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Analysis of two-operator boundary-domain integral equations for variable-coefficient Dirichlet and Neumann boundary value problems in 2D with general right-hand side

Tsegaye G. Ayele*[1] | Markos F. Yimer[2]

[1]Department of Mathematics, Addis Ababa University, P. O. Box 1176, Addis Ababa, Ethiopia
[2]Department of Mathematics, Addis Ababa University, P. O. Box 1176, Addis Ababa, Ethiopia

Correspondence
*Tsegaye G. Ayele
Department of Mathematics
Addis Ababa University
P.O. Box 1176 Addis Ababa
Email: tsegaye.ayele@aau.edu.et

Present Address
Department of Mathematics
Addis Ababa University
P.O.Box 1176
Addis Ababa-Ethiopia
Email: tsegaye.ayele@aau.edu.et

Summary

The Dirichlet and Neumann boundary value problems (BVPs) for the linear second-order scalar elliptic differential equation with variable coefficients in a bounded two-dimensional domain are considered. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\widetilde{H}^{-1}(\Omega)$, when neither classical nor canonical conormal derivatives of solutions are well defined. Using the two-operator approach and appropriate parametrix (Levi function) each problem is reduced to two different systems of two-operator boundary-domain integral equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solubility, solution uniquness/nonuniquness and invertibility BDIE system are analyzed in the appropriate Sobolev (Bessel potential) spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

KEYWORDS:
partial differential equations, variable coefficients, parametrix, boundary-domain integral equations, equivalence, solvability and invertibility

MSC: 35J25; 31A10; 45A05; 45P05; 45A05; 47G10; 47G30; 47G40

1 | INTRODUCTION

Partial differential equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermal conductivity, fluid flows through porous media, and other areas of physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of boundary value problems (BVPs) for such PDEs to explicit boundary integral equations (BIEs), which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce

BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further numerical solution of the latter, see e.g.,[1,2,3,4,5,6]. However, this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity). To overcome this difficulty, one can apply the so-called two-operator approach, formulated in[7] for a certain nonlinear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always choose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

For a function from the Sobolev space $H^1(\Omega)$, a classical conormal derivative in the sense of traces may not exist (see, e.g.,[8, Appendix A]). However, when this function satisfies a second order PDE with a right-hand side from $H^{-1}(\Omega)$, the generalized conormal derivative can be defined in the weak sense, associated with the first Green identity and an extension of the PDE right-hand side to $\widetilde{H}^{-1}(\Omega)$ (see, e.g.,[9, Lemma 4.3] and[10, Definition 3.1]). Since the extension is not unique, the conormal derivative appears to be an operator that is not unique, which is also nonlinear in u unless a linear relation between u and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the BDIEs. These difficulties are addressed in[8,11] presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann problems for the divergent-type PDE with a variable scalar coefficient and a general right-hand side. This needed a non-trivial generalization of the third Green identity and its conormal derivative for such functions, which extends the approach implemented in[1,2,6,12,13] for the PDE right-hand from $L_2(\Omega)$. In[14], using the two-operator approach in settings different from that in[15,16], a generalization of the two-operator third Green identity and its conormal derivative is derived and the two-operator BDIEs for variable-coefficient Dirichlet, Neumann and mixed BVPs are analyzed in 3D.

Nowadays, the theory of BDIEs in 3D is well developed, see[1,2,3,7,12], but the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs, see[17,18,19,20,21,22,23]. In this paper, we extend the results in[23], and consider the Dirichlet and Neumann boundary value problems for the linear second-order scalar elliptic differential equation with variable coefficient in a two-dimensional bounded domain. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\widetilde{H}^{-1}(\Omega)$ when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) is used to reduce each problem to two different systems of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs is proved. The BDIE systems are analysed in appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

2 | CONORMAL DERIVATIVES AND BOUNDARY VALUE PROBLEMS

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. Consider the scalar elliptic differential equation, which for sufficiently smooth function u has the following strong form,

$$Au(x) := A(x, \partial_x)u(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = \tilde{f}(x), \quad x \in \Omega, \tag{1}$$

where u is unknown function and $\tilde{f}$ is a given function in Ω. We assume that

$$a \in C^\infty(\mathbb{R}^2) \quad \text{and} \quad 0 < a_{\min} \le a(x) \le a_{\max} < \infty, \quad \forall\, x \in \mathbb{R}^2.$$

In what follows $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g.,[9,24]). We recall that H^s coincides with the Sobolev-Slobodetski spaces W_2^s for any nonnegative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^2)$,

$$\widetilde{H}^s(\Omega) := \{g : g \in H^s(\mathbb{R}^2),\ \mathrm{supp}(g) \subset \overline{\Omega}\}$$

while $H^s(\Omega)$ denotes the space of restriction on Ω of distributions from $H(\mathbb{R}^2)$,

$$H^s(\Omega) = \{r_{_\Omega} g : g \in H^s(\mathbb{R}^2)\}$$

where r_{Ω} denotes the restriction operator on Ω. We will also use the notation $g|_{\Omega} := r_{\Omega} g$. We denote by $H^s_{\partial\Omega}$ the following subspace of $H(\mathbb{R}^2)$ (and $\widetilde{H}(\Omega)$),

$$H^s_{\partial\Omega} := \{g : g \in H^s(\mathbb{R}^2), \ \mathrm{supp}(g) \subset \partial\Omega\}. \tag{2}$$

From the trace theorem (see, e.g.,[9,24,25] for $u \in H^1(\Omega)$, it follows that $\gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ = \gamma^+_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from Ω. Let also $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operator γ^+, i.e., $\gamma^+_{\partial\Omega}\gamma^{-1}_{\partial\Omega} w = \gamma^+\gamma^{-1} w = w$ for any $w \in H^{\frac{1}{2}}(\partial\Omega)$, and $(\gamma^{-1})^* : \widetilde{H}^{-1}(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous operator dual to γ^{-1}, i.e., $\langle (\gamma^{-1})^* \tilde{f}, w\rangle_{\partial\Omega} := \langle \tilde{f}, \gamma^{-1} w\rangle_{\Omega}$ for any $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $w \in H^{\frac{1}{2}}(\partial\Omega)$.

For $u \in H^2(\Omega)$, we denote by T^+_a the corresponding canonical (strong) conormal derivative operator on $\partial\Omega$ in the sense of traces,

$$T^+_a u := \sum_{i=1}^{2} a(x) n_i(x) \gamma^+ \frac{\partial u(x)}{\partial x_i} = a(x)\gamma^+ \frac{\partial u(x)}{\partial n(x)}, \tag{3}$$

where $n(x)$ is the outward to Ω unit normal vector at the point $x \in \partial\Omega$. However, the classical conormal derivative operator is generally, not well defined if $u \in H^1(\Omega)$, see, e.g.,[8, Appendix A].

For $u \in H^1(\Omega)$, the PDE Au in (1) is understood in the sense of distributions,

$$\langle Au, v\rangle_{\Omega} := -\mathcal{E}_a(u, v), \qquad \forall v \in \mathcal{D}(\Omega), \tag{4}$$

where

$$\mathcal{E}_a(u, v) := \int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x) dx$$

is a symmetric bilinear form and the duality brackets $\langle g, \cdot\rangle_{\Omega}$ denote the value of a linear functional (distribution) g, extending the usual L_2 inner product. Since the set $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$, the above formula defines a continuous operator $A : H^1(\Omega) \to H^{-1}(\Omega) = [\widetilde{H}^1(\Omega)]^*$,

$$\langle Au, v\rangle_{\Omega} := -\mathcal{E}_a(u, v), \quad \forall u \in H^1(\Omega), \ \forall v \in \widetilde{H}^1(\Omega). \tag{5}$$

Let us consider also the operator, $\check{A} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega) = [H^1(\Omega)]^*$,

$$\begin{aligned} \langle \check{A}u, v\rangle_{\Omega} &:= -\mathcal{E}_a(u, v) = -\int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x) dx = -\int_{\mathbb{R}^2} \mathring{E}[a\nabla u](x) \cdot \nabla V(x) dx \\ &= \langle \nabla \cdot \mathring{E}[a\nabla u], V\rangle_{\mathbb{R}^2} = \langle \nabla \cdot \mathring{E}[a\nabla u], v\rangle_{\Omega}, \quad \forall u \in H^1(\Omega), \quad \forall v \in H^1(\Omega) \end{aligned} \tag{6}$$

which is evidently continuous and can be written as

$$\check{A}u = \nabla \cdot \mathring{E}[a\nabla u]. \tag{7}$$

Here $V \in H^1(\mathbb{R}^2)$ is such that $r_{\Omega} V = v$ and $\mathring{E}$ denotes the operator of extension of the functions, defined in Ω, by zero outside Ω in $\mathbb{R}^2$. For any $u \in H^1(\Omega)$, the functional $\check{A}u$ belongs to $\widetilde{H}^{-1}(\Omega)$ and is the extension of the functional $Au \in H^{-1}(\Omega)$, which domain is thus extended from $\widetilde{H}^1(\Omega)$ to the domain $H^1(\Omega)$ for $\check{A}u$.

Inspired by the first Green identity for smooth functions, we can define *the generalized conormal derivative* (see, for example,[9, Lemma 4.3], [10, Definition 3.1] and [26, Lemma 2.2]).

Definition 1. Let $u \in H^1(\Omega)$ and $Au = r_{\Omega}\tilde{f}$ in Ω for some $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$. Then the generalized co-normal derivative $T^+_a(\tilde{f}, u) \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined as

$$\langle T^+_a(\tilde{f}, u), w\rangle_{\partial\Omega} := \langle \tilde{f}, \gamma^{-1} w\rangle_{\Omega} + \mathcal{E}_a(u, \gamma^{-1} w) = \langle \tilde{f} - \check{A}u, \gamma^{-1} w\rangle_{\Omega}, \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \quad \text{i.e.,} \quad T^+(\tilde{f}, u) := (\gamma^{-1})^*(\tilde{f} - \check{A}u). \tag{8}$$

By[9, Lemma 4.3] and[10, Theorem 3.2], we have the estimate

$$\|T^+_a(\tilde{f}, u)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C_1 \|u\|_{H^1(\Omega)} + C_2 \|\tilde{f}\|_{\widetilde{H}^{-1}(\Omega)}, \tag{9}$$

and for $u \in H^1(\Omega)$ such that $Au = r_{\Omega}\tilde{f}$ in Ω for some $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ the first Green identity holds in the following form:

$$\langle T^+_a(\tilde{f}, u), \gamma^+ v\rangle_{\partial\Omega} := \langle \tilde{f}, v\rangle_{\Omega} + \mathcal{E}_a(u, v) = \langle \tilde{f} - \check{A}u, v\rangle_{\Omega}, \qquad \forall v \in H^1(\Omega). \tag{10}$$

As follows from Definition 1, the generalised conormal derivative is nonlinear with respect to u for a fixed $\tilde{f}$, but linear with respect to the couple $(\tilde{f}, u)$, i.e.,

$$\alpha_1 T^+_a(\tilde{f}_1, u_1) + \alpha_2 T^+_a(\tilde{f}_2, u_2) = T^+_a(\alpha_1\tilde{f}_1, \alpha_1 u_1) + T^+_a(\alpha_2\tilde{f}_2, \alpha_2 u_2) = T^+_a(\alpha_1\tilde{f}_1 + \alpha_2\tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2) \tag{11}$$

for any real numbers α_1, α_2.

Let us also define some subspaces of $H^s(\Omega)$, cf. [10,13,27,28].

Definition 2. Let $s \in \mathbb{R}$ and $A_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ be a linear operator. For $t \geq -\frac{1}{2}$ we introduce the space

$$H^{s,t}(\Omega; A_*) := \{g \in H^s(\Omega) : \text{there exists } \tilde{f}_g \in \widetilde{H}^t(\Omega) \text{ such that } A_* g|_{\Omega} = \tilde{f}_g|_{\Omega}\}$$

endowed with the norm

$$\|g\|_{H^{s,t}(\Omega;A_*)} := \left(\|g\|^2_{H^s(\Omega)} + \|\tilde{f}_g\|^2_{\widetilde{H}^t(\Omega)}\right)^{\frac{1}{2}}$$

and the inner product

$$(g,h)_{H^{s,t}(\Omega;A_*)} = (g,h)_{H^s(\Omega)} + (\tilde{f}_g, \tilde{f}_h)_{\widetilde{H}^t(\Omega)}.$$

The distribution $\tilde{f}_g \in \widetilde{H}^t(\Omega)$, $t \geq -\frac{1}{2}$, in the above definition is an extension of the distribution $A_* g|_{\Omega} \in H^t(\Omega)$, and the extension is unique (if it does exist) since any distribution from the space $H^t(\mathbb{R}^2)$ with support in $\partial\Omega$ is identically zero if $t \geq -\frac{1}{2}$ (see, e.g., [9, Lemma 3.39] and [10, Theorem 2.10]). We denote this extension as an operator $\tilde{A}_*$, i.e., $\tilde{A}_* g = \tilde{f}_g$. The uniqueness implies that the norm $\|g\|_{H^{s,t}(\Omega;A_*)}$ is well defined.

