Asymptotic profiles of the endemic equilibrium of a diffusive SIS epidemic system with saturated incidence rate and spontaneous infection *

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Abstract

An SIS epidemic reaction-diffusion model with saturated incidence rate and spontaneous infection is considered. We establish the existence of endemic equilibrium by using a fixed point theorem. We mainly investigate the effects of diffusion and saturation on asymptotic profiles of the endemic equilibrium. Our analysis shows that the spontaneous infection can enhance persistence of infectious disease.

Keywords: SIS epidemic model; saturated incidence rate; spontaneous infection; endemic equilibrium; small/large diffusion; asymptotic profile

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1 Introduction

Understanding the development and extension of the diffusive susceptible-infected-susceptible (SIS) model is an important topic in the spatial transmission of a disease. Some important questions such as distinct dispersal rate may have different impacts on disease dynamics have been investigated and answered. One may refer to [2, 3, 10, 13, 33, 34].

To study the effect of spatial heterogeneity and the individual movement on the disease dynamics, Allen et al. in [1] considered the following SIS epidemic reaction-diffusion system:

$$\begin{cases} \dfrac{\partial \bar S}{\partial t} - d_S\Delta \bar S = -\beta(x)\dfrac{\bar S\bar I}{\bar S+\bar I} + \gamma(x)\bar I, & x\in\Omega,\ t>0,\\ \dfrac{\partial \bar I}{\partial t} - d_I\Delta \bar I = \beta(x)\dfrac{\bar S\bar I}{\bar S+\bar I} - \gamma(x)\bar I, & x\in\Omega,\ t>0,\\ \dfrac{\partial \bar S}{\partial \nu} = \dfrac{\partial \bar I}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ \bar S(x,0)=\bar S_0(x)\ge 0, \bar I(x,0)=\bar I_0(x)\ge,\not\equiv 0, & x\in\Omega. \end{cases} \tag{1.1}$$

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Here, $\bar{S}$ and $\bar{I}$ stand for the density of susceptible and infected population at location x and time t respectively; the habitat Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, and the Neumann boundary conditions mean that no individual flux across the boundary; the positive constants d_S and d_I represent the motility of susceptible and infected individuals respectively; the positive Hölder continuous functions $\beta(x)$ and $\gamma(x)$ denote the rates of disease transmission and recovery at location x, respectively; the infection mechanism $\bar{S}\bar{I}/(\bar{S}+\bar{I})$ is called standard incidence rate.

The main results of [1] concerned the existence, uniqueness and asymptotic profiles of the endemic equilibrium (the positive solution of the corresponding steady state system to (1.1)) as the diffusion rate d_S of the susceptible individuals approaches to zero. The global stability of the endemic equilibrium and asymptotic behavior of the endemic equilibrium were studied in [36, 37, 39]. In [40], Peng and Zhao considered the diffusive SIS model with spatially heterogeneous and temporally periodic disease transmission and recovery rates. For the SIS epidemic model with mass action (i.e. the infection mechanism term is $\beta\bar{S}\bar{I}$), the dynamics and asymptotic behaviors of steady states of the SIS epidemic model have been analyzed in [12, 45, 46]. In [44], Wang et al. investigated the SIS epidemic model with saturation (the infection mechanism term is $\bar{S}\bar{I}/(1+m\bar{I})$, the positive constant m is saturated incidence rate). In recent works [5–8, 23], there is growing interest in investigating SIS epidemic reaction-diffusion model in advective heterogeneous environments. The main purpose is to analyze how the diffusion and advection jointly affect the disease dynamics. One of the main features of above models is that the total number of susceptible and infected population is conserved. For the SIS epidemic reaction-diffusion model with vary total population, there have been many studies on the effect of vary total population on disease persistence, see [11, 16, 24, 26–29, 41]. These research show that the vary total population can enhance disease persistence.

In all above models, the spread of infectious diseases will be only occurred by direct contract between the infected and susceptible population. In [21, 22], the authors adapted the classic disease model to include the possibility for spontaneous (or "automatic") social infection (such as emotions, behaviors or ideas et al.). In this model, the disease infection is affected by both spontaneous infection and infected transmission. In the recent paper [43], Tong and Lei considered a diffusive SIS epidemic model with spontaneous infection, and investigated the effect of the spontaneous infection and spatial heterogeneity.

With these considerations, we are motivated to study SIS epidemic reaction-diffusion system subject to the saturated incidence rate and spontaneous infection in spatially heterogeneous environment:

$$\begin{cases} \dfrac{\partial \bar{S}}{\partial t} - d_S \Delta \bar{S} = -\beta(x)\dfrac{\bar{S}\bar{I}}{1+m\bar{I}} + \gamma(x)\bar{I} - \eta(x)\bar{S}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial \bar{I}}{\partial t} - d_I \Delta \bar{I} = \beta(x)\dfrac{\bar{S}\bar{I}}{1+m\bar{I}} - \gamma(x)\bar{I} + \eta(x)\bar{S}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial \bar{S}}{\partial \nu} = \dfrac{\partial \bar{I}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ \bar{S}(x,0) = \bar{S}_0(x) \geq 0, \bar{I}(x,0) = \bar{I}_0(x) \geq, \not\equiv 0, & x \in \Omega. \end{cases} \tag{1.2}$$

Here the spontaneous infection rate η depends on spatial location x, is positive and Hölder continuous function on $\overline{\Omega}$. It is clear that $\bar{S}\bar{I}/(1+m\bar{I})$ is a Lipschitz continuous function of S and I in the open first quadrant. Hence, while we defining it to be zero while either $S = 0$ or $I = 0$, it can be extended to entire first quadrant. Throughout the paper, it is assumed that initially, there is a positive number of infected individuals, i.e. $\int_\Omega \bar{I}_0(x)\mathrm{d}x > 0$. It is noticed that the total number of population of model (1.2) is conserved for all time $t > 0$ in the sense that

$$\int_\Omega \left[\bar{S}(x,t) + \bar{I}(x,t)\right] \mathrm{d}x = \int_\Omega \left[\bar{S}_0(x) + \bar{I}_0(x)\right] \mathrm{d}x = N, \quad \forall t > 0, \tag{1.3}$$

where the initial data $\bar{S}_0$ and $\bar{I}_0$ are assumed to be nonnegative continuous functions on $\overline{\Omega}$.

