Received 2020; | Revised 2020; | Accepted 2020

DOI: xxx/xxxx

ARTICLE TYPE

Analysis of Two-operator Boundary-Domain Integral Equations for Variable Coefficient BVPs with General Data

Tsegaye G. Ayele*

[1]Department of Mathematics, Addis Ababa University, P. O. Box 1176, Addis Ababa, Ethiopia

Correspondence
*Tsegaye G. Ayele,
Department of Mathematics
Addis Ababa University
P.O. Box 1176 Addis Ababa
Email: tsegaye.ayele@aau.edu.et

Summary

The Dirichlet, Neumann and mixed boundary value problems for the linear second-order scalar elliptic differential equation with variable coefficient in a bounded three-dimensional domain are considered. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\widetilde{H}^{-1}(\Omega)$, when neither classical nor canonical co-normal derivatives are well defined. Using the two-operator approach and appropriate parametrix (Levi function) each problem is reduced to different systems of boundary domain integral equations (BDIEs). Equivalence of the BDIEs to the original BVP, BDIE solvability, solution uniqueness/non- uniqueness, and as well as invertibility of the BDIE operators are analysed in appropriate Sobolev (Bessel potential) spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

KEYWORDS:
PDEs, variable coefficients, parametrix, integral equations, equivalence, unique solvability, invertibility

1 | INTRODUCTION

Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows trough porous media, and other areas of physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of boundary value problems (BVPs) for such PDEs to explicit boundary integral equations, which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further numerical solution of the latter, see e.g. Mikhailov [1,2,3,4] and Mikhailov et al [5,6] as well as references therein. However this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity).

To overcome this difficulty, one can apply the so-called two-operator approach, formulated in Mikhailov [7] for a certain non-linear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always chose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

The corresponding BVPs are well studied nowadays, see e.g., Lions and Magenes[8], Grisvard[9] and McLean[10], but this is not the case for the two-operator BDIEs associated with the BVPs. The BDIE analysis is useful for discretisation and numerical solution of the BDIE and thus of the associated BVP. To analyze the two-operator approach, in Ayele and Mikhailov[11,12] one of its linear versions is applied to the mixed (Dirichlet-Neumann) BVP for a linear second-order scalar elliptic variable-coefficient PDE with *square integrable right-hand side* and reduced it to four different two-operator BDIE systems. The BDIE systems are nonstandard systems of equations containing integral operators defined on the domain under consideration and potential type and pseudo-differential operators defined on open sub-manifolds of the boundary. Using results of[5], a rigorous analysis of the two-operator BDIEs was given in appropriate Sobolev spaces.

For a function from the Sobolev space $H^1(\Omega)$, a classical co-normal derivative in the sense of traces may not exist. However, when this function satisfies a second order PDE with a right-hand side from $H^{-1}(\Omega)$, the generalized co-normal derivative can be defined in the weak sense, associated with the first Green identity and an extension of the PDE right-hand side to $\widetilde{H}^{-1}(\Omega)$ (see[13,14], [10] Lemma 4.3, [15] Definition 3.1). Since the extension is non-unique, the co-normal derivative appears to be a non-unique operator, which is also non-linear in u unless a linear relation between u and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the boundary-domain integral equations. These difficulties are addressed in[13,14] presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann boundary value problems for the divergent-type PDE with a variable scalar coefficient and a general right-hand side from $H^{-1}(\Omega)$ extended when necessary to $\widetilde{H}^{-1}(\Omega)$. This needed a non-trivial generalization of the third Green identity and its co-normal derivative for such functions, which extends the approach implemented in[5,16,17,4,6] for the PDE right-hand from $L_2(\Omega)$. In this paper, using the two-operator approach in extended settings, different from the one in[11,12] and using the results in[13,18], we derive generalization of the two-operator third Green identity and its co-normal derivative and give a rigorous analysis of the two-operator BDIEs for Dirichlet, Neumann and mixed (Dirichlet-Neumann) problems in the appropriate Sobolev-Slobodetski (Bessel-potential) spaces. This paper extends our publication[18].