We will mostly use the operators A, B or Δ as A_* in the above definition. Note that since $Au - a\Delta u = \nabla a \cdot \nabla u \in L_2(\Omega)$, for $u \in H^1(\Omega)$, we have $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; \Delta)$.

Definition 3. For $u \in H^{1,-\frac{1}{2}}(\Omega; A)$, we define the canonical conormal derivative $T_a^+ u \in H^{-\frac{1}{2}}(\partial\Omega)$ as

$$\langle T_a^+ u, w\rangle_{\partial\Omega} := \langle \tilde{A}u, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_a(u, \gamma^{-1}w) = \langle \tilde{A}u - \check{A}u, \gamma^{-1}w\rangle_\Omega , \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \quad \text{i. e.,} \quad T_a^+ u := (\gamma^{-1})^*(\tilde{A}u - \check{A}u). \tag{12}$$

The canonical conormal derivative $T_a^+ u$ is independent of (non-unique) choice of the operator γ^{-1}, the operator $T_a^+ : H^{1,-\frac{1}{2}}(\Omega; A) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the following form,

$$\langle T_a^+ u, \gamma^+ v\rangle_{\partial\Omega} := \langle \tilde{A}u, v\rangle_\Omega + \mathcal{E}_a(u, v) , \qquad \forall v \in H^1(\Omega). \tag{13}$$

The operator $T_a^+ : H^{1,t}(\Omega; A) \to H^{-\frac{1}{2}}(\partial\Omega)$ in Definition 3 is continuous for $t \geq -\frac{1}{2}$. The canonical co-normal derivative is defined by the function u and the operator A and does not depend separately on the right-hand side $\tilde{f}$ (i.e. its behavior on the boundary), unlike the generalised co-normal derivative defined in (8), and the operator T_a^+ is linear. Note that the canonical co-normal derivative coincides with classical conormal derivative $T_a^+ u = a\frac{\partial u}{\partial n}$ if the latter does exist in the trace sense, see [10, Corollary 3.14 and Theorem 3.16].

Let $u \in H^{1,-\frac{1}{2}}(\Omega; A)$. Then Definitions 1 and 3 imply that the generalised co-normal derivative for arbitrary extension $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ of the distribution Au can be expressed as

$$\langle T_a^+(\tilde{f}, u), w\rangle_{\partial\Omega} := \langle T_a^+ u, w\rangle_{\partial\Omega} + \langle \tilde{f} - \check{A}u, \gamma^{-1}w\rangle_\Omega , \qquad \forall w \in H^{\frac{1}{2}}(\partial\Omega). \tag{14}$$

Let us consider the auxiliary linear elliptic partial differential operator B defined by

$$Bu(x) := B(x, \partial_x)u(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(x)\frac{\partial u(x)}{\partial x_i}\right), \tag{15}$$

where $b \in C^\infty(\overline{\Omega})$, $b(x) > 0$ for $x \in \overline{\Omega}$.

Since for $u \in H^{1,0}(\Omega, \Delta)$, $Au - Bu = (a-b)\Delta u + \nabla(a-b)\cdot\nabla u \in L_2(\Omega)$, we have, $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)$. Let $u \in H^1(\Omega)$ and $v \in H^{1,0}(\Omega; B)$. Then we write the first Green identity for operator B in the form

$$\mathcal{E}_b(u, v) + \int_\Omega u(x)Bv(x)dx = \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega} \tag{16}$$

where

$$\mathcal{E}_b(u, v) = \int_\Omega b(x)\nabla u(x)\cdot\nabla v(x)dx.$$

If, in addition, $Au = \tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, then according to the definition of $T_a^+(\tilde{f}, u)$, in (8), the *two-operator second Green identity* can be written as

$$\langle \tilde{f}, v\rangle_\Omega - \int_\Omega u(x)Bv(x)dx + \int_\Omega [a(x) - b(x)]\nabla u(x)\cdot\nabla v(x)dx = \langle T_a^+(\tilde{f}, u), \gamma^+ v\rangle_{\partial\Omega} - \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega}. \tag{17}$$

Moreover, for $u, v \in H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)$ (17) becomes

$$\int_{\Omega} [v(x)Au(x) - u(x)Bv(x)]dx + \int_{\Omega} [a(x) - b(x)]\nabla u(x) \cdot \nabla v(x)dx = \langle T_a^+ u, \gamma^+ v\rangle_{\partial\Omega} - \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega}. \tag{18}$$

3 | PARAMETRIX AND POTENTIAL TYPE OPERATORS

Definition 4. We will say, a function $P_b(x, y)$ of two variables $x, y \in \Omega$ is a parametrix (Levi function) for the operator $B(x; \partial_x)$ in $\mathbb{R}^2$ if (see, e.g.,[3,29,30,31,32])

$$B(x, \partial_x)P_b(x, y) = \delta(x - y) + R_b(x, y), \tag{19}$$

where δ is the Dirac-delta distribution, while $R(x, y)$ is a remainder possessing at most a weak singularity at $x = y$.

For some positive constant r_0, the parametrix and hence the corresponding remainder in 2D can be chosen as in[3],

$$P_b(x, y) = \frac{1}{2\pi b(y)} \ln\left(\frac{|x - y|}{r_0}\right), \tag{20}$$

$$R_b(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi b(y)|x - y|^2} \frac{\partial b(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2. \tag{21}$$

The parametrix $P_b(x, y)$ in (20) is the fundamental solution to the operator $B(y, \partial_x) := b(y)\Delta_x$ with “frozen" coefficient $b(x) = b(y)$, and

$$B(y, \partial_x)P_b(x, y) = \delta(x - y). \tag{22}$$

Let $b \in C^\infty(\mathbb{R}^2)$ and $b(x) > 0$ a.e. in $\mathbb{R}^2$. For some scalar function g the parametrix-based Newtonian and the remainder volume potential operators, corresponding to the parametrix (20) and the remainder (21) are given by

$$\mathbf{P}_b g(y) := \int_{\mathbb{R}^2} P_b(x, y)g(x)dx, \qquad y \in \mathbb{R}^2, \tag{23}$$

$$\mathcal{P}_b g(y) := \int_{\Omega} P_b(x, y)g(x)dx, \qquad y \in \Omega, \tag{24}$$

$$\mathcal{R}_b g(y) := \int_{\Omega} R_b(x, y)g(x)dx, \qquad y \in \Omega. \tag{25}$$

For $g \in H^s(\mathbb{R}^2),\ s \in \mathbb{R}$, (23) is understood as $\mathbf{P}_b g = \frac{1}{b}\mathbf{P}_\Delta g$, where the Newtonian potential operator $\mathbf{P}_\Delta$ for Laplacian Δ is well defined in terms of the Fourier transform (i.e., as pseudodifferential operator), on any space $H^s(\mathbb{R}^2)$. For $g \in \widetilde{H}^s(\Omega)$, and any $s \in \mathbb{R}$, definitions in (24) and (25) can be understood as

$$\mathcal{P}_b g = \frac{1}{b} r_\Omega \mathbf{P}_\Delta g,\ \mathcal{P}_b g = \frac{a}{b} r_\Omega \mathbf{P}_a g, \quad \text{and} \quad \mathcal{R}_b g = -\frac{1}{b} r_\Omega \nabla \cdot \mathbf{P}_\Delta(g\nabla b), \tag{26}$$

while for $g \in H^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}$, as (26) with g replaced by $\widetilde{E}g$, where $\widetilde{E} : H^s(\Omega) \to \widetilde{H}^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}$, is the unique extension operator related with the operator $\mathring{E}$ of extension by zero, cf.[10, Theorem 16].

For $y \notin \partial\Omega$, the single layer and the double layer surface potential operators, corresponding to the parametrix (20) are defined as

$$V_b g(y) := -\int_{\partial\Omega} P_b(x, y)g(x)dS_x, \tag{27}$$

$$W_b g(y) := -\int_{\partial\Omega} [T_b(x, n(x), \partial_x)P_b(x, y)]g(x)dS_x, \tag{28}$$

where g is some scalar density function. The integrals are understood in the distributional sense if g is not integrable, while V_Δ and W_Δ are the single layer and double layer potentials corresponding to the Laplacian Δ. The corresponding boundary integral (pseudodifferential) operators of direct surface values of the single and the double layer potentials,$\mathcal{V}_b$ and $\mathcal{W}_b$ for $y \in \partial\Omega$, are

$$\mathcal{V}_b g(y) := -\int_{\partial\Omega} P_b(x,y)g(x)dS_x, \tag{29}$$

$$\mathcal{W}_b g(y) := -\int_{\partial\Omega} T_b(x,n(x),\partial_x)P_b(x,y)g(x)dS_x, \tag{30}$$

where $\mathcal{V}_\Delta$ and $\mathcal{W}_\Delta$ are respectively the direct values of the single and double layer potentials corresponding to the Laplacian Δ.

We can also calculate at $y \in \partial\Omega$ the co-normal derivatives, associated with the operator A, of the single layer potential and of the double layer potential:

$$T_a^{\pm} V_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} V_b g(y), \tag{31}$$

$$\mathcal{L}_{ab}^{\pm} g(y) := T_a^{\pm} W_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} W_b g(y). \tag{32}$$

The direct value operators associated with (31) are

$$\mathcal{W}'_{ab} g(y) := -\int_{\partial\Omega} [T_a(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x, \tag{33}$$

$$\mathcal{W}'_b g(y) := -\int_{\partial\Omega} [T_b(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x. \tag{34}$$

From equations (23)-(34) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta = \frac{1}{2\pi}\log\left(\frac{|x-y|}{r_0}\right)$ of the Laplace operator Δ.

$$\mathbf{P}_a g = \frac{1}{a}\mathbf{P}_\Delta g, \quad \mathbf{P}_b g = \frac{1}{b}\mathbf{P}_\Delta g, \quad \mathcal{P}_a g = \frac{1}{a}\mathcal{P}_\Delta g, \quad \mathcal{P}_b g = \frac{1}{b}\mathcal{P}_\Delta g. \tag{35}$$

$$\frac{a}{b}V_a g = V_b g = \frac{1}{b}V_\Delta g; \qquad \frac{a}{b}W_a\left(\frac{bg}{a}\right) = W_b g = \frac{1}{b}W_\Delta\left(bg\right), \tag{36}$$

$$\frac{a}{b}\mathcal{V}_a g = \mathcal{V}_b g = \frac{1}{b}\mathcal{V}_\Delta g; \qquad \frac{a}{b}\mathcal{W}_a\left(\frac{bg}{a}\right) = \mathcal{W}_b g = \frac{1}{b}\mathcal{W}_\Delta\left(bg\right), \tag{37}$$

$$\mathcal{W'}_{ab} g = \frac{a}{b}\mathcal{W'}_b g = \frac{a}{b}\left\{\mathcal{W'}_\Delta g + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathcal{V}_\Delta g\right\}, \tag{38}$$

$$\mathcal{L}_{ab}^{\pm} g = \frac{a}{b}\mathcal{L}_b^{\pm} g = \frac{a}{b}\left\{\mathcal{L}_\Delta(bg) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\gamma^{\pm} W_\Delta(bg)\right\}, \tag{39}$$

$$\hat{\mathcal{L}}_b g := T_\Delta^{+} W_\Delta(bg) = T_\Delta^{-} W_\Delta(bg) = \hat{\mathcal{L}}_\Delta(bg) \quad \text{on} \quad \partial\Omega. \tag{40}$$

It is taken into account that b and its derivatives are continuous in $\mathbb{R}^2$ and

$$\mathcal{L}_\Delta(bg) := \mathcal{L}_\Delta^{+}(bg) = \mathcal{L}_\Delta^{-}(bg)$$

by the Liapunov-Tauber theorem. Hence,

$$\Delta(bV_b g) = 0, \ \Delta(bW_b g) = 0 \quad \text{in} \quad \Omega, \quad \forall g \in H^s(\partial\Omega) \quad (\forall s \in \mathbb{R}), \tag{41}$$

$$\Delta(b\mathcal{P}_b g) = g \quad \text{in}\ \Omega, \quad \forall g \in \widetilde{H}^s(\Omega) \quad (\forall s \in \mathbb{R}). \tag{42}$$

The mapping properties of the operators (23)-(34) follow from relations (35)-(40) and are described in detail in[15, Appendix A]. Particularly, we have the following jump relations:

Theorem 1. For $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then there hold the following relations on $\partial\Omega$,

$$\gamma^{\pm} V_b g_1 = \mathcal{V}_b g_1, \tag{43}$$

$$\gamma^{\pm} W_b g_2 = \mp\frac{1}{2}g_2 + \mathcal{W}_b g_2, \tag{44}$$

$$T_a^{\pm} V_b g_1 = \pm\frac{1}{2}\frac{a}{b}g_1 + \mathcal{W'}_{ab} g_1. \tag{45}$$

4 | THE TWO-OPERATOR THIRD GREEN IDENTITY AND INTEGRAL RELATIONS

Applying some limiting procedures (see, e.g.,[29, S.3.8] and[30]), we obtain the parametrix based third Green identities.