We are mainly interested in non-negative equilibrium solutions of (1.2), that is, the non-negative solutions of the following system:

$$\begin{cases} -d_S \Delta S = -\beta(x)\dfrac{SI}{1+mI} + \gamma(x)I - \eta(x)S, & x \in \Omega, \\ -d_I \Delta I = \beta(x)\dfrac{SI}{1+mI} - \gamma(x)I + \eta(x)S, & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Here, $S(x)$ and $I(x)$ denote the density of susceptible and infected individuals, respectively, at $x \in \bar{\Omega}$. By the strong maximum principle, it is clear that only solutions $(S(x), I(x))$ satisfying $S(x) > 0$ and $I(x) > 0$ on $\bar{\Omega}$ for all $x \in \overline{\Omega}$. An *endemic equilibrium* (EE) of (1.2) is a componentwise positive solution $(S(x), I(x))$ of (1.4).

The aim of the current paper is to investigate the effect of population movement, saturated incidence rate, environmental heterogeneity and spontaneous infection on the persistence and extinction of disease. The main focus on the asymptotic behavior of the endemic equilibrium when the small or large dispersal rate of susceptible or infected hosts and the large saturation. Our results show that the spontaneous infection can enhance persistence of infectious disease.

The rest of this paper is organized as follows. In Section 2, we derive the existence of endemic equilibrium by using a fixed point theorem. In Section 3, we analyze the asymptotic profiles of EE as the diffusion coefficient d_S or d_I goes to zero or infinity and the saturated coefficient m tends infinity. Section 4 is devoted to a brief discussion of the obtained results.

2 Existence of EE

In this section, we consider the existence of endemic equilibrium (i.e. positive solution of (1.4)). From now on, we use EE to represent the *endemic equilibrium.* In the rest of the paper, for notational convenience, we denote

$$F^* = \max_{x\in\overline{\Omega}} F(x), \quad F_* = \min_{x\in\overline{\Omega}} F(x),$$

for $F = \beta, \gamma, \eta$.

To achieve the aim, we first consider the uniform boundedness for solutions to (1.2). Actually, it follows from the conservation law (1.3) that any solution $(\bar{S}(x,t), \bar{I}(x,t))$ satisfies L^1 bound uniformly for all $t \in [0, \infty)$. Furthermore, applying [15, Lemma 2.1] (see also [40, Lemma 3.1]), one can obtain the uniform bounds of $\|S(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|I(\cdot,t)\|_{L^\infty(\Omega)}$ for all $t \geq 0$. Indeed, we are able to state the following result.

Proposition 2.1. *There exists a positive constant C independent of the initial data $(\bar{S}_0,\, \bar{I}_0)$ with $\int_\Omega (\bar{S}_0 + \bar{I}_0)\,\mathrm{d}x = N$, such that for the solution $(\bar{S}, \bar{I})$ of* (1.2) *satisfies*

$$\|\bar{S}(\cdot,t)\|_{L^\infty(\Omega)} + \|\bar{I}(\cdot,t)\|_{L^\infty(\Omega)} \le C, \quad \forall t \in [0,\infty).$$

Next, we will establish the existence of EE of (1.2). The main technique in the following proof is Hale-Lopes fixed point theorem [47, Lemma 1.3.6], which is a consequence of [18, Lemmas 2.6.5 and 2.6.6] or [19, Theorems 5 and 6].

Theorem 2.2. *System* (1.2) *admits at least one EE.*

Proof. We set

$$\mathbb{X} = \left\{ \boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in C(\overline{\Omega}, \mathbb{R}^2_+) :\ \int_0^L (\varphi_1 + \varphi_2)\,\mathrm{d}x = N \right\},$$

and decompose the state space $\mathbb{X}$:

$$\mathbb{X}_0 = \left\{ (\bar{S}_0, \bar{I}_0) \in \mathbb{X} :\ \bar{S}_0 \not\equiv 0 \text{ and } \bar{I}_0 \not\equiv 0 \right\}, \quad \partial\mathbb{X}_0 = \left\{ (\bar{S}_0, \bar{I}_0) \in \mathbb{X} :\ \bar{S}_0 \equiv 0 \text{ or } \bar{I}_0 \equiv 0 \right\}.$$

By the standard regularity theory for parabolic equations, for every initial value $(\bar{S}_0, \bar{I}_0) \in \mathbb{X}$, system (1.2) generates a semiflow, denoted by $\Phi(t) :\ \mathbb{X} \mapsto \mathbb{X}$:

$$\Phi(t)(\bar{S}_0, \bar{I}_0) = (\bar{S}(x,t), \bar{I}(x,t)), \quad (\bar{S}_0, \bar{I}_0) \in \mathbb{X},\ t \ge 0,$$

where $(\bar{S}(x,t), \bar{I}(x,t)) \in \mathbb{X}$ is the unique solution of (1.2) with $(\bar{S}(x,0), \bar{I}(x,0)) = (\bar{S}_0, \bar{I}_0)$. It is clear that $\Phi(t)\mathbb{X} \subseteq \mathbb{X}$ for all $t \ge 0$. By Proposition 2.1, we know that Φ is point-dissipative and eventually bounded in $\mathbb{X}_0$. Again from Proposition 2.1, standard parabolic theory and embedding theorems ensure that $\Phi(t) : \mathbb{X}_0 \mapsto \mathbb{X}_0$ is continuous and compact for any $t > 0$. Denote by $\omega((\bar{S}_0, \bar{I}_0))$ the omega limit set of the orbit $\gamma^+((\bar{S}_0, \bar{I}_0)) := \{\Phi(t)(\bar{S}_0, \bar{I}_0) : t \ge 0\}$. We have the following claim.
Claim: $\omega((\bar{S}_0, \bar{I}_0)) \cap \partial\mathbb{X}_0 = \emptyset$, $\forall\, (\bar{S}_0, \bar{I}_0) \in \mathbb{X}$.

We argue by contradiction. Suppose that $\omega((\bar{S}_0, \bar{I}_0)) \cap \partial\mathbb{X}_0 \neq \emptyset$. Then, for any arbitrary $\epsilon > 0$, there exists some $(\bar{S}^*_0, \bar{I}^*_0) \in \mathbb{X}$ and $T_1 > 0$ such that the unique solution $\Phi(\bar{S}^*_0, \bar{I}^*_0) = (\bar{S}^*(x,t), \bar{I}^*(x,t))$ satisfies

$$\lim_{t\to\infty} \bar{I}^*_0(x,t) < \epsilon \ \text{ or } \ \lim_{t\to\infty} \bar{S}^*_0(x,t) < \epsilon, \quad \forall x \in \overline{\Omega}.$$

If $\lim_{t\to\infty} \bar{I}^*_0(x,t) < \epsilon$ for any $x \in \overline{\Omega}$, then there exists $T_2 > 0$ such that

$$0 \le \bar{I}^*(x,t) < \epsilon, \quad x \in \overline{\Omega},\ \forall\, t \ge T_2. \tag{2.1}$$