2 | CO-NORMAL DERIVATIVES AND BOUNDARY VALUE PROBLEMS

Let Ω be an open bounded three-dimensional region of $\mathbb{R}^3$. For simplicity, we assume that the boundary $\partial\Omega$ is simply connected, closed, infinitely smooth surface. Moreover, $\partial\Omega = \overline{\partial\Omega_D} \bigcup \overline{\partial\Omega_N}$ where $\partial\Omega_D$ and $\partial\Omega_N$ are open, non-empty, non-intersecting, simply connected sub-manifolds of $\partial\Omega$ with an infinitely smooth boundary curve $\overline{\partial\Omega_D} \bigcap \overline{\partial\Omega_N} \in C^\infty$. Let $a \in C^\infty(\overline{\Omega})$, $a(x) > 0$ for $x \in \overline{\Omega}$. Let also $\partial_j = \partial_{x_j} := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. We consider the scalar elliptic differential equation, which for sufficiently smooth u has the following strong form,

$$Au(x) := A(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \quad x \in \Omega, \tag{1}$$

where u is unknown function and f is a given function in Ω.

In what follows $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g.,[8,10]). We recall that H^s coincides with the Sobolev-Slobodetski spaces W_2^s for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^3)$,

$$\widetilde{H}^s(\Omega) := \{g : g \in H^s(\mathbb{R}^3),\ \mathrm{supp}(g) \subset \overline{\Omega}\}$$

while $H^s(\Omega)$ denotes the space of restriction on Ω of distributions from $H(\mathbb{R}^3)$,

$$H^s(\Omega) = \{r_\Omega g : g \in H^s(\mathbb{R}^3)\}$$

where r_Ω denotes the restriction operator on Ω. We will also use the notation $g|_\Omega := r_\Omega g$. We denote by $H^s_{\partial\Omega}$ the following subspace of $H(\mathbb{R}^3)$ (and $\widetilde{H}(\Omega)$),

$$H^s_{\partial\Omega} := \{g : g \in H^s(\mathbb{R}^3),\ \mathrm{supp}(g) \subset \partial\Omega\}. \tag{2}$$

From the trace theorem (see, e.g.,[8,19,10]) for $u \in H^1(\Omega)$, it follows that $\gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ = \gamma^+_{\partial\Omega}$ are the trace operators on $\partial\Omega$ from Ω. Let also $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operator γ^+, i.e., $\gamma^+_{\partial\Omega}\gamma^{-1}_{\partial\Omega} w = \gamma^+\gamma^{-1} w = w$ for any $w \in H^{\frac{1}{2}}(\partial\Omega)$, and $(\gamma^{-1})^* : \widetilde{H}^{-1}(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous operator dual to $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, i.e., $\langle(\gamma^{-1})^*\tilde{f}, w\rangle_{\partial\Omega} := \langle\tilde{f}, \gamma^{-1}w\rangle_\Omega$ for any $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $w \in H^{\frac{1}{2}}(\partial\Omega)$.

For $u \in H^2(\Omega)$, we denote by T_a^+ the corresponding canonical (strong) co-normal derivative operator on $\partial\Omega$ in the sense of traces,

$$T_a^+ u := \sum_{i=1}^{3} a(x) n_i(x) \gamma^+ \frac{\partial u(x)}{\partial x_i} = a(x)\gamma^+ \frac{\partial u(x)}{\partial n(x)},$$

where $n(x)$ is the outward (to Ω) unit normal vector at the point $x \in \partial\Omega$. However the classical co-normal derivative operator is generally, not well defined if $u \in H^1(\Omega)$, (see, e.g.[13] Appendix A).

For $u \in H^1(\Omega)$, the PDE Au in (1) is understood in the sense of distributions,

$$\langle Au, v\rangle_\Omega := -\mathcal{E}_a(u,v), \qquad \forall v \in \mathcal{D}(\Omega), \tag{3}$$

where

$$\mathcal{E}_a(u,v) := \int_\Omega a(x)\nabla u(x).\nabla v(x)dx$$

and the duality brackets $\langle g, \cdot\rangle_\Omega$ denote the value of a linear functional (distribution) g, extending the usual L_2 inner product.