Theorem 2. (i) If $u \in H^1(\Omega)$, then the following third Green identity holds,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \gamma^+ u = \mathcal{P}_b \check{A} u \quad \text{in} \quad \Omega, \tag{46}$$

where the operator $\check{A}$ is defined in (7), and for $u \in C^1(\overline{\Omega})$,

$$\mathcal{P}_b \check{A} u(y) := \langle \check{A} u, P_b(.,y) \rangle_\Omega = -\mathcal{E}_a(u, P_b(.,y)) = -\int_\Omega a(x) \nabla u(x) \cdot \nabla_x P_b(x,y) dx \tag{47}$$

and

$$\mathcal{Z}_b u = -\int_\Omega [a(x) - b(x)] \nabla_x P_b(x,y) \cdot \nabla u(x) dx = \frac{1}{b(y)} \sum_{j=1}^{2} \partial_j \mathcal{P}_\Delta \left[(a-b) \partial_j u \right] \quad \text{in} \quad \Omega. \tag{48}$$

(ii) If $Au = r_\Omega \tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, then recalling the definition of $T_a^+(\tilde{f}, u)$, in (8), we arrive at the generalised two-operator third Green identity in the following form,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+(\tilde{f}, u) + W_b \gamma^+ u = \mathcal{P}_b \tilde{f} \quad \text{in} \quad \Omega, \tag{49}$$

where it was taken into account that

$$\langle T_a^+(\tilde{f}, u), P_b(x,y) \rangle_{\partial\Omega} = -V_b T_a^+(\tilde{f}, u), \quad \langle \tilde{f}, P_b(x,y) \rangle_\Omega = \mathcal{P}_b \tilde{f}. \tag{50}$$

Proof. (i) Let first $u \in D(\overline{\Omega})$. Let $y \in \Omega$, $B_\epsilon(y) \subset \Omega$ be a ball centred at y with sufficiently small radius ϵ, and $\Omega_\epsilon := \Omega \setminus \overline{B_\epsilon}(y)$. For the fixed y, evidently, $P_b(.,y) \in D(\overline{\Omega_\epsilon}) \subset H^{1,0}(A; \Omega_\epsilon)$ and has the coinciding classical and cannonical co-normal derivatives on $\partial\Omega_\epsilon$. Then from (20) and the first Green identity (16) employed for Ω_ϵ with $v = P_b(.,y)$ we obtain

$$-\int_{\partial B_\epsilon(y)} T_x^+ P_b(x,y) \gamma^+ u(x) ds_x - \int_{\partial\Omega} T_x P_b(x,y) \gamma^+ u(x) ds_x + \int_{\Omega_\epsilon} u(x) R_b(x,y) dx = -\int_{\Omega_\epsilon} b(x) \nabla u(x) \cdot \nabla_x P_b(x,y) dx,$$

which we rewrite as

$$-\int_{\partial B_\epsilon(y)} T_x^+ P_b(x,y) \gamma^+ u(x) ds_x - \int_{\partial\Omega} T_x P_b(x,y) \gamma^+ u(x) ds_x - \int_{\Omega_\epsilon} [a(x) - b(x)] \nabla u(x) \cdot \nabla_x P_b(x,y) dx$$
$$+ \int_{\Omega_\epsilon} u(x) R_b(x,y) dx = -\int_{\Omega_\epsilon} a(x) \nabla u(x) \cdot \nabla_x P_b(x,y) dx. \tag{51}$$

Taking the limit as $\epsilon \to 0$, (51) reduces to the third Green identity (46)–(47) for any $u \in \mathcal{D}(\overline{\Omega})$. Taking into account the density of $\mathcal{D}(\overline{\Omega})$ in $H^1(\Omega)$, and the mapping properties of the integral potentials, see Appendix, we obtain that (46)–(47) hold true also for any $u \in H^1(\Omega)$. (ii) Let $\{\tilde{f}_k\} \in \mathcal{D}(\Omega)$ be a sequence of converging to $\tilde{f}$ in $\widetilde{H}^{-1}(\Omega)$ as $k \to \infty$. Then, according to[8, Theorem B.1] there exists a sequence $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ converging to u in $H^1(\Omega)$ such that $Au_k = r_\Omega \tilde{f}_k$ and $T_a^+(u_k) = T_a^+(\tilde{f}_k, u_k)$ converge to $T_a^+(\tilde{f}, u)$ in $H^{-\frac{1}{2}}(\partial\Omega)$. For such u_k by (47) and (8), we have

$$\begin{aligned} \mathcal{P}_b \check{A} u_k(y) &= \frac{1}{b(y)} \nabla_y \cdot \int_\Omega a(x) P_\Delta(x,y) \nabla u_k(x) dx = -\frac{1}{b(y)} \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} a(x) \nabla u_k(x) P_\Delta(x,y) dx = -\lim_{\epsilon \to 0} \mathcal{E}_{\Omega_\epsilon}(u_k, P_b(.,y)) \\ &= -\lim_{\epsilon \to 0} \left[\int_{\Omega_\epsilon} \tilde{f}_k P_b(x,y) dx - \int_{\partial B_\epsilon(y)} P_b(x,y) T_a^+ u_k(x) dS(x) \right] + \lim_{\epsilon \to 0} \int_{\partial\Omega} P_b(x,y) T_a^+ u_k(x) dS(x) = \mathcal{P}_b \tilde{f}_k + V_b T_a^+ u_k(y). \end{aligned} \tag{52}$$

Taking the limits as $k \to \infty$ in (52), we obtain $\mathcal{P}_b \check{A} u(y) = \mathcal{P}_b \tilde{f} + V_b T_a^+(\tilde{f}, u)$, which substitution to (46) gives (49). □

Below we state and prove[20, Corollary 3] for completeness.

Corollary 1. Using the Gauss divergence theorem, we can rewrite $\mathcal{Z}_b u(y)$ in the form that does not involve derivatives of u,

$$\mathcal{Z}_b u(y) := \left[\frac{a(y)}{b(y)} - 1\right] u(y) + \hat{\mathcal{Z}}_b u(y), \tag{53}$$

$$\hat{\mathcal{Z}}_b u(y) := \frac{a(y)}{b(y)} W_a \gamma^+ u(y) - W_b \gamma^+ u(y) + \frac{a(y)}{b(y)} \mathcal{R}_a u(y) - \mathcal{R}_b u(y), \tag{54}$$

which allows to call $\mathcal{Z}_b$ integral operator in spite of its integro-differential representation (48).

Proof. As in the proof of Theorem 2 item (i), let first $u \in D(\overline{\Omega})$. Let $y \in \Omega$, $\underline{B_\epsilon}(y) \subset \Omega$ be a ball centred at y with sufficiently small radius ϵ, and $\Omega_\epsilon := \Omega \setminus \overline{B_\epsilon}(y)$. For the fixed y, evidently, $P_b(., y) \in D(\overline{\Omega_\epsilon}) \subset H^{1,0}(A; \Omega_\epsilon)$ and has the coinciding classical and cannonical co-normal derivatives on $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(y)$. Next, let us denote $\mathcal{Z}_b^\epsilon u(y) := -\int_{\Omega_\epsilon} [a(x) - b(x)] \nabla_x P_b(x, y) \cdot \nabla u(x) dx$, which can be rewritten as

$$\mathcal{Z}_b^\epsilon u(y) = \int_{\Omega_\epsilon} \left[\nabla\big(a(x) - b(x)\big) \cdot \nabla_x P_b(x, y)\right] u(x) dx - \int_{\Omega_\epsilon} \nabla\left[\big(a(x) - b(x)\big) u(x)\right] \cdot \nabla_x P_b(x, y) dx.$$

Observe that

$$\begin{aligned} I_1(y, \epsilon) = \int_{\Omega_\epsilon} \left[\nabla\big(a(x) - b(x)\big) \cdot \nabla_x P_b(x, y)\right] u(x) dx &= \int_{\Omega_\epsilon} \left[\nabla a(x) \cdot \nabla_x P_b(x, y)\right] u(x) \mathrm{d}x - \int_{\Omega_\epsilon} \left[\nabla b(x) \cdot \nabla_x P_b(x, y)\right] u(x) \mathrm{d}x \\ &= \frac{a(y)}{b(y)} \int_{\Omega_\epsilon} \left[\nabla a(x) \cdot \nabla_x P_a(x, y)\right] u(x) \mathrm{d}x - \int_{\Omega_\epsilon} \left[\nabla b(x) \cdot \nabla_x P_b(x, y)\right] u(x) \mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} I_2(y, \epsilon) = -\int_{\Omega_\epsilon} \nabla\left[\big(a(x) - b(x)\big) u(x)\right] \cdot \nabla_x P_b(x, y) dx &= \int_{\Omega_\epsilon} [a(x) - b(x)] u(x) \Delta_x P_b(x, y) dx - \int_{\partial\Omega_\epsilon} [a(x) - b(x)] \gamma^+ u(x) \nabla_x P_b(x, y) \cdot n(x) dS_x \\ &= -\frac{a(y)}{b(y)} \int_{\partial\Omega} a(x) \nabla_x P_a(x, y) \cdot n(x) \gamma^+ u(x) dS_x + \int_{\partial\Omega} b(x) \nabla_x P_b(x, y) \cdot n(x) \gamma^+ u(x) dS_x \\ &\quad - \frac{a(y)}{b(y)} \int_{\partial B(y,\epsilon)} a(x) \nabla_x P_a(x, y) \cdot n(x) u(x) dS_x + \int_{\partial B_\epsilon(y)} b(x) \nabla_x P_b(x, y) \cdot n(x) u(x) dS_x \\ = \frac{a(y)}{b(y)} W_a \gamma^+ u(y) - W_b \gamma^+ u(y) - \frac{1}{b(y)} \int_{\partial B_\epsilon(y)} a(x) \nabla_x P_\Delta(x, y) \cdot n(x) \gamma^+ u(x) dS_x &+ \frac{1}{b(y)} \int_{\partial B_\epsilon(y)} b(x) \nabla_x P_\Delta(x, y) \cdot n(x) \gamma^+ u(x) dS_x. \end{aligned}$$

Taking the limit as $\epsilon \to 0$ we obtain

$$\mathcal{Z}_b u(y) = \lim_{\epsilon \to 0} \mathcal{Z}_b^\epsilon u(y) = \lim_{\epsilon \to 0} \left[I_1(y, \epsilon) + I_2(y, \epsilon)\right] = \frac{a(y)}{b(y)} \mathcal{R}_a u(y) - \mathcal{R}_b u(y) + \frac{a(y)}{b(y)} W_a \gamma^+ u(y) - W_b \gamma^+ u(y) + \left[\frac{a(y)}{b(y)} - 1\right] u(y)$$

which is as in Eqs. (53) and (54). □

Note that the operator $\mathcal{Z}_b$ does not vanish unless operators A and B are equal. For some functions $\tilde{f}$, Ψ, Φ let us consider a more general "indirect" integral relation, associated with (49).