Hence when $t > T_2$, $\bar{S}^*(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial \bar{S}}{\partial t} - d_S \Delta \bar{S} \le \gamma^* \epsilon - \eta_* \bar{S}, & x \in \Omega,\ t > T_2, \\ \dfrac{\partial \bar{S}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > T_2, \end{cases}$$

then the comparison principle of parabolic equations yields that

$$\bar{S}^*(x,t) \le \tilde{S}^* \mathrm{e}^{-\eta_* t} + (1-\mathrm{e}^{-\eta_* t})\gamma^* \epsilon/\eta_*, \quad x\in\Omega,\ \forall\, t\ge T_2, \tag{2.2}$$

where $\tilde{S}^* = \sup_{x\in\overline{\Omega}} \bar{S}(x,T_2)$. From (2.1) and (2.2), it can be concluded that there exists $T_3 \ge T_2$ such that $\bar{S}^*(x,t)+\bar{I}^*(x,t) \le (2+\frac{\gamma^*}{\eta_*})\epsilon$ for $(x,t)\in\overline{\Omega}\times[T_3,\infty)$. Choose ϵ small enough such that

$$\int_\Omega \left[\bar{S}^*(x,t)+\bar{I}^*(x,t)\right]\mathrm{d}x < N, \quad \forall\, t > T_3.$$

This contradicts the conservation law (1.3).

Similarly, if $\lim\limits_{t\to\infty} \bar{S}_0^*(x,t) < \epsilon$ for any $x\in\overline{\Omega}$, then there exists $T_4>0$ such that $0\le \bar{S}^*(x,t)<\epsilon$ for $(x,t)\in\overline{\Omega}\times[T_4,\infty)$, and $\bar{I}^*(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial \bar{I}}{\partial t} - d_I\Delta \bar{I} \le \Big(\dfrac{\beta^*}{m}+\eta^*\Big)\epsilon - \gamma_*\bar{I}, & x\in\Omega,\ t>T_4,\\ \dfrac{\partial \bar{I}}{\partial \nu} = 0, & x\in\partial\Omega,\ t>T_4, \end{cases}$$

then $\bar{I}^*(x,t) \le \tilde{I}^*\mathrm{e}^{-\gamma_* t} + (1-\mathrm{e}^{-\gamma_* t})(\frac{\beta^*+m\eta^*}{m})\epsilon$ for $(x,t)\in\overline{\Omega}\times[T_4,\infty)$, where $\tilde{I}^* = \sup_{x\in\overline{\Omega}} \bar{I}(x,T_4)$. By a similar argument, we can derive a contradiction to the conservation law (1.3).

The above claim implies that $\omega((\bar{S}_0,\bar{I}_0))\subset \mathbb{X}_0$. From the above arguments, we can deduce that the semiflow Φ has a compact attractor A of $\mathbb{X}$ and $A\subset\mathbb{X}_0$. Thus Φ has a fixed point in A by the Hale-Lopes fixed point theorem [47, Lemma 1.3.6], and hence system (1.2) admits at least one positive EE. □

3 Asymptotic profiles of EE

In this section, we discuss the asymptotic behavior of EE of (1.2), which is the positive solution to the elliptic system (1.4). From (1.3) it easily follows that the conservation law

$$\int_\Omega [S(x)+I(x)]\,\mathrm{d}x = N. \tag{3.1}$$

We start with the L^1-lower estimate for positive solutions to (1.4), which turn out to be independent of d_S and d_I.

Lemma 3.1. *Let (S,I) be any positive solution of (1.4). For any $d_S, d_I, m>0$, we have*

$$\int_\Omega S(x)\,\mathrm{d}x \ge \frac{m\gamma_* N}{\beta^*+m(\eta^*+\gamma_*)}, \quad \int_\Omega I(x)\,\mathrm{d}x \ge \frac{\eta_* N}{\eta_*+\gamma^*}. \tag{3.2}$$

Proof. Integrating the S-equation of (1.4) over Ω, we can get

$$\int_\Omega \gamma(x) I\,\mathrm{d}x = \int_\Omega \beta(x)\frac{SI}{1+mI}\,\mathrm{d}x + \int_\Omega \eta(x) S\,\mathrm{d}x, \tag{3.3}$$

which means that

$$\gamma^* \int_\Omega I \, \mathrm{d}x \geq \eta_* \int_\Omega S \, \mathrm{d}x.$$

Due to the conservation law (3.1), it is clear that

$$\int_\Omega I \, \mathrm{d}x \geq \frac{\eta_* N}{\eta_* + \gamma^*}.$$

Again from (3.3), one can get

$$\gamma_* \int_\Omega I \, \mathrm{d}x \leq \left(\frac{\beta^*}{m} + \eta^*\right) \int_\Omega S \, \mathrm{d}x.$$

Thus, it follows from the conservation law (3.1) that

$$\int_\Omega S \, \mathrm{d}x \geq \frac{m\gamma_* N}{\beta^* + m(\eta^* + \gamma_*)}.$$

□

3.1 The case of $d_S \to 0$

In this subsection, we are concerned with the asymptotic profile of the EE as $d_S \to 0$. Our main result reads as follows.

Theorem 3.2. *Fix $d_I, m > 0$, and let $d_S \to 0$, then every positive solution (S, I) of (1.4), up to a subsequence of d_S, satisfies*

$$(S(x), I(x)) \to (\Phi_S(x), \Phi_I(x)) \quad \textit{uniformly on } \overline{\Omega},$$

where

$$\Phi_S(x) = U\left(x, \Phi_I(x)\right) := \frac{\gamma(x)\Phi_I(1 + m\Phi_I)}{\beta(x)\Phi_I + \eta(x)(1 + m\Phi_I)},$$

and Φ_I is a positive constant satisfies

$$\int_\Omega \left[\Phi_I + \frac{\gamma(x)\Phi_I(1 + m\Phi_I)}{\beta(x)\Phi_I + \eta(x)(1 + m\Phi_I)}\right] \mathrm{d}x = N. \tag{3.4}$$

Proof. In this proof, we always assume that C is a positive constant independent of d_S and may vary from place to place. For the sake of clarity, we divide our proof into four steps.

Step 1. L^p-norm bound of (S, I) for any $p \geq 1$. From the the conservation law (3.1), we know that the L^1-bounds of S and I. We next estimate the L^p-norm bound of (S, I) for any $p > 1$.