Since the set $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$, the above formula defines a continuous operator $A : H^1(\Omega) \to H^{-1}(\Omega) = [\widetilde{H}^1(\Omega)]^*$,

$$\langle Au, v\rangle_\Omega := -\mathcal{E}_a(u,v), \quad \forall u \in H^1(\Omega),\ \forall v \in \widetilde{H}^1(\Omega).$$

Let us consider also the operator, $\check{A} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega) = [H^1(\Omega)]^*$,

$$\langle \check{A}u, v\rangle_\Omega := -\mathcal{E}_a(u,v) = -\int_\Omega a(x)\nabla u(x).\nabla v(x)dx = -\int_{\mathbb{R}^3} \mathring{E}[a\nabla u](x).\nabla V(x)dx = \langle \nabla.\mathring{E}[a\nabla u], V\rangle_{\mathbb{R}^3} = \langle \nabla.\mathring{E}[a\nabla u], v\rangle_\Omega,$$

$\forall u \in H^1(\Omega)$, $\quad \forall v \in H^1(\Omega)$, which is evidently continuous and can be written as

$$\check{A}u = \nabla.\mathring{E}[a\nabla u]. \tag{4}$$

Here $V \in H^1(\mathbb{R}^3)$ is such that $r_\Omega V = v$ and $\mathring{E}$ denotes the operator of extension of the functions, defined in Ω, by zero outside Ω in $\mathbb{R}^3$. For any $u \in H^1(\Omega)$, the functional $\check{A}u$ belongs to $\widetilde{H}^{-1}(\Omega)$ and is the extension of the functional $Au \in H^{-1}(\Omega)$, which domain is thus extended from $\widetilde{H}^1(\Omega)$ to the domain $H^1(\Omega)$ for $\check{A}u$.

Inspired by the first Green identity for smooth functions, we can define *the generalized co-normal derivative* (cf., for example,[13] Definition 2.3[10] Lemma 4.3,[15] Definition 3.1,[20] Lemma 2.2).

Definition 1. Let $u \in H^1(\Omega)$ and $Au = r_\Omega \tilde{f}$ in Ω for some $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$. Then the generalized co-normal derivative $T_a^+(\tilde{f}, u) \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined as

$$\langle T_a^+(\tilde{f},u), w\rangle_{\partial\Omega} := \langle \tilde{f}, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_a(u, \gamma^{-1}w) = \langle \tilde{f} - \check{A}u, \gamma^{-1}w\rangle_\Omega\,, \quad \forall w \in H^{\frac{1}{2}}(\Omega), \tag{5}$$

that is, $T^+(\tilde{f}, u) := (\gamma^{-1})^*(\tilde{f} - \check{A}u)$.

By[10] Lemma 4.3,[15] Theorem 5.3, we have the estimate

$$\|T_a^+(\tilde{f},u)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C_1\|u\|_{H^1(\Omega)} + C_2\|\tilde{f}\|_{\widetilde{H}^{-1}(\Omega)}, \tag{6}$$

and for $u \in H^1(\Omega)$ such that $Au = r_\Omega \tilde{f}$ in Ω for some $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ the first Green identity holds in the following form,

$$\langle T_a^+(\tilde{f},u), \gamma^+ v\rangle_{\partial\Omega} := \langle \tilde{f}, v\rangle_\Omega + \mathcal{E}_a(u,v) = \langle \tilde{f} - \check{A}u, v\rangle_\Omega\,, \qquad \forall v \in H^1(\Omega). \tag{7}$$

As follows from Definition 1, the generalized co-normal derivative is nonlinear with respect to u for a fixed $\tilde{f}$, but linear with respect to the couple $(\tilde{f}, u)$, i.e.,

$$\alpha_1 T_a^+(\tilde{f}_1, u_1) + \alpha_2 T_a^+(\tilde{f}_2, u_2) = T_a^+(\alpha_1\tilde{f}_1, \alpha_1 u_1) + T_a^+(\alpha_2\tilde{f}_2, \alpha_2 u_2) = T_a^+(\alpha_1\tilde{f}_1 + \alpha_2\tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2) \tag{8}$$

for any complex numbers α_1, α_2.

Let us also define some subspaces of $H^s(\Omega)$, cf.[21,9,15,17].