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi = \mathcal{P}_b \tilde{f} \quad \text{in} \quad \Omega. \tag{55}$$

Lemma 1. Let $u \in H^1(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ satisfy (55). Then

$$Au = r_\Omega \tilde{f} \quad \text{in} \quad \Omega, \tag{56}$$

$$r_\Omega V_b(\Psi - T_a^+(\tilde{f}, u)) - r_\Omega W_b(\Phi - \gamma^+ u) = 0 \quad \text{in} \quad \Omega, \tag{57}$$

$$\gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b \Psi - \frac{1}{2}\Phi + \mathcal{W}_b \Phi = \gamma^+ \mathcal{P}_b \tilde{f} \qquad \text{on} \quad \partial\Omega, \tag{58}$$

$$T_a^+(\tilde{f}, u) + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \frac{a}{2b}\Psi - \mathcal{W}'_{ab}\Psi + \mathcal{L}_{ab}^+ \Phi = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) \quad \text{on} \quad \partial\Omega, \tag{59}$$

where

$$\mathcal{R}_*^b \tilde{f}(y) := -\sum_{j=1}^{3} \partial_j[(\partial_j b)\mathcal{P}_b \tilde{f}]. \tag{60}$$

Proof. Subtracting (55) from identity (46), we obtain

$$V_b\Psi(y) - W_b(\Phi - \gamma^+ u)(y) = \mathcal{P}_b[\check{A}u(y) - \tilde{f}](y), \quad y \in \Omega. \tag{61}$$

Multiplying equality (61) by $b(y)$, applying the Laplace operator Δ and taking into account equations (41) and (42), we get $r_\Omega \tilde{f} = r_\Omega(\check{A}u) = Au$ in Ω. This means $\tilde{f}$ is an extension of the distribution $Au \in H^{-1}(\Omega)$ to $\widetilde{H}^{-1}(\Omega)$, and u satisfies (56). Then (8) implies

$$\mathcal{P}_b[\check{A}u - \tilde{f}](y) = \langle \check{A}u - \tilde{f}, P_b(.,y)\rangle_\Omega = -\langle T_a^+(\tilde{f},u), P_b(.y)\rangle_{\partial\Omega} = V_b T_a^+(\tilde{f},u), \quad y \in \Omega. \tag{62}$$

Substituting (62) into (61) leads to (57). Equation (58) follows from (55) and jump relations in (43) and (44). To prove (59), let us first remark that for $u \in H^1(\Omega)$, we have $H^1(\Omega; A) = H^1(\Omega; \Delta) = H^1(\Omega; B)$ and

$$B\mathcal{P}_b \tilde{f} = \tilde{f} + \mathcal{R}_*^b \tilde{f} \quad \text{in } \Omega, \tag{63}$$

due to (56), which implies $B(\mathcal{P}_b \tilde{f} - u) = \mathcal{R}_*^b \tilde{f}$ in Ω, with $\mathcal{R}_*^b \tilde{f}$ given by (60), and thus $\mathcal{R}_*^b \tilde{f} \in L_2(\Omega)$. Then $B(\mathcal{P}_b \tilde{f} - u)$ can be canonically extended (by zero) to

$$\widetilde{B}(\mathcal{P}_b \tilde{f} - u) = \mathring{E}\mathcal{R}_*^b \tilde{f} \in \widetilde{H}^0(\Omega) \subset \widetilde{H}^{-1}(\Omega).$$

Thus there exists a canonical co-normal derivative $T_b^+(\mathcal{P}_b \tilde{f} - u)$ written as (see, e.g.,[8, Eq. (4.14)], [11, Eq. (4.23)].)

$$T_b^+(\mathcal{P}_b \tilde{f} - u) = T_b^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_b^+(\tilde{f}, u), \tag{64}$$

and hence

$$T_a^+\big(\mathcal{P}_b \tilde{f} - u\big) = \frac{a}{b} T_b^+\big(\mathcal{P}_b \tilde{f} - u\big) = \frac{a}{b}\big[T_b^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_b^+(\tilde{f},u)\big] = T_a^+\big(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}\big) - T_a^+(\tilde{f},u). \tag{65}$$

From (55) it follows that $\mathcal{P}_b \tilde{f} - u = \mathcal{Z}_b u + \mathcal{R}_b u - V_b\Psi + W_b\Phi$ in Ω. Substituting this on the left-hand side of (64) and taking into account (39) and the jump relation (45), we arrive at (59). □

Remark 1. If $\tilde{f} \in \widetilde{H}^{-\frac{1}{2}}(\Omega) \subset \widetilde{H}^{-1}(\Omega)$, then $\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f} \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$ as well, which implies $\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f} = \tilde{A}\mathcal{P}_b \tilde{f}$ and

$$T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) = T_a^+(\tilde{B}\mathcal{P}_b \tilde{f}, \mathcal{P}_b \tilde{f}) = T_a^+ \mathcal{P}_b \tilde{f}. \tag{66}$$

Furthermore, if the hypotheses of Lemma 1 are satisfied, then (56) implies $u \in H^{1,-\frac{1}{2}}(\Omega; A)$ and $T_a^+(\tilde{f},u) = T_a^+(\tilde{A}u, u) = T_a^+ u$. Henceforth (59), takes the familiar form, cf.[15, equation (3.23)],

$$T_a^+ u + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \frac{a}{2b}\Psi - \mathcal{W}'_{ab}\Psi + \mathcal{L}_{ab}^+\Phi = T_a^+ \mathcal{P}_b \tilde{f} \quad \text{on} \quad \partial\Omega.$$

Remark 2. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and a sequence $\{\phi_i\} \in \widetilde{H}^{-1}(\Omega)$ converge to $\tilde{f}$ in $\widetilde{H}^{-1}(\Omega)$. By the continuity of operators[8, C.1 and C.2], estimate (9) and relation (66) for ϕ_i, we obtain that

$$T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) = \lim_{i\to\infty} T_a^+(\phi_i + \mathring{E}\mathcal{R}_*^b \phi_i, \mathcal{P}_b \phi_i) = \lim_{i\to\infty} T_a^+ \mathcal{P}_b \phi_i.$$

in $H^{-\frac{1}{2}}(\partial\Omega)$, cf. also[8, Theorem B.1].

Lemma 1 and the third Green identity (49) imply, the following assertion.

Corollary 2. If $u \in H^1(\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ are such that $Au = r_\Omega \tilde{f}$ in Ω, then

$$\frac{1}{2}\gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b T_a^+(\tilde{f},u) + \mathcal{W}_b \gamma^+ u = \gamma^+ \mathcal{P}_b \tilde{f} \quad \text{on} \quad \partial\Omega, \tag{67}$$

$$\left(1 - \frac{a}{2b}\right) T_a^+(\tilde{f},u) + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} T_a^+(\tilde{f},u) + \mathcal{L}_{ab}^+ \gamma^+ u = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) \quad \text{on} \quad \partial\Omega. \tag{68}$$

Note that if $\mathcal{P}_b$ is not only the parametrix but also the fundamental solution of the operator B, then the remainder operator $\mathcal{R}_b$ vanishes in (49) and (67)-(68) (and everywhere in the paper), while the operator $\mathcal{Z}_b$ stays unless $A = B$. The following statement is proved in[8, Lemma 4.6].

Theorem 3. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE $Au = r_\Omega \tilde{f}$ in Ω if and only if it is a solution of BDIDE (49).

Proof. If $u \in H^1(\Omega)$ solves PDE $Au = r_\Omega \tilde{f}$ in Ω, then it satisfies (49). On the other hand, if u solves BDIDE (49), then using Lemma 1 for $\Psi = T_a^+(\tilde{f}, u)$, $\Phi = \gamma^+ u$ completes the proof. □

5 | INVERTIBILITY OF SINGLE LAYER POTENTIAL OPERATOR

The boundary integral operator, $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator of index zero (see, e.g.,[9, Theorem 7.6]). Thus the first relation in (37) leads to the same result for the single layer potential $\mathcal{V}_b$. For the case of 3D, Lemma 3.2(i) in[15] asserts that for $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$, if $V_b\Psi^* = 0$ in Ω, then $\Psi^* = 0$ in Ω. Implying the invertibility of single layer potential operator V_b mapping from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. But this is not the case for 2D. It is well-known (see, e.g.,[33, Remark 1.42(ii)] and[34, Theorem 6.22]) that for some 2D domains the kernel of the operator $\mathcal{V}_\Delta$ is nontrivial, thus due to the first relation in (37), the kernel of operator $\mathcal{V}_b$ is nontrivial as well for the same domains. To ensure the invertibility of the single layer potential operator in 2D, for $s \in \mathbb{R}$, let us define the subspace of $H^s(\partial\Omega)$, (cf. e.g.,[34, p. 147]),

$$H^s_{**}(\partial\Omega) := \{g \in H^s(\partial\Omega) : \langle g, 1\rangle_{\partial\Omega} = 0\}. \tag{69}$$

The following result is proved in[18, Theorem 4], see also[35, Theorem 1].

Theorem 4. If $\psi \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$ satisfies $\mathcal{V}_b\psi = 0$ on $\partial\Omega$, then $\psi = 0$.

Following[9, Theorem 8.15], there exists a unique real-valued distribution $\psi_{eq} \in H^{-\frac{1}{2}}(\partial\Omega)$ called equilibrium density for $\partial\Omega$ such that $\mathcal{V}_\Delta\psi_{eq}$ is constant on $\partial\Omega$, and $(1, \psi_{eq})_{\partial\Omega} = 1$. For $n = 2$ the constant $\mathcal{V}_\Delta\psi_{eq}$ is not always positive and one introduces the *logarithmic capacity*, $\mathrm{Cap}_{\partial\Omega}$ using the relation

$$\mathcal{V}_\Delta\psi_{eq} = \frac{1}{2\pi}\ln\left(\frac{r_0}{\mathrm{Cap}_{\partial\Omega}}\right),$$

for some positive constant r_0 as in equation (20). In particular $\mathcal{V}_\Delta\psi_{eq} = 0$ if and only if $r_0 = \mathrm{Cap}_{\partial\Omega}$. The following statement is proved in[9, Theorem 8.16].

Theorem 5. Let r_0 be some positive constant.

(i) The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, is $H^{-\frac{1}{2}}(\partial\Omega)$- elliptic, i.e., $\langle\mathcal{V}_\Delta\psi, \psi\rangle_{\partial\Omega} \geq c\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)}$ for all $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, if and only if $r_0 > \mathrm{Cap}_{\partial\Omega}$.

(ii) The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, has a bounded inverse if and only if $r_0 \neq \mathrm{Cap}_{\partial\Omega}$.

The following theorem ensures the invertibility of the single layer potential operator $\mathcal{V}_b$ in 2D.

Theorem 6. Let $\Omega \subset \mathbb{R}^2$ with $r_0 > \mathrm{diam}(\Omega)$. Then the single layer potential $\mathcal{V}_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.

Proof. Since $\mathrm{Cap}_{\partial\Omega} \leq \mathrm{diam}(\Omega)$, (see,[36, p.553, properties 1 and 3]), then $r_0 > \mathrm{diam}(\Omega)$ implies $r_0 > \mathrm{Cap}_{\partial\Omega}$. For the case $a = b$ the assertion is proved in[18, Theorem 5]. Due to the first relation in (37) and Theorem 5(ii) follows the invertibility of the single layer potential operator $\mathcal{V}_b$ for the case $a \neq b$ as well (see also[35, Theorem 2]). □

As in[20] we shall restrict ourselves to Theorem 6 while discussing about the invertibility of single layer potential V_b in 2D. Similar results can be obtained using Theorem 4 as well. The proof to the following result is due to[35, Lemma 1] and[14, Lemma 2].

Lemma 2. (i) Let $r_0 > \mathrm{diam}(\Omega)$. If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_\Omega V_b\Psi^* = 0$ in Ω, then $\Psi^* = 0$.

(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $r_\Omega W_b\Phi^* = 0$ in Ω, then $\Phi^* = 0$.

6 | TWO-OPERATOR BDIE SYSTEMS FOR DIRICHLET PROBLEM

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. We shall derive and investigate the two-operator BDIE systems for the following Dirichlet problem: *for* $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $f \in H^{-1}(\Omega)$, *find a function* $u \in H^1(\Omega)$ *satisfying*

$$Au = f \qquad \text{in} \quad \Omega, \tag{70}$$

$$\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial\Omega. \tag{71}$$

Here equation (70) is understood in the distributional sense (4) and the Dirichlet boundary condition (71) is understood in the trace sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma:

Theorem 7. The Dirichlet problem (70)-(71) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (\mathcal{A}^D)^{-1}(f, \varphi_0)^T$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{\frac{1}{2}}(\partial\Omega) \times H^{-1}(\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \times H^{-1}(\Omega)$, of the Dirichlet problem (70)-(71), is continuous.

6.1 | BDIE system formulation to the Dirichlet problem

Following[8], for $u \in H^1(\Omega)$, we shall reduce the Dirichlet problem (70)-(71) with $f \in H^{-1}(\Omega)$ in to two different *segregated two-operator* BDIE systems.

Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ be an extension of $f \in H^{-1}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists, see,[8, Lemma 2.15 and Theorem 2.16]. We represent in (49), (67) and (68) the generalized conormal derivative and the trace of the function u as

$$T^+(\tilde{f}, u) = \psi, \quad \gamma^+ u = \varphi_0$$

respectively, and will regard the new unknown function $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ as formally segregated of u. Thus we will look for the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

BDIE system (D1). To reduce BVP (70)-(71) to one of BDIE systems we will use equation (49) in Ω and equation (67) on $\partial\Omega$. Then we arrive at the system of BDIEs (D1),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi = \mathcal{F}_1^{D1} \qquad \text{in} \quad \Omega, \tag{72}$$

$$\gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b \psi = \mathcal{F}_2^{D1} \qquad \text{on} \quad \partial\Omega, \tag{73}$$

where

$$\mathcal{F}^{D1} := \begin{bmatrix} \mathcal{F}_1^{D1} \\ \mathcal{F}_2^{D1} \end{bmatrix} = \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \varphi_0 \end{bmatrix} \quad \text{and} \quad F_0 := \mathcal{P}_b \tilde{f} - W_b \varphi_0. \tag{74}$$

For $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusions $\mathcal{F}^{D1} = F_0 \in H^1(\Omega)$ if $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and due to the mapping properties of operators involved in (74), we have the inclusion $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Remark 3. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Then $\mathcal{F}^{D1} = 0$ if and only if $(\tilde{f}, \varphi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{D1} = 0$, that is, $F_0 = \mathcal{P}_b \tilde{f} - W_b \varphi_0 = 0$ in Ω and $\gamma^+ F_0 - \varphi_0 = 0$ on $\partial\Omega$. Multiplying the first relation by b, we get $\mathcal{P}_\Delta \tilde{f} - W_\Delta(b\varphi_0) = 0$ in Ω. Taking into account that $bW_b(\varphi_0) = W_\Delta(b\varphi_0)$ is harmonic and applying Laplace operator gives $\tilde{f} = 0$ in $\mathbb{R}^2$, and hence $W_b \varphi_0 = 0$ in Ω. Then by Lemma 2(ii), $\varphi_0 = 0$ on $\partial\Omega$. $\square$

BDIE system (D2). To obtain a segregated BDIE system of *the second kind*, we will use equation (49) in Ω and equation (68) on $\partial\Omega$. Then we arrive at the system, (D2), of BDIEs,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi = \mathcal{P}_b \tilde{f} - W_b \varphi_0 \quad \text{in} \quad \Omega, \tag{75}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} \psi = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \mathcal{L}_{ab}^+ \varphi_0 \quad \text{on} \quad \partial\Omega, \tag{76}$$

where

$$\mathcal{F}^{D2} := \begin{bmatrix} \mathcal{F}_1^{D2} \\ \mathcal{F}_2^{D2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} - W_b \varphi_0 \\ T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \mathcal{L}_{ab}^+ \varphi_0 \end{bmatrix}. \tag{77}$$

Due to the mapping properties of operators involved in (77), we have the inclusion $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. In similar way as in Remark 3, we can prove the following statement.