Rewrite the I-equation of (1.4) as

$$\begin{cases} -d_I \Delta I + \gamma(x) I = \left[\beta(x) \dfrac{I}{1 + mI} + \eta(x)\right] S, & x \in \Omega, \\ \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

It follows from (3.1) and the elliptic L^1-estimate (see, e.g., [4, Corollary 12] or [35, Lemma 2.2]) that $\|I\|_{W^{1,q}(\Omega)} \le C$ for all $1 \le q < N/(N-1)$ (or for all $q \ge 1$ if $N = 1$). The Sobolev embedding theorem $W^{1,q}(\Omega) \hookrightarrow L^{p_1}(\Omega)$ [17] shows that $\|I\|_{L^{p_1}(\Omega)} \le C$ for all $1 < p_1 \le Nq/(N-q)$. Since q can be close to $N/(N-1)$, it is evident that

$$\|I\|_{L^{p_1}(\Omega)} \le C \quad \text{for all } 1 < p_1 < \frac{N}{N-2}. \tag{3.5}$$

Note that (3.5) holds for all $1 < p_1 < \infty$ if $N \le 2$.

Multiplying the S-equation of (1.4) by S^k for any $k > 0$ and integrating over Ω, we have

$$d_S k \int_\Omega S^{k-1} |\nabla S|^2 \,\mathrm{d}x = \int_\Omega \gamma(x) I S^k \,\mathrm{d}x - \int_\Omega \beta(x) \frac{I}{1+mI} S^{k+1} \,\mathrm{d}x - \int_\Omega \eta(x) S^{k+1} \,\mathrm{d}x.$$

It is clear that

$$\eta_* \int_\Omega S^{k+1} \,\mathrm{d}x \le \gamma^* \int_\Omega I S^k \,\mathrm{d}x. \tag{3.6}$$

Using Hölder inequality for (3.6) with (3.5) and taking $k_1 = 1/q_1$ with $q_1 = 1 + 1/(p_1 - 1) = p_1/(p_1 - 1)$ (note that $1/p_1 + 1/q_1 = 1$), we conclude that

$$\eta_* \int_\Omega S^{k_1+1} \,\mathrm{d}x \le \gamma^* \int_\Omega I S^{k_1} \,\mathrm{d}x \le \gamma^* \left(\int_\Omega I^{p_1} \,\mathrm{d}x \right)^{\frac{1}{p_1}} \left(\int_\Omega S \,\mathrm{d}x \right)^{\frac{1}{q_1}} \le C.$$

Thus we obtain

$$\|S\|_{L^{k_1+1}(\Omega)} \le C. \tag{3.7}$$

By (3.6) and Hölder inequality again, it can be concluded that

$$\eta_* \int_\Omega S^{k_2+1} \,\mathrm{d}x \le \gamma^* \int_\Omega I S^{k_2} \,\mathrm{d}x \le \gamma^* \left(\int_\Omega I^{p_1} \,\mathrm{d}x \right)^{\frac{1}{p_1}} \left(\int_\Omega S^{k_2 q_1} \,\mathrm{d}x \right)^{\frac{1}{q_1}},$$

where $k_2 = (k_1 + 1)/q_1 = 1/q_1 + 1/q_1^2$. Taking (3.7) into account, we obtain

$$\eta_* \int_\Omega S^{k_2+1} \,\mathrm{d}x \le \gamma^* \left(\int_\Omega I^{p_1} \,\mathrm{d}x \right)^{\frac{1}{p_1}} \left(\int_\Omega S^{k_1+1} \,\mathrm{d}x \right)^{\frac{1}{q_1}} \le C,$$

which implies that

$$\|S\|_{L^{k_2+1}(\Omega)} \le C.$$

Repeating the above iteration and taking

$$k_\infty = \frac{1}{q_1} + \frac{1}{q_1^2} + \frac{1}{q_1^3} + \cdots = \frac{1}{q_1 - 1} = p_1 - 1,$$

then we can conclude that

$$\|S\|_{L^{k_\infty+1}(\Omega)} = \|S\|_{L^{p_1}(\Omega)} \le C. \tag{3.8}$$

By (3.8) and the elliptic L^p-theory for I-equation, $\|I\|_{W^{2,p_1}(\Omega)} \leq C$. From the Sobolev embedding theorem $W^{2,p_1}(\Omega) \hookrightarrow L^{p_2}(\Omega)$ for all $1 < p_2 \leq Np_1/(N-2p_1)$, and the fact of p_1 can be close to $N/(N-2)$, we deduce that

$$\|I\|_{L^{p_2}(\Omega)} \leq C, \quad \text{for all } 1 < p_2 < \frac{N}{N-4} \text{ (or for all } 1 < p_2 < \infty \text{ if } N \leq 4).$$

From the similar arguments to the L^{p_1}-estimate of S, one can show

$$\|S\|_{L^{p_2}(\Omega)} \leq C.$$

By standard bootstrapping arguments, we can eventually obtain

$$\|S\|_{L^p(\Omega)} \leq C, \quad \|I\|_{L^p(\Omega)} \leq C, \text{ for all } 1 \leq p < \infty. \tag{3.9}$$

Step 2. Lower bounds of (S, I). Set $S(x_1) = \min_{x\in\overline{\Omega}} S(x)$, then it follows from the maximum principle [32, Proposition 2.2] for the S-equation of (1.4) that

$$\gamma(x_1)I(x_1) \leq \frac{\beta(x_1)S(x_1)I(x_1)}{1+mI(x_1)} + \eta(x_1)S(x_1),$$

this gives

$$\min_{x\in\overline{\Omega}} I(x) \leq \frac{\beta^* + m\eta^*}{m\gamma_*} \min_{x\in\overline{\Omega}} S(x). \tag{3.10}$$

Using a useful lemma from [17, Lemma 8.18] (see also [30, Lemma 2.2] or [38]) with $q = 1$ for I-equation, we conclude that

$$\min_{x\in\overline{\Omega}} I \geq C\|I\|_{L^1(\Omega)}. \tag{3.11}$$

Thanks to (3.2) and (3.11), we have

$$\min_{x\in\overline{\Omega}} I(x) \geq C. \tag{3.12}$$

Hence, (3.10) and (3.12) yields

$$S(x) \geq C, \quad I(x) \geq C, \quad \forall\, x \in \overline{\Omega}. \tag{3.13}$$