Definition 2. Let $s \in \mathbb{R}$ and $A_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ be a linear operator. For $t \ge -\frac{1}{2}$ we introduce the space

$$H^{s,t}(\Omega; A_*) := \{g : g \in H^s(\Omega) : A_* g|_\Omega = \tilde{f}_g|_\Omega\,, \quad \tilde{f}_g \in \widetilde{H}^t(\Omega)\}$$

endowed with the norm

$$\|g\|^2_{H^{s,t}(\Omega;A_*)} := \|g\|^2_{H^s(\Omega)} + \|\tilde{f}_g\|^2_{\widetilde{H}^t(\Omega)}$$

and the inner product

$$(g,h)_{H^{s,t}(\Omega;A_*)} = (g,h)_{H^s(\Omega)} + (\tilde{f}_g, \tilde{f}_h)_{\widetilde{H}^t(\Omega)}$$

We will mostly use the operator A or Δ as A_* in the above definition. Note that since $Au - a\Delta u = \nabla a.\nabla u \in L_2(\Omega)$, for $u \in H^1(\Omega)$, we have $H^{1,0}(\Omega;A) = H^{1,0}(\Omega;\Delta)$.

Definition 3. For $u \in H^{1,-\frac{1}{2}}(\Omega;A)$, we define the canonical co-normal derivative $T_a^+ u \in H^{-\frac{1}{2}}(\partial\Omega)$ as

$$\langle T_a^+ u, w\rangle_{\partial\Omega} := \langle \tilde{A}u, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_a(u,\gamma^{-1}w) = \langle \tilde{A}u - \check{A}u, \gamma^{-1}w\rangle_\Omega\,, \quad \forall w \in H^{\frac{1}{2}}(\Omega), \tag{9}$$

that is, $T_a^+ u := (\gamma^{-1})^*(\tilde{A}u - \check{A}u)$.

The canonical co-normal derivative $T_a^+ u$ is independent of (non-unique) choice of the operator γ^{-1}, the operator $T_a^+ : H^{1,-\frac{1}{2}}(\Omega;A) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the following form,

$$\langle T_a^+ u, \gamma^+ v\rangle_{\partial\Omega} := \langle \tilde{A}u, v\rangle_\Omega + \mathcal{E}_a(u,v)\,, \qquad \forall v \in H^1(\Omega). \tag{10}$$

The operator $T_a^+ : H^{1,t}(\Omega;A) \to H^{-\frac{1}{2}}(\partial\Omega)$ in Definition 3 is continuous for $t \geq -\frac{1}{2}$. The canonical co-normal derivative is defined by the function u and the operator A and does not depend separately on the right-hand side $\tilde{f}$ (i.e. its behavior on the boundary), unlike the generalized co-normal derivative defined in (5), and the operator T_a^+ is linear. Note that the canonical co-normal derivative coincides with classical co-normal derivative $T_a^+ u = a\frac{\partial u}{\partial n}$ if the latter does exist in the trace sense, see,[15] Corollary 3.14 and Theorem 3.16.

Let $u \in H^{1,-\frac{1}{2}}(\Omega;A)$. Then Definitions 1 and 3 imply that the generalized co-normal derivative for arbitrary extension $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ of the distribution Au can be expressed as

$$\langle T_a^+(\tilde{f},u), w\rangle_{\partial\Omega} := \langle T_a^+ u, w\rangle_{\partial\Omega} + \langle \tilde{f} - \check{A}u, \gamma^{-1}w\rangle_\Omega\,, \qquad \forall w \in H^{\frac{1}{2}}(\Omega). \tag{11}$$

Let us consider the auxiliary linear elliptic partial differential operator B defined by

$$Bu(x) := B(x,\partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(b(x)\frac{\partial u(x)}{\partial x_i}\right), \tag{12}$$

where $b \in C^\infty(\overline{\Omega})$, $b(x) > 0$ for $x \in \overline{\Omega}$.

Note that since for $u \in H^1(\Omega)$, $Au - Bu = (a-b)\Delta u + \nabla(a-b)\nabla u \in L_2(\Omega)$, we have, $H^{1,0}(\Omega;A) = H^{1,0}(\Omega;\Delta) = H^{1,0}(\Omega;B)$.