Remark 4. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Then $\mathcal{F}^{D2} = 0$ if and only if $(\tilde{f}, \varphi_0) = 0$.

6.2 | BDIE systems equivalence to the Dirichlet problem

Theorem 8. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in H^{-1}(\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ is such that $r_\Omega \tilde{f} = f$. Then

(i) If $u \in H^1(\Omega)$ solves the BVP (70)-(71), then the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Omega)$, where

$$\psi = T_a^+(\tilde{f}, u), \qquad \text{on } \partial\Omega, \tag{78}$$

solves the BDIE systems (D1) and (D2).

(ii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1) and $r_0 > \operatorname{diam}(\Omega)$, then this solution is unique and solves BDIEs (D2), while u solves the Dirichlet problem (70)-(71), and ψ satisfies (78).

(iii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), then this solution is unique and solves BDIEs (D1), while u solves the Dirichlet problem (70)-(71), and ψ satisfies (78).

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to BVP (70)–(71). Due to Theorem 7 it is unique. Setting ψ by (78) evidently implies, $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. From Theorem 3 and relations (67)–(68) follows that the couple (u, ψ) satisfies the BDIE systems (D1) and (D2), with the right-hand sides (74) and (77) respectively, which completes the proof of item (i).

Let now the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D1) or (D2). Due to Theorem 3, the hypothesis of Lemma 1 are satisfied implying that u solves PDE (70) in Ω, while relations in (56) and (57) also hold.

(ii) Let the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D1). Taking trace of (72) on $\partial\Omega$ and subtracting (73) from it we obtain

$$\gamma^+ u = \varphi_0 \qquad \text{on } \partial\Omega, \tag{79}$$

i.e. u satisfies the Dirichlet condition (71). (72) and Lemma 1 with $\Psi = \psi$, $\Phi = \varphi_0$ imply that $V_b\Psi^* - W_b\Phi^* = 0$, in Ω, where $\Psi^* = \psi - T_a^+(\tilde{f}, u)$ and $\Phi^* = \varphi_0 - \gamma^+ u$. Due to (79), $\Phi^* = 0$. Then Lemma 2(i) implies $\Psi^* = 0$, which proves condition (78). Thus u obtained from the solution of BDIE system (D1) solves the Dirichlet problem and hence, by item (i) of the theorem, (u, ψ) solve also BDIE system (D2). Let now the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D2). Taking generalized conormal derivative of (75) and subtracting (76) from it, we get $\psi = T_a^+(\tilde{f}, u)$ on $\partial\Omega$. Then substituting this in (57) gives $W_b(\varphi_0 - \gamma^+ u) = 0$ in Ω and Lemma 2(ii) then implies $\varphi_0 = \gamma^+ u$ on $\partial\Omega$.

Due to (74), the BDIE system (72)-(73) with zero right-hand side can be considered as obtained for $\tilde{f} = 0$, $\varphi_0 = 0$, where $\tilde{f} \in \widetilde{H}(\Omega)$ is an extension of $f \in H^{-1}(\Omega)$, i.e., $f = r_\Omega \tilde{f}$, implying that its solution is given by a solution of the homogeneous problem (70)-(71), which is zero by Theorem 7. This implies uniqueness of the solution of the the inhomogeneous BDIE system (72)-(73). Similar arguments work for the BDIE system (75)-(76). □

6.3 | BDIE system operators invertibility for the Dirichlet problem

The BDIE systems (D1) and (D2) can be written as

$$\mathfrak{D}^1 \mathcal{U}^D = \mathcal{F}^{D1} \quad \text{and} \quad \mathfrak{D}^2 \mathcal{U}^D = \mathcal{F}^{D2},$$

respectively. Here $\mathcal{U}^D := (u,\psi)^T \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$,

$$\mathfrak{D}^1 := \begin{bmatrix} I+\mathcal{Z}_b+\mathcal{R}_b & -V_b \\ \gamma^+\mathcal{Z}_b+\gamma^+\mathcal{R}_b & -\mathcal{V}_b \end{bmatrix}, \tag{80}$$

$$\mathfrak{D}^2 := \begin{bmatrix} I+\mathcal{Z}_b+\mathcal{R}_b & -V_b \\ T_a^+\mathcal{Z}_b+T_a^+\mathcal{R}_b & \left(1-\frac{a}{2b}\right)I-\mathcal{W}'_{ab} \end{bmatrix}, \tag{81}$$

while $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ are given by (74) and (77) respectively. Due to the mapping properties of the operators participating in the definitions of the operators $\mathfrak{D}^1$ and $\mathfrak{D}^2$ as well as the right-hand sides $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ (see, e.g.,[1,6] and the Appendix in[8]), we have $\mathcal{F}^{D1}\in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$, $\mathcal{F}^{D2}\in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$, while the operators

$$\mathfrak{D}^1 : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{82}$$

$$\mathfrak{D}^2 : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \tag{83}$$

are continuous. Due to Theorem 8(ii)-(iii), operators (82) and (83) are injective.

Lemma 3. For any couple $(\mathcal{F}_1,\mathcal{F}_2)\in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde f_{**},\Phi_*)\in \widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\mathcal{F}_1 = \mathcal{P}_b\tilde f_{**} - W_b\Phi_* \tag{84}$$

$$\mathcal{F}_2 = T_a^+(\tilde f_{**}+\mathring{E}\mathcal{R}_*^b\tilde f_{**},\mathcal{P}_b\tilde f_{**}) - \mathcal{L}_{ab}^+\Phi_* \tag{85}$$

Moreover, $(\tilde f_{**},\Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1,\mathcal{F}_2)$ with $\mathcal{C}_{**} : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\to \widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ a linear continuous operator given by

$$\tilde f_{**} = \check{\Delta}(b\mathcal{F}_1)+\gamma^*(\mathcal{F}_2+(\gamma^+\mathcal{F}_1)\partial_n b) \tag{86}$$

$$\Phi_* = \frac{1}{b}\Big(-\frac{1}{2}I+\mathcal{W}_\Delta\Big)^{-1}\gamma^+\Big\{-b\mathcal{F}_1+\mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1)+\gamma^*\Big(\frac{b}{a}\mathcal{F}_2+(\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big]\Big\} \tag{87}$$

where $\check{\Delta}(b\mathcal{F}_1) = \nabla\cdot\check{E}\nabla(b\mathcal{F}_1)$.

Proof. Let us first assume that there exist $(\tilde f_{**},\Phi_*)\in\widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (84)-(85) and find their expression in terms of $\mathcal{F}_1$ and $\mathcal{F}_2$. Let us rewrite (84) as

$$\mathcal{F}_1-\mathcal{P}_b\tilde f_{**} = -W_b\Phi_* \quad \text{in } \Omega. \tag{88}$$

Multiplying (88) by b and applying Laplacian to it, we obtain,

$$\Delta(b\mathcal{F}_1-\mathcal{P}_\Delta\tilde f_{**}) = \Delta(b\mathcal{F}_1)-\tilde f_{**} = -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \tag{89}$$

which means

$$\Delta(b\mathcal{F}_1) = r_\Omega\tilde f_{**} \quad \text{in } \Omega, \tag{90}$$

and $b\mathcal{F}_1-\mathcal{P}_\Delta\tilde f_{**}\in H^{1,0}(\Omega,\Delta)$ and hence $\mathcal{F}_1-\mathcal{P}_b\tilde f_{**}\in H^{1,0}(\Omega,B) = H^{1,0}(\Omega,A)$. The latter imply that the canonical co-normal derivatives $T_b^+(\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})$ and $T_a^+(\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})$ are well defined and can be also written in terms of their generalized co-normal derivatives

$$\begin{aligned}
\frac{b}{a}T_a^+(\mathcal{F}_1-\mathcal{P}_b\tilde f_{**}) &= T_b^+(\mathcal{F}_1-\mathcal{P}_b\tilde f_{**}) = T_b^+(\tilde B(\mathcal{F}_1-\mathcal{P}_b\tilde f_{**}),\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})\\
&= T_b^+(\mathring{E}\nabla\cdot(b\nabla(\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})),\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})\\
&= T_b^+(\mathring{E}\Delta(b\mathcal{F}_1-\mathcal{P}_\Delta\tilde f_{**})-\mathring{E}\nabla\cdot((\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})\nabla b),\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})\\
&= T_b^+(-\mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)-\mathring{E}\mathcal{R}_*^b\tilde f_{**},\mathcal{F}_1-\mathcal{P}_b\tilde f_{**})
\end{aligned}$$

and therefore,

$$T_a^+(\mathcal{F}_1-\mathcal{P}_b\tilde f_{**}) = T_a^+(-\mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)-\mathring{E}\mathcal{R}_*^b\tilde f_{**},\ \mathcal{F}_1-\mathcal{P}_b\tilde f_{**}) \tag{91}$$

where (63) and (90) were taken into account. Applying the co-normal derivative operator T_a^+ to both sides of equation (88), substituting their (91), taking into account (11), we obtain,

$$T_a^+(\tilde f_{**}-\mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b),\ \mathcal{F}_1)-T_a^+(\tilde f_{**}+\mathring{E}\mathcal{R}_*^b\tilde f_{**},\mathcal{P}_b\tilde f_{**}) = -\mathcal{L}_{ab}^+\Phi_*,\ \text{on } \partial\Omega. \tag{92}$$

Subtracting this from (85), we get,

$$\mathcal{F}_2 = T_a^+\big(\tilde{f}_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big) \quad \text{on } \partial\Omega. \tag{93}$$

Due to (90), we can represent

$$\tilde{f}_{**} = \check{\Delta}(b\mathcal{F}_1) + \tilde{f}_{1*} = \nabla\cdot\mathring{E}\nabla(b\mathcal{F}_1) - \gamma^*\Psi_{**} \tag{94}$$

where $\tilde{f}_{1*} \in H^{-1}_{\partial\Omega}$ is defined by (2) and hence, due to e.g.[10, Theorem 2.10] can be in turn represented as $\tilde{f}_{1*} = -\gamma^*\Psi_{**}$, with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$. Then (90) is satisfied and

$$\begin{aligned}
&\frac{b}{a}T_a^+\big(\tilde{f}_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big) = T_b^+\big(\tilde{f}_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big)\\
&= (\gamma^{-1})^*[\tilde{f}_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b) - \check{B}\mathcal{F}_1] = (\gamma^{-1})^*[\tilde{f}_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b) - \nabla\cdot\mathring{E}(b\nabla\mathcal{F}_1)]\\
&= (\gamma^{-1})^*[\nabla\cdot\mathring{E}\nabla(b\mathcal{F}_1) - \nabla\cdot\mathring{E}(b\nabla\mathcal{F}_1) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)]\\
&(\gamma^{-1})^*[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)] = -\Psi_{**} - (\gamma^+\mathcal{F}_1)\partial_n b
\end{aligned}$$

for which

$$T_a^+(\tilde{f}_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1) = \frac{a}{b}\left[-\Psi_{**} - (\gamma^+\mathcal{F}_1)\partial_n b\right] \tag{95}$$

because

$$\begin{aligned}
&\langle(\gamma^{-1})^*[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], w\rangle_{\partial\Omega}\\
&= \langle[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], \gamma^{-1}w\rangle_{\Omega}\\
&= \langle[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b), \gamma^{-1}w\rangle_{\mathbb{R}^2} - \gamma^*\Psi_{**} - \langle\mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], \gamma^{-1}w\rangle_{\Omega}\\
&= -\langle[\mathring{E}(\mathcal{F}_1\nabla b), \nabla(\gamma^{-1}w)\rangle_{\mathbb{R}^2} - \gamma^*\Psi_{**} + \langle(\mathcal{F}_1\nabla b), \nabla(\gamma^{-1}w)\rangle_{\Omega}\\
&- \langle n\cdot\gamma^+(\mathcal{F}_1\nabla b), \gamma^+\gamma^- w\rangle_{\Omega} = -\langle(\gamma^+(\mathcal{F}_1)\nabla b), w\rangle_{\partial\Omega} - \Psi_{**}.
\end{aligned} \tag{96}$$

Hence (93) reduces to

$$\Psi_{**} = -\frac{b}{a}\mathcal{F}_2 - (\gamma^+\mathcal{F}_1)\partial_n b = -T_b^+\mathcal{F}_1 - (\gamma^+\mathcal{F}_1)\partial_n b, \tag{97}$$

and (94) to (86).