Step 3. Convergence of I. By (3.9), the standard L^p-theory (see, e.g., [17]) for I-equation shows that

$$\|I\|_{W^{2,p}(\Omega)} \leq C \quad \text{for all } p > 1.$$

Taking p to be sufficiently large, we deduce from the Sobolev embedding theorem that

$$\|I\|_{C^{1+\alpha}(\Omega)} \leq C \text{ for some } 0 < \alpha < 1.$$

Thus, there exists a subsequence of $d_S \to 0$, denoted by $d_i := d_{S,i}$, satisfying $d_i \to 0$ as $i \to \infty$, and a corresponding positive solution $(S_i, I_i) := (S_{d_i}, I_{d_i})$ of (1.4) with $d_S = d_i$, such that

$$I_i \to \Phi_I \quad \text{uniformly on } \overline{\Omega}, \text{ as } i \to \infty, \tag{3.14}$$

where $\Phi_I \in C^1(\overline{\Omega})$ and $\Phi_I > 0$ on $\overline{\Omega}$ due to (3.12).
Step 4. Convergence of S. We consider the S-equation:

$$\begin{cases} -d_i \Delta S_i = -\beta(x)\dfrac{S_i I_i}{1+mI_i} + \gamma(x) I_i - \eta(x) S_i, & x \in \Omega, \\ \dfrac{\partial S_i}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

In light of (3.14), for arbitrarily small $\epsilon > 0$, we have for sufficiently large i such that

$$0 < \Phi_I(x) - \epsilon \le I_i(x) \le \Phi_I(x) + \epsilon \ \text{ for all } x \in \overline{\Omega}.$$

Thus, for all large i, it is evident that

$$\begin{aligned} -\beta(x)\frac{S_i I_i}{1+mI_i} + \gamma(x) I_i - \eta(x) S_i &\le -\beta(x)\frac{S_i\left(\Phi_I - \epsilon\right)}{1+m\left(\Phi_I+\epsilon\right)} + \gamma(x)\left(\Phi_I+\epsilon\right) - \eta(x) S_i \\ &= \frac{\left[w^\epsilon\left(x, \Phi_I\right) - S_i\right] g^\epsilon(x, \Phi_I)}{1+m\left(\Phi_I+\epsilon\right)}, \end{aligned}$$

where

$$\begin{aligned} w^\epsilon\left(x, \Phi_I\right) &= \frac{\gamma(x)(\Phi_I+\epsilon)[1+m(\Phi_I+\epsilon)]}{\beta(x)(\Phi_I-\epsilon)+\eta(x)[1+m(\Phi_I+\epsilon)]}, \\ g^\epsilon(x, \Phi_I) &= \beta(x)(\Phi_I-\epsilon)+\eta(x)[1+m(\Phi_I+\epsilon)]. \end{aligned}$$

For given large i, resorting to the auxiliary problem

$$\begin{cases} -d_i \Delta u = \dfrac{\left[w^\epsilon\left(x, \Phi_I\right) - u\right] g^\epsilon(x, \Phi_I)}{1+m\left(\Phi_I+\epsilon\right)}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.15}$$

one can assert that S_i is a subsolution of (3.15) and any sufficiently large positive constant C satisfies $S_i \le C$ is a supersolution of (3.15). Then, (3.15) admits at least a positive solution, denoted by u_i, which satisfies $S_i \le u_i \le C$ on $\overline{\Omega}$. By the maximum principle [32, Proposition 2.2], it is observed that

$$\min_{x\in\overline{\Omega}} w^\epsilon\left(x, \Phi_I(x)\right) \le \min_{x\in\overline{\Omega}} u_i(x) \le u_i(x) \le \max_{x\in\overline{\Omega}} u_i(x) \le \max_{x\in\overline{\Omega}} w^\epsilon\left(x, \Phi_I(x)\right), \quad \forall x \in \overline{\Omega}.$$

Using the singular perturbation theory technique [31, Lemma 2.1] (see also [14, Lemma 2.4]), we can deduce that any positive solution u_i of (3.15) fulfills

$$u_i \to w^\epsilon\left(x, \Phi_I(x)\right) \ \text{ uniformly on } \overline{\Omega}, \text{ as } i \to \infty.$$

Observe that $S_i \le u_i \le C$ on $\overline{\Omega}$. Hence, we can assert that

$$\limsup_{i\to\infty} S_i(x) \le w^\epsilon\left(x, \Phi_I(x)\right) \ \text{ uniformly on } \overline{\Omega}. \tag{3.16}$$

Similarly, for all large i, it follows that

$$\begin{aligned}-\beta(x)\frac{S_iI_i}{1+mI_i}+\gamma(x)I_i-\eta(x)S_i &\geq -\beta(x)\frac{S_i\left(\Phi_I+\epsilon\right)}{1+m\left(\Phi_I-\epsilon\right)}+\gamma(x)\left(\Phi_I-\epsilon\right)-\eta(x)S_i\\ &=\frac{\left[w_\epsilon\left(x,\Phi_I\right)-S_i\right]g_\epsilon(x,\Phi_I)}{1+m\left(\Phi_I-\epsilon\right)},\end{aligned}$$

where

$$\begin{aligned}w_\epsilon\left(x,\Phi_I\right)&=\frac{\gamma(x)(\Phi_I-\epsilon)[1+m(\Phi_I-\epsilon)]}{\beta(x)(\Phi_I+\epsilon)+\eta(x)[1+m(\Phi_I-\epsilon)]},\\ g_\epsilon(x,\Phi_I)&=\beta(x)(\Phi_I+\epsilon)+\eta(x)[1+m(\Phi_I-\epsilon)].\end{aligned}$$

Similarly, we can get

$$\liminf_{i\to\infty}S_i(x)\geq w_\epsilon\left(x,\Phi_I(x)\right)\ \text{ uniformly on }\overline{\Omega}. \tag{3.17}$$

Note that

$$\lim_{\epsilon\to0}w_\epsilon\left(x,\Phi_I(x)\right)=\lim_{\epsilon\to0}w^\epsilon\left(x,\Phi_I(x)\right)=\frac{\gamma(x)\Phi_I(1+m\Phi_I)}{\beta(x)\Phi_I+\eta(x)(1+m\Phi_I)}:=U\left(x,\Phi_I(x)\right),$$

it then follows from (3.16) and (3.17) that

$$S_i(x)\to U\left(x,\Phi_I(x)\right)=\Phi_S(x)\ \text{ uniformly on }\overline{\Omega},\ \text{ as } i\to\infty. \tag{3.18}$$

Finally, we determine Φ_I. Substituting (3.14) and (3.18) into the right-hand side of the I-equation in (1.4), we conclude that

$$\beta(x)\frac{S_iI_i}{1+mI_i}-\gamma(x)I_i+\eta(x)S_i\to\beta(x)\frac{\Phi_S\Phi_I}{1+m\Phi_I}-\gamma(x)\Phi_I+\eta(x)\Phi_S=0$$

uniformly on $\overline{\Omega}$, as $i\to\infty$. Clearly, Φ_I fulfills

$$-d_I\Delta\Phi_I=0,\ x\in\Omega;\quad \frac{\partial\Phi_I}{\partial\nu}=0,\ x\in\partial\Omega.$$

Hence, Φ_I must be a positive constant due to (3.13). It follows from the conservation law (3.1) that

$$\int_\Omega\left(\Phi_S+\Phi_I\right)\mathrm{d}x=N.$$

Thus, it holds

$$\int_\Omega\left[\Phi_I+\frac{\gamma(x)\Phi_I(1+m\Phi_I)}{\beta(x)\Phi_I+\eta(x)(1+m\Phi_I)}\right]\mathrm{d}x=N.$$

The proof is complete. □

3.2 The case of $d_I \to 0$

This subsection is devoted the investigation of the asymptotic behavior of the positive solutions of (1.4) as $d_I \to 0$. Our result can be stated as follows.