Let $u \in H^1(\Omega)$ and $v \in H^{1,0}(\Omega;B)$. Then we write the first Green identity for operator B in the form

$$\mathcal{E}_b(u,v) + \int_\Omega u(x)Bv(x)dx = \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega} \tag{13}$$

with

$$\mathcal{E}_b(u,v) = \int_\Omega b(x)\nabla u(x).\nabla v(x)dx.$$

If, in addition, $Au = \tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, then according to the definition of $T_a^+(\tilde{f},u)$, in (5), the *two-operator second Green identity* can be written as

$$\langle \tilde{f}, v\rangle_\Omega - \int_\Omega u(x)Bv(x)dx + \int_\Omega [a(x)-b(x)]\nabla u(x)\cdot\nabla v(x)dx = \langle T_a^+(\tilde{f},u), \gamma^+ v\rangle_{\partial\Omega} - \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega}. \tag{14}$$

If, moreover $u, v \in H^{1,0}(\Omega;A) = H^{1,0}(\Omega;B)$ then (14) becomes

$$\int_\Omega [v(x)Au(x) - u(x)Bv(x)]dx + \int_\Omega [a(x)-b(x)]\nabla u(x)\cdot\nabla v(x)dx = \langle T_a^+ u, \gamma^+ v\rangle_{\partial\Omega} - \langle T_b^+ v, \gamma^+ u\rangle_{\partial\Omega}. \tag{15}$$

3 | PARAMETRIX AND POTENTIAL TYPE OPERATORS

3.1 | Parametrix

Definition 4. A function $P_b(x,y)$ of two variables $x, y \in \Omega$ is a parametrix (Levi function) for the operator $B(x;\partial_x)$ in $\mathbb{R}^3$ if (see, e.g.,[22,23,1])

$$B(x,\partial_x)P_b(x,y) = \delta(x-y) + R_b(x,y), \tag{16}$$

where $\delta(.)$ is the Dirac distribution and $R_b(x,y)$ possesses a weak (integrable) singularity at $x = y$, i.e.,

$$R_b(x,y) = \mathcal{O}(|x-y|^{-\varkappa}) \quad \text{with } \varkappa < 3. \tag{17}$$

It is easy to see that for the operator $B(x;\partial_x)$ defined by the right-hand side of (12), the function

$$P_b(x,y) = \frac{1}{b(y)}P_\Delta(x,y) = \frac{-1}{4\pi b(y)|x-y|}, \qquad x,y \in \mathbb{R}^3, \tag{18}$$

is a parametrix, while the corresponding remainder function is

$$R_b(x,y) = \nabla b(x).\nabla_x P_b(x,y) = -\frac{\nabla b(x).\nabla_y P_\Delta(x,y)}{b(y)} = \frac{(x-y).\nabla b(x)}{4\pi b(y)|x-y|^3}, \qquad x,y \in \mathbb{R}^3, \tag{19}$$

which satisfies estimate (17) with $\varkappa = 2$, due to smoothness of the function $b(x)$. Here the function $P_\Delta(x,y) = -(4\pi|x-y|)^{-1}$ is the fundamental solution of the Laplace operator. Evidently, the parametrix $P_b(x,y)$ given by (18) is the fundamental solution to the operator $B(y,\partial_x) := b(y)\Delta(\partial_x)$ with “frozen" coefficient $b(x) = b(y)$, and

$$B(y,\partial_x)P_b(x,y) = \delta(x-y).$$

3.2 | Volume potentials

Let $b \in C^\infty(\mathbb{R}^3)$ and $b(x) > 0$ a.e. in $\mathbb{R}^3$. For some scalar function g the parametrix-based Newtonian and the remainder volume potential operators, corresponding to the parametrix (18) and the remainder (19) are given by

$$\mathbf{P}_b g(y) := \int_{\mathbb{R}^3} P_b(x,y)g(x)dx \tag{20}$$

$$\mathcal{P}_b g(y) := \int_{\Omega} P_b(x,y)g(x)dx \tag{21}$$

$$\mathcal{R}_b g(y) := \int_{\Omega} R_b(x,y)g(x)dx. \tag{22}$$

For $g \in H^s(\Omega),\ s \in \mathbb{R}$, (20) is understood as $\mathbf{P}_b g = \frac{1}{b}\mathbf{P}_\Delta g$, where the Newtonian potential operator $\mathbf{P}_\Delta$ for Laplacian Δ is well defined in terms of the Fourier transform (i.e., as pseudo-differential operator), on any space $H^s(\mathbb{R}^3)$. For $g \in \widetilde{H}^s(\Omega)$, and any $s \in \mathbb{R}$, definitions in (21) and (22) can be understood as