Now (88) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{F}_\Delta \quad \text{in } \Omega, \tag{98}$$

where

$$\mathcal{F}_\Delta := -b\mathcal{F}_1 + \mathcal{P}_\Delta\tilde{f}_{**} = -b\mathcal{F}_1 + \mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big] \tag{99}$$

is harmonic function in Ω due to (89). The trace of Eq. (99) gives

$$-\frac{1}{2}b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+\mathcal{F}_\Delta \quad \text{on } \partial\Omega. \tag{100}$$

It is well known that the operator $\left[-\frac{1}{2}I + \mathcal{W}_\Delta\right]$ is an isomorphism (see,e.g.,[34, Lemmas 6.10 and 6.11]), this implies

$$\begin{aligned}
\Phi_* &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\mathcal{F}_\Delta\\
&= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\Big\{-b\mathcal{F}_1 + \mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big]\Big\},
\end{aligned}$$

which is Eq. (87). Relations (86), (87) can be written as $(\tilde{f}_{**}, \Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{C}_{**} : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions $\tilde{f}_{**}$ and Φ_*, given by (86) and (87), satisfy equations (84) and (85). Indeed, Φ_* given by (87) satisfies equation (100) and thus $\gamma^+W_\Delta(a\Phi_*) = \gamma^+\mathcal{F}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and $\mathcal{F}_\Delta$ are harmonic functions, this implies (98)-(99) and by (86) also (84). Finally, (86) implies by (95) that (93) is satisfied, and adding (92) to it leads to (85). Let us prove that the operator C_{**} is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (84)-(85) with $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. Then (90) implies that $r_\Omega\tilde{f}_{**} = 0$ in Ω, that is $\tilde{f}_{**} \in H^{-1}_{\partial\Omega} \subset \widetilde{H}^{-1}(\Omega)$. Hence (93) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \qquad \text{on } \partial\Omega. \tag{101}$$

By the first Green identity (10), this gives,

$$0 = \langle T_a^+(\tilde{f}_{**}, 0), \gamma^+ v\rangle_{\partial\Omega} = \langle \tilde{f}_{**}, v\rangle_\Omega, \quad \forall v \in H^1(\Omega), \tag{102}$$

which implies $\tilde{f}_{**} = 0$ in $\mathbb{R}^2$. Finally, (87) gives $\Phi_* = 0$. Hence any solution of non-homogeneous linear system (84) – (85) has only one solution, which implies the uniqueness of the operator $\mathcal{C}_{**}$. □

The following assertion is[5, Lemma 19] generalized to a wider space in 2D.

Lemma 4. For any couple $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\tilde{\mathcal{F}}_1 = \mathcal{P}_b \tilde{f}_{**} - W_b \Phi_* \tag{103}$$

$$\tilde{\mathcal{F}}_2 = \gamma^+(\mathcal{P}_b \tilde{f}_{**} - W_b \Phi_*) \tag{104}$$

Moreover, $(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ with $\tilde{\mathcal{C}}_{**} : H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ a linear continuous operator is given by

$$\tilde{f}_{**} = \check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2)\partial_n b) \tag{105}$$

$$\Phi_* = \frac{1}{b}\Big(-\frac{1}{2} I + \mathcal{W}_\Delta\Big)^{-1} \Big(-b\tilde{\mathcal{F}}_2 + \gamma^+ \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2)\partial_n b)]\Big) \tag{106}$$

where $\check{\Delta}(b\tilde{\mathcal{F}}_1) = \nabla \cdot \check{E}\nabla(b\tilde{\mathcal{F}}_1)$.

Proof. Let us first assume that there exist $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (103)-(104) and find their expression in terms of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$. Let us re write (103) as

$$\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**} = -W_b \Phi_* \quad \text{in } \Omega. \tag{107}$$

Multiplying (107) by b and applying Laplacian to it, we obtain,

$$\Delta(b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) = \Delta(b\tilde{\mathcal{F}}_1) - \tilde{f}_{**} = -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \tag{108}$$

which means

$$\Delta(b\tilde{\mathcal{F}}_1) = r_\Omega \tilde{f}_{**} \quad \text{in } \Omega, \tag{109}$$

and $b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta \tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)$, while $\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)$. The latter imply that the canonical co-normal derivatives $T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$ and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$ are well defined and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**}) = \frac{b}{a} T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$.

Due to (109) and using $\tilde{f}_{1*} = -\gamma^* \Psi_{**}$ with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$ as in (97), we can represent

$$\tilde{f}_{**} = \check{\Delta}(b\tilde{\mathcal{F}}_1) + \tilde{f}_{1*} = \nabla \cdot \mathring{E}\nabla(b\tilde{\mathcal{F}}_1) - \gamma^* \Psi_{**} \tag{110}$$

where $\tilde{f}_{1*} \in H^{-1}_{\partial\Omega}$. Then (109) is satisfied. Replacing $\mathcal{F}_2$ by $T_a^+(\tilde{\mathcal{F}}_1, u)$ in Lemma 3, Eq. (97) yields,

$$\Psi_{**} = -\frac{b}{a} T_a^+ \tilde{\mathcal{F}}_1 - (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b = -T_b^+ \tilde{\mathcal{F}}_1 - \tilde{\mathcal{F}}_2 \partial_n b \tag{111}$$

and (110) reduces to (105). Now (107) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{Q}_\Delta \quad \text{in } \Omega, \tag{112}$$

where

$$\mathcal{Q}_\Delta := -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta \tilde{f}_{**} = -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b)] \tag{113}$$

is harmonic function in Ω due to (108). The trace of equation (113) gives

$$-\frac{1}{2} b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+ \mathcal{Q}_\Delta \quad \text{on } \partial\Omega. \tag{114}$$

By similar argument as in Lemma 3, the operator $-\frac{1}{2} I + \mathcal{W}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is an isomorphism this implies

$$\begin{aligned}
\Phi_* &= \frac{1}{b}\Big(-\frac{1}{2} I + \mathcal{W}_\Delta\Big)^{-1} \gamma^+ \mathcal{Q}_\Delta \\
&= \frac{1}{b}\Big(-\frac{1}{2} I + \mathcal{W}_\Delta\Big)^{-1} \gamma^+ \{-b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b)]\} \\
&= \frac{1}{b}\Big(-\frac{1}{2} I + \mathcal{W}_\Delta\Big)^{-1} \big(-b\tilde{\mathcal{F}}_2 + \gamma^+ \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b)]\big)
\end{aligned}$$

which is Eq. (106). Relations (105), (106) can be written as $(\tilde{f}_{**}, \Phi_*) = \tilde{C}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$, where $\tilde{C}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions $\tilde{f}_{**}$ and Φ_*, given by (105) and (106), satisfy equations (103) and (104). Indeed, Φ_* given by (106) satisfies equation (114) and thus $\gamma^+ W_\Delta(a\Phi_*) = \gamma^+ \mathcal{Q}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and $\mathcal{Q}_\Delta$ are harmonic functions, this implies (112)-(113) and by (105) also (103) while (104) follows from Eqs.(105) and (112).

Let us prove that the operator $\tilde{C}_{**}$ is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (103)-(104) with $\tilde{\mathcal{F}}_1 = 0$ and $\tilde{\mathcal{F}}_2 = 0$. Then (109) implies that $r_\Omega \tilde{f}_{**} = 0$ in Ω, that is $\tilde{f}_{**} \in H^{-1}_{\partial\Omega} \subset \widetilde{H}^{-1}(\Omega)$. Hence (93) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \qquad \text{on } \partial\Omega. \tag{115}$$

By the first Green identity (10), this gives relation (102), which implies $\tilde{f}_{**} = 0$ in $\mathbb{R}^2$. Finally, (106) gives $\Phi_* = 0$. Hence any solution of nonhomogeneous linear system $(103) - (104)$ has only one solution, which implies the uniqueness of the operator $\tilde{C}_{**}$. □

Theorem 9. Let $r_0 > \operatorname{diam}(\Omega)$. The operators (82) and (83) are continuous and continuously invertible.

Proof. The continuity of operators (82) and (83) is proved above. To prove the invertibility of operator (82), let us consider the BDIE system (D1) with arbitrary right-hand side

$$\mathcal{F}_*^{D1} = (\mathcal{F}_{*1}^{D1}, \mathcal{F}_{*2}^{D1})^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Take $\tilde{\mathcal{F}}_1 = \mathcal{F}_{*1}^{D1}$ and $\Phi_* = \gamma^+ \mathcal{F}_{*1}^{D1} - \mathcal{F}_{*2}^{D1}$ in Lemma 4, to obtain the representation of $\mathcal{F}_*^{D1}$ as:

$$\mathcal{F}_{*1}^{D1} = \tilde{\mathcal{F}}_1 \qquad \mathcal{F}_{*2}^{D1} = \gamma^+ \tilde{\mathcal{F}}_1 - \Phi_*$$

where the couple

$$(\tilde{f}_*, \Phi_*) = \tilde{C}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{116}$$

is unique and the operator

$$\tilde{C}_{**} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{117}$$

is linear and continuous. If $r_0 > \operatorname{diam}(\Omega)$, then taking into account[8, Remark 5.3] and applying Theorem 7 with $f = r_\Omega \tilde{f} = r_\Omega \tilde{f}_*$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D1) is uniquely solvable and its solution is: $\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega \tilde{f}, \varphi_0)^T$, $\quad \mathcal{U}_2 = \gamma^+ \mathcal{U}_1 - \varphi_0$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (70)–(71), is continuous. Representation (116) and continuity of the operator (117) imply invertibility of (82). To prove the invertibility of operator (83), let us consider the BDIE system (D2) with arbitrary right-hand side

$$\mathcal{F}_*^{D2} = (\mathcal{F}_{*1}^{D2}, \mathcal{F}_{*2}^{D2})^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega).$$

Take $\mathcal{F}_1 = \mathcal{F}_{*1}^{D2}$ and $\mathcal{F}_2 = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_{*2}^{D2}$ in Lemma 3 to represent $\mathcal{F}_*^{D2}$ as

$$\mathcal{F}_{*1}^{D2} = \mathcal{F}_1 \qquad \mathcal{F}_{*2}^{D2} = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_2$$

and the couple

$$(\tilde{f}_{**}, \Phi_*) = \tilde{C}_{**}(\mathcal{F}_1, \mathcal{F}_2) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is unique and the operator

$$\tilde{C}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{118}$$

is linear and continuous. Taking into account[8, Remark 5.3] and applying Theorem 7 with $\tilde{f} = \tilde{f}_{**}$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D2) is uniquely solvable and its solution is: $\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega \tilde{f}, \varphi_0)^\top$, $\quad \mathcal{U}_2 = T_a^+(r_\Omega \tilde{f}, \mathcal{U}_1)$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (70)–(71), is continuous. Representation (116) and continuity of the operator (118) imply invertibility of (83). □

7 | TWO-OPERATOR BDIE SYSTEMS FOR NEUMANN PROBLEM

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. We shall derive and investigate the two-operator BDIE systems for the following Neumann problem: *for* $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ *and* $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, *find a function* $u \in H^1(\Omega)$ *satisfying*

$$Au = r_\Omega \tilde{f} \qquad \text{in} \quad \Omega, \tag{119}$$

$$T_a^+(\tilde{f}, u) = \psi_0 \qquad \text{on} \quad \partial\Omega. \tag{120}$$

Here Eq. (119) is understood in the distributional sense (4) and the Neumann boundary condition (120) in the weak sense (10). The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

Theorem 10. (i) The homogeneous Neumann problem (119)-(120) admits only linearly independent solution $u^0 = 1$ in $H^1(\Omega)$.