Theorem 3.3. *Fix $d_S, m > 0$ and let $d_I \to 0$, then every positive solution (S, I) of (1.4) satisfies (up to a subsequence of $d_I \to 0$)*

$$(S(x), I(x)) \to (\Psi_S(x), \Psi_I(x)) \quad \text{uniformly on } \overline{\Omega},$$

where $\Psi_S(x)$ is a positive constant satisfies

$$\int_\Omega \left[\frac{\sqrt{\left[(\beta + m\eta)\Psi_S - \gamma\right]^2 + 4m\gamma\eta\Psi_S} + \left[(\beta + m\eta)\Psi_S - \gamma\right]}{2m\gamma} + \Psi_S \right] = N, \tag{3.19}$$

and $\Psi_I(x)$ satisfies

$$\Psi_I(x) = V\left(x, \Psi_S(x)\right) := \frac{1}{2m\gamma} \left\{ \left[(\beta + m\eta)\Psi_S - \gamma\right] + \sqrt{\left[(\beta + m\eta)\Psi_S - \gamma\right]^2 + 4m\gamma\eta\Psi_S} \right\}.$$

Proof. The proof will be divided into three steps.
Step 1. *A priori* estimates of (S, I). In what follows, we always assume that C is a positive constant independent of small d_I and may vary from line to line. By the similar analysis to step 1 in the proof of Theorem 3.2, we first consider the S-equation of (1.4). The elliptic L^1-theory (see, e.g., [4, Corollary 12] or [35, Lemma 2.2]) and Sobolev imbedding theorem yield that

$$\|S\|_{L^{p_1}(\Omega)} \le C \text{ for all } 1 < p_1 < \frac{N}{N-2} \text{ (or for all } 1 < p_1 < \infty \text{ if } N \le 2).$$

Next, we establish the L^{p_1}-norm estimate of I. Multiplying the I-equation of (1.4) by I^k for any $k > 0$ and integrating over Ω, we can conclude that

$$\gamma_* \int_\Omega I^{k+1}\, \mathrm{d}x \le \left(\frac{\beta^*}{m} + \eta^* \right) \int_\Omega S I^k\, \mathrm{d}x.$$

Repeating the similar iteration in step 1 of the proof to Theorem 3.2, one have L^{p_1}-norm estimate of I. By the bootstrap argument, we can obtain

$$\|S\|_{L^p(\Omega)} \le C, \quad \|I\|_{L^p(\Omega)} \le C, \quad \forall 1 \le p < \infty. \tag{3.20}$$

We next consider the lower bound of (S, I). Set $I(x_2) = \min_{x \in \overline{\Omega}} I(x)$, one can apply [32, Proposition 2.2] again to the I-equation of (1.4) to assert that

$$\gamma(x_2) I(x_2) \ge \frac{\beta(x_2) S(x_2) I(x_2)}{1 + m I(x_2)} + \eta(x_2) S(x_2),$$

which implies that

$$\min_{x \in \overline{\Omega}} I(x) \ge \frac{\eta_*}{\gamma^*} \min_{x \in \overline{\Omega}} S(x). \tag{3.21}$$

Making use of [17, Theorem 8.18] (see also [30, Lemma 2.2] or [38]) with $q = 1$ for S-equation, we conclude that

$$\min_{x\in\overline{\Omega}} S \geq C\|S\|_{L^1(\Omega)}. \tag{3.22}$$

From (3.2), (3.21) and (3.22), we can deduce that

$$S(x) \geq C,\ I(x) \geq C, \quad \forall x \in \overline{\Omega}. \tag{3.23}$$

Step 2. Convergence of S. In view of (3.20), the elliptic L^p-theory and Sobolev embedding theorem guarantee that

$$\|S\|_{C^{1+\alpha}(\Omega)} \leq C \quad \text{for some } \alpha \in (0,1).$$

Therefore, there exists a subsequence of $d_I \to 0$, denoted by $d_j := d_{I,j} \to 0$, satisfying $d_j \to 0$ as $j \to \infty$, and a corresponding positive solution (S_j, I_j) of (1.4) with $d_I = d_j$ satisfies

$$S_j \to \Psi_S \text{ on } C^1(\overline{\Omega}),\ \text{ as } j \to \infty, \tag{3.24}$$

where $\Psi_S \in C^1(\overline{\Omega})$ and $\Psi_S \geq C > 0$ due to (3.23).
Step 3. Convergence of I. Consider the equation satisfied by I_j:

$$\begin{cases} -d_I \Delta I_j = \beta(x)\dfrac{S_j I_j}{1+mI_j} - \gamma(x) I_j + \eta(x) S_j, & x \in \Omega, \\ \dfrac{\partial I_j}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.25}$$

By (3.24), given any small $\epsilon > 0$, we have for sufficiently large j such that

$$0 < \Psi_S(x) - \epsilon \leq S_j(x) \leq \Psi_S(x) + \epsilon \ \text{ for all } x \in \overline{\Omega}. \tag{3.26}$$

Thus, it holds

$$\begin{aligned} \beta(x)\frac{S_j I_j}{1+mI_j} + \eta(x) S_j - \gamma(x) I_j &\leq \beta(x)\frac{I_j(\Psi_S+\epsilon)}{1+mI_j} - \gamma(x) I_j + \eta(x)(\Psi_S+\epsilon) \\ &= \frac{\left[H_+^{1,\epsilon}(x,\Psi_S(x)) - I_j\right]\left[I_j - H_-^{1,\epsilon}(x,\Psi_S(x))\right]}{1+mI_j}, \end{aligned}$$

where

$$\begin{aligned} H_\pm^{1,\epsilon}(x,\Psi_S(x)) =& \frac{1}{2m\gamma(x)}\left[\big(\beta(x)+m\eta(x)\big)(\Psi_S+\epsilon) - \gamma(x)\right] \\ &\pm \frac{1}{2m\gamma(x)}\sqrt{\left[\big(\beta(x)+m\eta(x)\big)(\Psi_S+\epsilon)-\gamma(x)\right]^2 + 4m\gamma(x)\eta(x)(\Psi_S+\epsilon)}, \end{aligned}$$

with $H_-^{1,\epsilon}(x,\Psi_S(x)) < 0 < H_+^{1,\epsilon}(x,\Psi_S(x))$ on $\overline{\Omega}$. For all large j, we consider the following auxiliary elliptic problem:

$$\begin{cases} -d_j \Delta v = \dfrac{\left[H_+^{1,\epsilon}(x,\Psi_S(x)) - v\right]\left[v - H_-^{1,\epsilon}(x,\Psi_S(x))\right]}{1+mv}, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.27}$$

