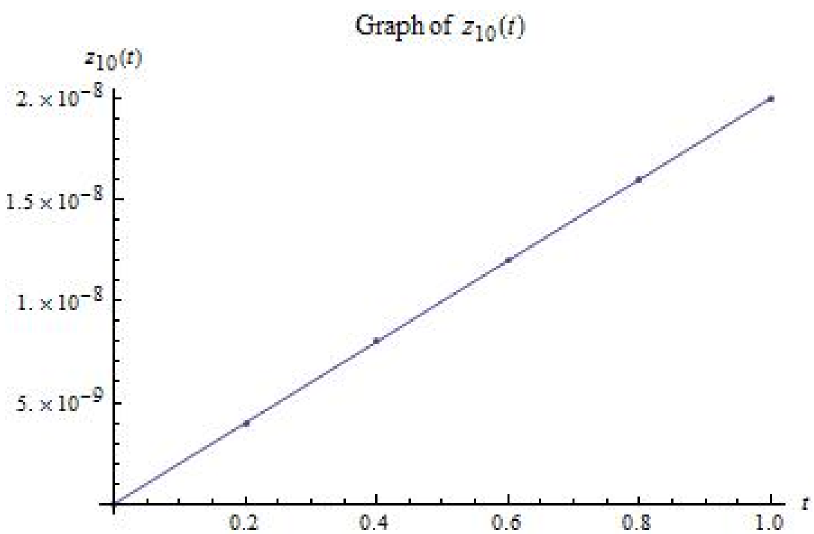

Figure 2

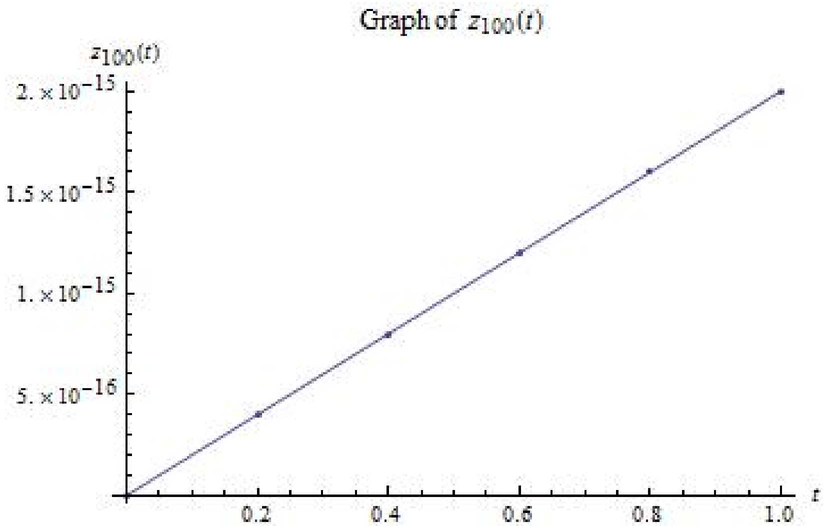

Figure 3

From figures $(1-3)$ we observe that for $0 \leq t \leq 1$ as the value of n increases then the values of $z_n(t)$ decreases. Also we observe that $z_1(t) \leq 0.4$, $z_{10}(t) \leq 2 \times 10^{-8}$, $z_{100}(t) \leq 2 \times 10^{-15}$ for $0 \leq t \leq 1$. Thus we conclude that $z_n(t) \to 0$ as $n \to \infty$ for $0 \leq t \leq 1$.

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