(ii) The nonhomogeneous Neumann problem (119)-(120) is solvable if and only if the following solvability condition is satisfied.

$$\langle \tilde{f}, u^0 \rangle_\Omega - \langle \psi_0, \gamma^+ u^0 \rangle_{\partial\Omega} = 0 \tag{121}$$

7.1 | BDIE system formulation for the Neumann problem

We explore different possibilities of reducing the Neumann problem (119)–(120) with $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to two different *segregated* boundary-domain integral equations (BDIE) systems. Corresponding formulations for the Neumann problem for $u \in H^{1,0}(\Omega, \Delta)$ with $f \in L_2(\Omega)$ in 2D were introduced and analysed in [19]. Let us represent in (49), (67) and (68) the generalised co-normal derivative and the trace of the function u as

$$T_a^+(\tilde{f}, u) = \psi_0, \quad \gamma^+ u = \varphi,$$

and will regard the new unknown function $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ as formally segregated of u. Thus we will look for the couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

BDIE system (N1). To reduce BVP (119)-(120) to a BDIE system in this section we will use equation (49) in Ω and equation (68) on $\partial\Omega$. Then we arrive at the following system, (N1), of two boundary-domain integral equations for the couple of unknowns,(u, φ),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1^{N1} \qquad \text{in} \quad \Omega, \tag{122}$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ \varphi = \mathcal{F}_2^{N1} \qquad \text{on} \quad \partial\Omega, \tag{123}$$

where

$$\mathcal{F}^{N1} := \begin{bmatrix} \mathcal{F}_1^{N1} \\ \mathcal{F}_2^{N1} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} + V_b \psi_0 \\ T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \psi_0 + \frac{a}{2b}\psi_0 + \mathcal{W}'_{ab}\psi_0 \end{bmatrix}. \tag{124}$$

Due to the mapping properties of operators involved in (124) we have $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}_0^N := \mathcal{P}_b \tilde{f} + V_b \psi_0 \in H^1(\Omega)$.

Remark 5. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \operatorname{diam}(\Omega)$. Then $\mathcal{F}^{N1} = 0$ if and only if $(\tilde{f}, \psi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{N1} = 0$, that is, $\mathcal{P}_b \tilde{f} + V_b \psi_0 = 0$ in Ω and $T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \psi_0 + \frac{a}{2b}\psi_0 + \mathcal{W}'_{ab}\psi_0 = 0$ on $\partial\Omega$. Multiplying the first relation by b gives $\mathcal{P}_\Delta \tilde{f} + V_\Delta \psi_0 = 0$ in Ω. Further, taking into account that $bV_b(\psi_0) = V_\Delta(\psi_0)$ is harmonic and applying Laplace operator we get $\tilde{f} = 0$ in $\mathbb{R}^2$ and hence $V_b \psi_0 = 0$ in Ω. Then due to Lemma 2(i), we get $\psi_0 = 0$ on $\partial\Omega$. □

BDIE system (N2). To obtain a segregated BDIE system of *the second kind*, we will use equation (49) in Ω and equation (67) on $\partial\Omega$. Then we arrive at the following system, (D2), of boundary-domain integral equation systems,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{P}_b \tilde{f} + V_b \psi_0 \qquad \text{in} \quad \Omega, \tag{125}$$

$$\frac{1}{2}\varphi + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u + \mathcal{W}_b \varphi = \gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0 \quad \text{on} \quad \partial\Omega, \tag{126}$$

where

$$\mathcal{F}^{N2} := \begin{bmatrix} \mathcal{F}_1^{N2} \\ \mathcal{F}_2^{N2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} + V_b \psi_0 \\ \gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0 \end{bmatrix} \tag{127}$$

Due to the mapping properties of operators involved in (127), we have the inclusion $\mathcal{F}_1^{N2} = \mathcal{P}_b \tilde{f} + V_b \psi_0 \in H^1(\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Remark 6. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \operatorname{diam}(\Omega)$. Then $\mathcal{F}^{N2} = 0$ if and only if $(\tilde{f}, \psi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{N2} = 0$, that is, $\mathcal{P}_b \tilde{f} + V_b \psi_0 = 0$ in Ω and $\gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0$ on $\partial\Omega$. Multiplying the first relation by b gives $\mathcal{P}_\Delta \tilde{f} + V_\Delta \psi_0 = 0$ in Ω. Further, taking into account that $bV_b(\psi_0) = V_\Delta(\psi_0)$ is harmonic and applying Laplace operator we get $\tilde{f} = 0$ in $\mathbb{R}^2$ and hence $V_b \psi_0 = 0$ in Ω. Then due to Lemma 2(i), we get $\psi_0 = 0$ on $\partial\Omega$. □

7.2 | BDIE systems equivalence to the Neumann problem

Theorem 11. Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ satisfying condition (121).

(i) If a function $u \in H^1(\Omega)$ solves the BVP (119)-(120) , then the couple (u, φ), where

$$\varphi = \gamma^+ u \tag{128}$$

solves the BDIE systems (N1) and (N2).

(ii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (N1), then the u solves BDIE system (N2) and u solves the Neumann problem (119)-(120) and φ satisfies (128).

(iii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (N2) and $r_0 > \operatorname{diam}(\Omega)$, then the u solves BDIE system (N2) and u solves the Neumann problem (119)-(120) and φ satisfies (128).

(iv) The homogeneous BDIE systems (N1) and (N2) have unique linearly independent solution $\mathcal{U}_0 = (u^0, \varphi^0)^T$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Condition (121) is necessary and sufficient for solvability of the non-homogeneous BDIE systems (N1) and (N2) in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to the Neumann BVP (119)–(120). It immediately follows from Theorem 49 and relations (67)–(68) that the couple (u, φ) with $\varphi = \gamma^+ u$ satisfies the BDIE systems (N1) and (N2), which proves item (i).

(ii) Let now a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N1) or (N2). Due to the first equations in the BDIE systems, the hypotheses of Lemma 1 are satisfied implying that u is a solution of equation (119) in Ω, and and equations (56)-(59) hold for $\Psi = \psi_0$ and $\Phi = \varphi$.

If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve the system (N1) then subtracting (59) from (123) gives $T_a^+(\tilde{f}, u) = \psi_0$ on $\partial\Omega$. Thus Neumann (120) is satisfied. Further, from (56) we derive $W_b(\gamma^+ u - \varphi) = 0$ in Ω, where $\gamma^+ u = \varphi$ on $\partial\Omega$ by Lemma 2 completing item (ii).

(iii) Let now couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N2). Further, taking the trace of (125) on $\partial\Omega$ and comparing the results with (126), we easily derive that $\gamma^+ u = \varphi$ on $\partial\Omega$. Lemma 1 for equation (125) implies that u is a solution of equation (119), while equations (56)-(59) hold for $\Psi = \psi_0$ and $\Phi = \varphi$. Further, from (56) we derive

$$V_b(\psi_0 - T_a(\tilde{f}, u)) = 0 \quad \text{in} \quad \Omega,$$

whence $T_a(\tilde{f}, u) = \psi_0$ on $\partial\Omega$ due to Lemma 2 (i) and u solves Neumann problem (119)-(120) which completes the proof of item (iii).

(iv) Theorem 10 along with items (i) and (ii) imply the claims of item (iii) for BDIE system (N2) and (N1). □

7.3 | Properties of BDIE system operators for the Neumann problem

BDIE systems (N1) and (N2) can be written respectively, as

$$\mathfrak{R}^1 \mathcal{U}^N = \mathcal{F}^{N1}, \qquad \mathfrak{R}^2 \mathcal{U}^N = \mathcal{F}^{N2}, \tag{129}$$

where $\mathcal{U}^N = (u, \varphi)^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D)$, while $\mathcal{F}^{N1}$ and $\mathcal{F}^{N2}$ are given by Eqs. (124) and (127) respectively. Due to the mapping properties of potentials in (124) and (127), $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

$$\mathfrak{R}^1 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ T_a^+ \mathcal{Z}_b + T_a^+ \mathcal{R}_b & \mathcal{L}_{ab}^+ \end{bmatrix}, \qquad \mathfrak{R}^2 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ \gamma^+ \mathcal{Z}_b + \gamma^+ \mathcal{R}_b & \frac{1}{2} I + \mathcal{W}_b \end{bmatrix}.$$

Due to the mapping properties of potentials in (124) and (127), the right hand sides of BDIE systems (N1)and (N2) are such that $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Theorem 12. The operators

$$\mathfrak{R}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{130}$$

$$\mathfrak{R}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{131}$$

are continuous. They have one-dimensional null spaces, $\ker \mathfrak{R}^1 = \ker \mathfrak{R}^2$, in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, spanned over the element $(u^0, \varphi^0) = (1, 1)$.

Proof. The mapping properties of the potentials imply continuity of the operators (130) and (131). The claims that $\ker \mathfrak{R}^1$ and $\ker \mathfrak{R}^2$ are one-dimensional and the couple $(u^0, \varphi^0) = (1, 1)$ belong to $\ker \mathfrak{R}^1 = \ker \mathfrak{R}^2$ directly follows from Theorem 11(iii). □

To describe in more details the range of operators (130) and (131), i.e., to give more information about the co-kernels of these operators, we will need several auxiliary assertions. First of all, let us remark that for any $v \in H^{s-\frac{3}{2}}(\partial\Omega)$, $s < \frac{3}{2}$, the single layer potential can be defined as follows:

$$V_b v(y) := -\langle \gamma P_b(., y), v \rangle_{\partial\Omega} = -\langle P_b(., y), \gamma^* v \rangle_{\mathbb{R}^3} = -\mathbf{P}_b \gamma^* v(y), \quad y \in \mathbb{R}^2 \setminus \partial\Omega. \tag{132}$$

where $\gamma^* : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-2}_{\partial\Omega}$, $s < \frac{3}{2}$, is the operator adjoined to the trace operator $\gamma : H^{2-s}(\mathbb{R}^3) \to H^{\frac{3}{2}-s}(\partial\Omega)$, and the space $H^s_{\partial\Omega}$ is defined by (2).

Lemma 5. Let $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, $s > \frac{1}{2}$ and $r_0 > \operatorname{diam}(\Omega)$. If

$$r_\Omega \mathbf{P}_b \tilde{f} = 0 \quad \text{in } \Omega, \tag{133}$$

then $\tilde{f} = 0$ in $\mathbb{R}^2$.

Proof. Multiplying (133) by b, taking into account the first relation in (35) and applying the Laplace operator, we obtain $r_\Omega \tilde{f} = 0$, which means $\tilde{f} \in H^{s-2}_{\partial\Omega}$. If $s \geq \frac{3}{2}$, then $\tilde{f} = 0$ by [10, Theorem 2.10]. If $\frac{1}{2} < s < \frac{3}{2}$, then by the same theorem there exists $v \in H^{s-\frac{3}{2}}(\partial\Omega)$ such that $\tilde{f} = \gamma^* v$. This gives $\mathbf{P}_b \tilde{f} = \mathbf{P}_b \gamma^* v = -V_b v$ in $\mathbb{R}^2$. Then (133) reduces to $V_b v = 0$ in Ω, which by Lemma 2(i) (for $s = 1$, which can be generalized to $\frac{1}{2} < s < \frac{3}{2}$) implies $v = 0$ on $\partial\Omega$ and thus $\tilde{f} = 0$ in $\mathbb{R}^2$. □

Theorem 13. Let $\frac{1}{2} < s < \frac{3}{2}$ and $r_0 > \operatorname{diam}(\Omega)$. The operator

$$\mathbf{P}_b : \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega) \tag{134}$$

and its inverse

$$(\mathbf{P}_b)^{-1} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega) \tag{135}$$

are continuous and

$$(\mathbf{P_b})^{-1} g = \left[\Delta \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+) - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+\right](bg) \qquad \text{in} \quad \mathbb{R}^2, \quad \forall g \in H^s(\Omega). \tag{136}$$

Proof. The continuity of equation (134) follows from [1, Theorem 3.8]. By Lemma 5 operator (134) is injective. Let us prove its surjectivity. To this end, for arbitrary $g \in H^s(\Omega)$ let us consider the following equation with respect to $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$,

$$\mathbf{P}_\Delta \tilde{f} = g \quad \text{in} \quad \Omega. \tag{137}$$

Let $g_1 \in H^s(\Omega)$ be the (unique) solution of the following Dirichlet problem:

$$\Delta g_1 = 0 \text{ in } \Omega, \quad \gamma^+ g_1 = \gamma^+ g,$$

which by[35, Theorem 2] the single layer potential $\mathcal{V}_\Delta^{-1}$ exists and due to[27] or[10, Lemma 2.6] can be particularly presented as $g_1 = V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+ g$. Let $g_0 := g - g_1$. Then $\in H^s(\Omega)$ and $\gamma^+ g_0 = 0$ and thus g_0 can be uniquely extended to $\mathring{E} g_0 \in \widetilde{H}^s(\Omega)$, where $\mathring{E}$ is the operator of extension by zero outside Ω . Thus by (132), equation (137) takes form

$$r_\Omega \mathbf{P}_\Delta[\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g] = g_0 \quad \text{in } \Omega. \tag{138}$$

Any solution $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ of the corresponding equation on $\mathbb{R}^2$

$$\mathbf{P}_\Delta[\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g] = \mathring{E} g_0 \quad \text{in } \mathbb{R}^2, \tag{139}$$

solves (138). If $\tilde{f}$ solves (139) then acting with the Laplace operator on (139), we obtain

$$\tilde{f} = \tilde{Q} g := \Delta \mathring{E} g_0 - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g = \Delta \mathring{E}(g - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+ g) - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g \quad \text{in} \quad \mathbb{R}^2. \tag{140}$$

On the other hand, substituting $\tilde{f}$ given by (140) to (139) and taking into account that $\mathbf{P}_\Delta \Delta \tilde{h} = \tilde{h}$ for any $\tilde{h} \in \widetilde{H}^s(\Omega)$, $s \in \mathbb{R}$, we obtain that $\tilde{Q} g$ is indeed a solution of equation (139) and thus (138). By Lemma 5 the solution of (139) is unique, which means that the operator $\tilde{Q}$ is inverse to operator (134), i.e., $\tilde{Q} = (r_\Omega \mathbf{P}_b)^{-1}$. Since Δ is a continuous operator from $\widetilde{H}^s(\Omega)$ to $\widetilde{H}^{s-2}(\Omega)$, equation (86) implies that operator $(r_\Omega \mathbf{P})_b^{-1} = \tilde{Q} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is continuous. The relations $\mathbf{P}_b = \frac{1}{b}\mathbf{P}_\Delta$ and $b(x) > c > 0$ then imply invertibility of operator (134) and anstatz (136). □

Theorem 14. The co-kernel of operator (130) is spanned over the functional

$$g^{*1} := ((\gamma^+)^* \partial_n b, 1)^\top \tag{141}$$

in $\widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, i.e., $g^{*1}(\mathcal{F}_1, \mathcal{F}_2) = \langle (\gamma^+ \mathcal{F}_1)\partial_n b + \mathcal{F}_2, \gamma^+ u^0 \rangle_{\partial\Omega}$, where $u^0 = 1$.