It is clear that I_j is a subsolution of (3.27) and any sufficiently large positive constant C satisfies $I_j \le C$ is a supersolution of (3.27). Therefore, (3.27) admits at least one positive solution, denoted by v_j, which satisfies $I_j \le v_j \le C$ on $\overline{\Omega}$. Applying the maximum principle [1, Proposition 2.2] to the system (3.27), one can see that

$$\min_{x\in\overline{\Omega}} H_+^{1,\epsilon}(x,\Psi_S(x)) \le \min_{x\in\overline{\Omega}} v_j(x) \le v_j(x) \le \max_{x\in\overline{\Omega}} v_j(x) \le \max_{x\in\overline{\Omega}} H_+^{1,\epsilon}(x,\Psi_S(x)) \text{ for all } x\in\overline{\Omega}.$$

From the similar singular perturbation theory technique [31, Lemma 2.1] (see also [14, Lemma 2.4]), we obtain

$$v_j(x) \to H_+^{1,\epsilon}(x,\Psi_S(x)) \text{ uniformly on } \overline{\Omega}, \text{ as } j\to\infty.$$

Since I_j is a subsolution of (3.27), we have

$$\limsup_{j\to\infty} I_j(x) \le H_+^{1,\epsilon}(x,\Psi_S(x)) \text{ uniformly on } \overline{\Omega}. \tag{3.28}$$

Similarly, from (3.26), for all large j, we have

$$\begin{aligned}\beta(x)\frac{S_jI_j}{1+mI_j} + \eta(x)S_j - \gamma(x)I_j &\ge \beta(x)\frac{I_j(\Psi_S-\epsilon)}{1+mI_j} + \eta(x)(\Psi_S-\epsilon) - \gamma(x)I_j \\ &= \frac{\left[H_+^{2,\epsilon}(x,\Psi_S(x)) - I_j\right]\left[I_j - H_-^{2,\epsilon}(x,\Psi_S(x))\right]}{1+mI_j},\end{aligned}$$

where

$$\begin{aligned}H_\pm^{2,\epsilon}(x,\Psi_S(x)) =& \frac{1}{2m\gamma(x)}\left[\big(\beta(x)+m\eta(x)\big)(\Psi_S-\epsilon)-\gamma(x)\right] \\ &\pm \frac{1}{2m\gamma(x)}\sqrt{\left[\big(\beta(x)+m\eta(x)\big)(\Psi_S-\epsilon)-\gamma(x)\right]^2 + 4m\gamma(x)\eta(x)(\Psi_S-\epsilon)},\end{aligned}$$

with $H_-^{2,\epsilon}(x,\Psi_S(x)) < 0 < H_+^{2,\epsilon}(x,\Psi_S(x))$ on $\overline{\Omega}$. Similarly, for fixed large j, we consider the following elliptic problem:

$$\begin{cases} -d_j\Delta v = \dfrac{\left[H_+^{2,\epsilon}(x,\Psi_S(x)) - v\right]\left[v - H_-^{2,\epsilon}(x,\Psi_S(x))\right]}{1+mv}, & x\in\Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega. \end{cases} \tag{3.29}$$

Observe that 0 and I_j form a pair of sub-supersolution of (3.29), it can be concluded from the above analysis that

$$\liminf_{j\to\infty} I_j(x) \ge H_+^{2,\epsilon}(x,\Psi_S(x)) \quad \text{uniformly on } \overline{\Omega}. \tag{3.30}$$

Notice that

$$\begin{aligned}\lim_{\epsilon\to0} H_+^{1,\epsilon}(x,\Psi_S(x)) &= \lim_{\epsilon\to0} H_+^{2,\epsilon}(x,\Psi_S(x)) \\ &= \frac{1}{2m\gamma(x)}\left[\big(\beta(x)\Psi_S + m\eta(x)\big)\Psi_S - \gamma(x)\right] \\ &\quad + \frac{1}{2m\gamma(x)}\sqrt{\left[\big(\beta(x)\Psi_S + m\eta(x)\big)\Psi_S - \gamma(x)\right]^2 + 4m\gamma(x)\eta(x)\Psi_S} \\ &:= V(x,\Psi_S(x)),\end{aligned}$$

it follows from (3.28) and (3.30) that

$$I_j(x) \to V\left(x, \Psi_S(x)\right) = \Psi_I(x) \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } j \to \infty. \tag{3.31}$$

By (3.24) and (3.31), it is clear that

$$-\beta(x)\frac{S_j I_j}{1+mI_j} - \eta(x)S_j + \gamma(x)I_j \to -\beta(x)\frac{\Psi_S\Psi_I}{1+m\Psi_I} - \eta(x)\Psi_S + \gamma(x)\Psi_I = 0$$

uniformly on $\overline{\Omega}$, as $j \to \infty$. Thus, we can obtain

$$-d_I\Delta\Psi_S = 0,\ x \in \Omega; \quad \frac{\partial \Psi_S}{\partial \nu} = 0,\ x \in \partial\Omega.$$

As a result, Ψ_S is a positive constant due to (3.23). It follows from the conservation law (3.1) that (3.19) holds. This completes the proof. □

3.3 The case of $d_S \to \infty$ or $d_I \to \infty$

In this subsection, we discuss the asymptotic behavior of the EE when $d_S \to \infty$ or $d_I \to \infty$. Our main results read as follows.