Proof. The proof follows from the proof of[8, Theorem 6.7] and Lemma 3. Indeed, let us consider the first equation in (129), i.e. the equation $\mathfrak{R}^1 \mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, representing the BDIE system (N1)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \tag{142}$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ \varphi = \mathcal{F}_2 \quad \text{on} \quad \partial\Omega, \tag{143}$$

with arbitrary right hand side $(\mathcal{F}_1, \mathcal{F}_2)^T \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. By Lemma 3 the right hand side of the system has the form (84)-(85), i.e., system (142)-(143) reduces to

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b(\varphi + \Phi_*) = \mathcal{P}_b \tilde{f}_{**} \qquad \text{in} \quad \Omega, \tag{144}$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+(\varphi + \Phi_*) = T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{P}_b \tilde{f}_{**}) \quad \text{on} \quad \partial\Omega, \tag{145}$$

where the couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is given by (84)-(85). Up to the notations (144)-(145) is the same as in (124) with $\psi_0 = 0$. Then Theorems 11(iii) and 13 imply that the BDIE system (144)-(145) and hence (142)-(143) is solvable if and only if

$$\begin{aligned}
\langle \tilde{f}_{**}, u^0 \rangle_\Omega &= \langle (\check{\Delta} b \mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1)\partial_n b), u^0 \rangle_\Omega \\
&= \langle (\nabla \cdot \check{E} \nabla (b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1)\partial_n b), u^0 \rangle_{\mathbb{R}^2} \\
&= \langle (\nabla \cdot \check{E} \nabla (b\mathcal{F}_1, \nabla u^0 \rangle_{\mathbb{R}^2} + \langle (\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1)\partial_n b), \gamma^+ u^0 \rangle_{\partial\Omega} \\
&= \langle (\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1)\partial_n b), \gamma^+ u^0 \rangle_{\partial\Omega} = 0
\end{aligned} \tag{146}$$

where we took into account that $\nabla u^0 = 0$ in $\mathbb{R}^2$. Thus the functional g^{*1} defined by (141) generates the necessary and sufficient solvability condition for the first equation in (129). Hence g^{*1} is basis of the co-kernel of $\mathfrak{R}^1$.

□

Theorem 15. Let $r_0 > \operatorname{diam}(\Omega)$. Then the co-kernel of operator (131) is spanned over

$$g^{*2} := \begin{pmatrix} -b\gamma^{+*}(\frac{1}{2} + \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+ u^0 \\ -b(\frac{1}{2} - \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+ u^0 \end{pmatrix} \tag{147}$$

in $\widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, i.e.,

$$g^{*2}(\mathcal{F}_1, \mathcal{F}_2) = \left\langle -b\gamma^{+*}\left(\frac{1}{2} + \mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+ u^0, \mathcal{F}_1 \right\rangle_\Omega + \left\langle -b\left(\frac{1}{2} - \mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+ u^0, \mathcal{F}_2 \right\rangle_{\partial\Omega},$$

where $u^0 = 1$.

Proof. The proof follows from the proof of[8, Theorem 6.8],[35, Theorem 2] and Lemma 3. Indeed, let us consider the first equation in (129), i.e. the equation $\mathfrak{R}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, representing the BDIE system (N1)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \tag{148}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u + \mathcal{W}_b\varphi = \mathcal{F}_2 \quad \text{on} \quad \partial\Omega, \tag{149}$$

with arbitrary $(\mathcal{F}_1, \mathcal{F}_2)^T \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u,\varphi) \in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$.

Introducing the new variable , $\varphi' = \varphi - (\mathcal{F}_2 - \gamma^+\mathcal{F}_1)$, BDIE system (148)-(149) takes the form

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi = \mathcal{F}_1' \quad \text{in} \quad \Omega, \tag{150}$$

$$\frac{1}{2}\varphi' + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u + \mathcal{W}_b\varphi' = \mathcal{F}_2' \quad \text{on} \quad \partial\Omega, \tag{151}$$

where

$$\mathcal{F}_1' = \mathcal{F}_1 - W_b(\mathcal{F}_2 - \gamma^+\mathcal{F}_1) \in H^1(\Omega).$$

Let us recall that $\mathcal{P}_b = r_\Omega \mathbf{P}_b : \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega)$ and then by Theorem 13, the operator $\mathcal{P}_b^{-1} = (\mathbf{P}_b)^{-1} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is continuous for $\frac{1}{2} < s < \frac{3}{2}$, while $\mathcal{V}_\Delta^{-1}$ exists due to[35, Theorem 2]. Hence we always represent $\mathcal{F}_1 = \mathcal{P}_b\tilde{f}_*$, with

$$\tilde{f}_* = [\Delta\mathring{E}(I - r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^*\mathcal{V}_\Delta^{-1}\gamma^+](b\mathcal{F}_1') \in \widetilde{H}^{-1}(\Omega).$$

For $\mathcal{F}_1' = \mathcal{P}_b\tilde{f}_*$, the right hand side of BDIE system (150)-(151) is the same as in (127) with $f = \tilde{f}_*$ and $\psi_0 = 0$. Then Theorems 11(iii) implies that the BDIE system (150)-(151) and hence (148)-(149) is solvable if and only if

$$\begin{aligned}
\langle \tilde{f}_*, u^0\rangle_\Omega &= \langle [\Delta\mathring{E}(I - r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+](b\mathcal{F}_1'), u^0\rangle_{\mathbb{R}^2}\\
&= \langle \mathring{E}(I - r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+)(b\mathcal{F}_1'), \Delta u^0\rangle_{\mathbb{R}^2} - \langle (\gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+)(b\mathcal{F}_1'), u^0\rangle_{\mathbb{R}^2}\\
&= -\langle \gamma^+(b\mathcal{F}_1'), \mathcal{V}_\Delta^{-1}\gamma^+u^0\rangle_{\partial\Omega}\\
&= -\left\langle \frac{1}{2}[\gamma^+(b\mathcal{F}_1) + (b\mathcal{F}_2)] - \mathcal{W}_\Delta[b(\mathcal{F}_2 - \gamma^+\mathcal{F}_1)], \mathcal{V}_\Delta^{-1}\gamma^+u^0\right\rangle_{\partial\Omega}\\
&= \left\langle -b\gamma^{+*}\left(\frac{1}{2} + \mathcal{W}_\Delta'\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_1\right\rangle_\Omega + \left\langle -b\left(\frac{1}{2} + \mathcal{W}_\Delta'\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_2\right\rangle_{\partial\Omega} = 0.
\end{aligned} \tag{152}$$

Thus the functional g^{*2} defined by (147) generates the necessary and sufficient solvability condition of the equation $\mathfrak{R}^2\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$. Hence g^{*2} is basis of the cokernel of $\mathfrak{R}^2$. □

7.4 | Perturbed segregated BDIE systems for Neumann problem

Theorem 11 implies, that even when the solvability condition (121) is satisfied, the solutions of both BDIE systems, (N1) and (N2), are not unique. By Theorem 12, in turn, the BDIE left hand side operators, $\mathfrak{R}^1$ and $\mathfrak{R}^2$, have non-zero kernels and thus are not invertible. To find a solution (u,φ) from uniquely solvable BDIE system with continuously invertible left hand side operators, let us consider, following[37], some BDIE systems obtained form (N1) and (N2) by finite-dimensional operator perturbations (cf. [14]for the three-dimensional case). Below we use the notations $\mathcal{U} = (u,\varphi)^\top$ and $|\partial\Omega| := \int_{\partial\Omega} dS$.

7.4.1 | Perturbation of BDIE system (N1)

Let us introduce the perturbed counterparts of the BDIE system (N1),

$$\hat{\mathfrak{R}}^1\mathcal{U}^N = \mathcal{F}^{N1}, \tag{153}$$

where

$$\hat{\mathfrak{R}}^1 := \hat{\mathfrak{R}}^1 + \mathring{\mathfrak{R}}^1 \text{ and } \mathring{\mathfrak{R}}^1\mathcal{U}^N(y) := g^0(\mathcal{U}^N)\mathcal{G}^1(y) = \frac{1}{|\partial\Omega|}\int\limits_{|\partial\Omega|}\varphi(x)dS\begin{pmatrix}0\\1\end{pmatrix},$$

that is,

$$g^0(\mathcal{U}^N) := \frac{1}{|\partial\Omega|}\int\limits_{\partial\Omega}\varphi(x)dS, \quad \mathcal{G}^1(y) := \begin{pmatrix}0\\1\end{pmatrix}.$$

For the functional g^{*1} given by (141) in Theorem 14, $g^{*1}(\mathcal{G}^1) = |\partial\Omega|$, while $g^0(\mathcal{U}^0) = 1$. Hence[8, Theorem D.1 in Appendix] implies the following assertion.

Theorem 16. (i) The operator $\hat{\mathfrak{R}}^1 : H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertable.

(ii) If condition $g^{*1}(\mathcal{F}^{N1}) = 0$ or condition (121) for $\mathcal{F}^{N1}$ in form (130) is satisfied, then the unique solution of perturbed BDIDE system (153) gives a solution of original BDIE system (N1) such that

$$g^0(\mathcal{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+ u dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi dS = 0.$$

7.4.2 | Perturbation of BDIE system (N2)

Let us introduce the perturbed counterparts of the BDIE system (N2)

$$\hat{\mathfrak{R}}^2\mathcal{U} = \mathcal{F}^{N2}, \tag{154}$$

where

$$\hat{\mathfrak{R}}^2 := \mathfrak{R}^2 + \mathring{\mathfrak{R}}^2 \quad \text{and} \quad \mathring{\mathfrak{R}}^2\mathcal{U}(y) := g^0(\mathcal{U})\mathcal{G}^2(y) = \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds\begin{pmatrix} b^{-1}(y)\\ \gamma^+ b^{-1}(y)\end{pmatrix},$$

that is,

$$g^0(\mathcal{U}) := \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds, \quad \mathcal{G}^2(y) := \begin{pmatrix} b^{-1}(y)u^0(y)\\ \gamma^+[b^{-1}u^0](y)\end{pmatrix}.$$

For the functional g^{*2} given by (147) in Theorem 15, since the operator $\mathcal{V}_\Delta^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned} g^{*2}(\mathcal{G}^2) &= \Big\langle -b\gamma^{+*}\Big(\frac{1}{2}+\mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, b^{-1}u^0\Big\rangle_\Omega \\ &\quad + \Big\langle -b\Big(\frac{1}{2}-\mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+\Big(b^{-1}u^0\Big)\Big\rangle_{\partial\Omega} \\ &= -\langle(\frac{1}{2}+\mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 + (\frac{1}{2}-\mathcal{W}')\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \\ &= -\langle\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \\ &\le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = -C|\partial\Omega|^2 < 0. \end{aligned} \tag{155}$$

Due to (155) and $g^0(\mathcal{U}^0) = 1$, Theorem[8, Theorem D.1] implies the following assertion.

Theorem 17. (i) The operator $\hat{\mathfrak{R}}^2 : H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertable.

(ii) If condition $g^{*2}(\mathcal{F}^2) = 0$ or condition (121) for $\mathcal{F}^{N2}$ in form (131) is satisfied, then the unique solution of perturbed BDIDE system (154) gives a solution of original BDIE system (N2) such that

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+ u dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi dS = 0.$$

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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