Theorem 3.4. *Fix $d_I, m > 0$ and let $d_S \to \infty$, then every positive solution (S, I) of (1.4) satisfies (up to a subsequence of $d_S \to \infty$)*

$$(S, I) \to (S^\infty, I^\infty) \quad \textit{uniformly on } \overline{\Omega},$$

where S^∞ is a positive constant and $I^\infty > 0$ on $\overline{\Omega}$, and (S^∞, I^∞) solves

$$\begin{cases} -d_I\Delta I^\infty = \beta(x)\dfrac{S^\infty I^\infty}{1+mI^\infty} - \gamma(x)I^\infty + \eta(x)S^\infty & x \in \Omega, \\ \dfrac{\partial I^\infty}{\partial \nu} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega (S^\infty + I^\infty)\,\mathrm{d}x = N. \end{cases} \tag{3.32}$$

Proof. From the estimates of (S, I) in steps 1 and 2 in the proof of Theorem 3.2, it is clear that the estimates (3.9) and (3.13) hold, and the positive constants C are independent of $d_S \geq 1$. Now, we rewrite the S-equation of (1.4) as

$$\begin{cases} -\Delta S = \dfrac{1}{d_S}\left[-\dfrac{\beta(x)SI}{1+mI} + \gamma(x)I - \eta(x)S\right], & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Then the elliptic L^p-theory and Sobolev embedding theorem guarantee that there exists a subsequence of $d_{S,k}$, labeled by d_k, with $d_k \to \infty$ as $k \to \infty$, such that the corresponding positive solution (S_k, I_k) of (1.4) for $d_S = d_k$ satisfies $S_k \to S^\infty$ in $C^1(\overline{\Omega})$ as $k \to \infty$. It is evident that S^∞ solves

$$-\Delta S^\infty = 0, x \in \Omega; \quad \frac{\partial S^\infty}{\partial \nu} = 0, x \in \partial\Omega.$$

Hence, S^∞ is a positive constant due to (3.13). As before, resorting to the I-equation of (1.4) and passing to a further subsequence if necessary, we have

$$I_k \to I^\infty \quad \text{in } C^1(\overline{\Omega}), \quad \text{as } k \to \infty,$$

and $I^\infty > 0$ on $\overline{\Omega}$ due to (3.13). It follows from standard elliptic regularity theory that $(S^\infty, I^\infty) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$, and (S^∞, I^∞) satisfies (3.32) due to (1.4) and (3.1). The proof is complete. $\square$

By a modifying argument of the proof of Theorem 3.4, we can obtain the asymptotic profile of EE as $d_I \to \infty$.

Theorem 3.5. *Fix $d_S, m > 0$ and let $d_I \to \infty$. Then every positive solution (S, I) of (1.4) satisfies (up to a subsequence of $d_I \to \infty$)*

$$(S, I) \to (S_\infty, I_\infty) \quad \textit{uniformly on } \overline{\Omega},$$

where I_∞ is a positive constant and $S_\infty > 0$ on $\overline{\Omega}$, and (S_∞, I_∞) solves

$$\begin{cases} -d_S \Delta S_\infty = -\dfrac{\beta(x) S_\infty I_\infty}{1 + m I_\infty} + \gamma(x) I_\infty - \eta(x) S_\infty, & x \in \Omega, \\ \dfrac{\partial S_\infty}{\partial \nu} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega (S_\infty + I_\infty)\, \mathrm{d}x = N. \end{cases}$$

3.4 The case of $m \to \infty$

This subsection is devoted to investigating the asymptotic behavior of positive solutions of (1.4) with $d_S, d_I > 0$ being fixed and $m \to \infty$.

Theorem 3.6. *Fix $d_S, d_I > 0$ and let $m \to \infty$. Then every positive solution (S, I) of (1.4) satisfies*

$$(S, I) \to (\hat{S}, \hat{I}), \quad \textit{uniformly on } \overline{\Omega},$$

where $(\hat{S}, \hat{I})$ solves the following system:

$$\begin{cases} -d_S \Delta \hat{S} = \gamma(x)\hat{I} - \eta(x)\hat{S}, & x \in \Omega, \\ -d_I \Delta \hat{I} = -\gamma(x)\hat{I} + \eta(x)\hat{S}, & x \in \Omega, \\ \dfrac{\partial \hat{S}}{\partial \nu} = \dfrac{\partial \hat{I}}{\partial \nu} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega (\hat{S} + \hat{I})\, \mathrm{d}x = N. \end{cases} \tag{3.33}$$

Proof. We first consider the estimates of S and I. Actually, again from the discussion in step 1 and step 2 in the proof of Theorem 3.2, we can conclude that for all sufficiently large m,

$$\|S\|_{W^{2,p}(\Omega)} \le C, \quad \|I\|_{W^{2,p}(\Omega)} \le C,$$

and

$$I(x) \ge C, \ S(x) \ge C, \quad \forall\, x \in \overline{\Omega}, \tag{3.34}$$

where the positive constants C are independent of large m. Thus, the elliptic L^p-theory and Sobolev imbedding theorem guarantee that there exists a subsequence of m, labeled by m_n, with $m_n \to \infty$ as $n \to \infty$, such that the corresponding positive solution (S_n, I_n) of (1.4) satisfies

$$S_n \to \hat{S}, \quad I_n \to \hat{I} \text{ uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty.$$

Thanks to (3.34), we know $\hat{S} > 0$ and $\hat{I} > 0$ on $\overline{\Omega}$. It is clear that $(\hat{S}, \hat{I})$ solves (3.33). The proof is complete. □

4 Discussion

In this paper, we have proposed an SIS epidemic reaction-diffusion model (1.2), which includes saturated incidence rate $SI/(1+mI)$ and spontaneous infection η. We are concerned with the existence of positive solutions to steady state system. Moreover, we also have analyzed the asymptotic profiles of positive solutions if the migration rate of the susceptible or infected population is small or large and the saturation rate is large. The main purpose of our present work is to investigate the effect of spontaneous infection on the qualitative behavior of (1.2). Similar questions are addressed for the model with the standard incidence rate $SI/(S+I)$ in [43] and the model with the saturated incidence rate without spontaneous infection (i.e. $\eta \equiv 0$ on $\overline{\Omega}$) in [44].

For the reaction-diffusion SIS model (1.2) without spontaneous infection ($\eta \equiv 0$ on $\overline{\Omega}$), the asymptotic profiles of the EE (when exist) were studied in [44]. When the diffusion rate of the susceptible individual tends to zero, it was proved that the infected individuals vanish on the entire habitat in the case ($N < \int_\Omega \gamma/\beta \, \mathrm{d}x$). However, as the diffusion rate d_S is small, our result (Theorem 3.2) shows that infected individuals always exist and distribute evenly throughout the habitat, and the susceptible distribute inhomogeneous on the entire habitat. For the model (1.2) without spontaneous infection, when the saturated incidence rate tends to infinity, it shows that the EE tends to the DFE, which means that the disease will be eliminated. In sharp contrast, for our model (1.2), we see from Theorem 3.6 that the susceptible and infective stays inhomogeneously on the whole habitat. Our study shows that the spontaneous infection can enhance persistence of infectious disease and the disease will becomes more threatening and harder to control.

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