$$\mathcal{P}_b g = \frac{1}{b}r_\Omega \mathbf{P}_\Delta g,\ \mathcal{P}_b g = \frac{1}{b}r_\Omega \mathbf{P}_\Delta g \ \text{ and } \quad \mathcal{R}_b g = -\frac{1}{b}r_\Omega \nabla.\mathbf{P}_\Delta(g\nabla b), \tag{23}$$

while for $g \in H^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}$, as (23) with g replaced by $\widetilde{E}g$, where $\widetilde{E} : H^s(\Omega) \to \widetilde{H}^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}$, is the unique extension operator related with the operator $\mathring{E}$ of extension by zero, cf.[15] Theorem 16. For $y \notin \partial\Omega$, the single layer and the double layer surface potential operators, corresponding to the parametrix (18) are defined as

$$V_b g(y) := -\int_{\partial\Omega} P_b(x,y)g(x)dS_x = \frac{1}{b}V_\Delta g(y), \tag{24}$$

$$W_b g(y) := -\int_{\partial\Omega} [T_b(x,n(x),\partial_x)P_b(x,y)]g(x)dS_x = \frac{1}{b}W_\Delta(bg)(y), \tag{25}$$

where g is some scalar density function, and the integrals are understood in the distributional sense if g is not integrable. The corresponding boundary integral (pseudo-differential) operators of direct surface values of the single layer potential $\mathcal{V}_b$ and the double layer potentials $\mathcal{W}_b$ for $y \in \partial\Omega$ are,

$$\mathcal{V}_b g(y) := -\int_{\partial\Omega} P_b(x,y)g(x)dS_x = \frac{1}{b}\mathcal{V}_\Delta g(y), \tag{26}$$

$$\mathcal{W}_b g(y) := -\int_{\partial\Omega} T_b(x,n(x),\partial_x)P_b(x,y)g(x)dS_x = \frac{1}{b}\mathcal{W}_\Delta(bg)(y) \tag{27}$$

We can also calculate at $y \in \partial\Omega$ the co-normal derivatives, associated with the operator A, of the single layer potential and of the double layer potential:

$$T_a^{\pm} V_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} V_b g(y), \tag{28}$$

$$\mathcal{L}_{ab}^{\pm} g(y) := T_a^{\pm} W_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} W_b g(y) =: \frac{a(y)}{b(y)} \mathcal{L}_b^{\pm} g(y) \tag{29}$$

The direct value operators associated with (28) are

$$\mathcal{W}'_{ab} g(y) := -\int_{\partial\Omega} [T_a(y, n(y), \partial_y) P_b(x, y)] g(x) dS_x = \frac{a(y)}{b(y)} \mathcal{W}'_b g(y), \tag{30}$$

$$\mathcal{W}'_b g(y) := -\int_{\partial\Omega} [T_b(y, n(y), \partial_y) P_b(x, y)] g(x) dS_x. \tag{31}$$

From equations (20)-(31) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta(x, y)$ of the Laplace operator Δ.

$$\mathbf{P}_a g = \frac{1}{a}\mathbf{P}_\Delta g, \quad \mathbf{P}_b g = \frac{1}{b}\mathbf{P}_\Delta g, \quad \mathcal{P}_a g = \frac{1}{a}\mathcal{P}_\Delta g, \quad \mathcal{P}_b g = \frac{1}{b}\mathcal{P}_\Delta g. \tag{32}$$

$$\frac{a}{b} V_a g = V_b g = \frac{1}{b} V_\Delta g; \qquad \frac{a}{b} W_a\left(\frac{bg}{a}\right) = W_b g = \frac{1}{b} W_\Delta (bg), \tag{33}$$

$$\frac{a}{b} \mathcal{V}_a g = \mathcal{V}_b g = \frac{1}{b} \mathcal{V}_\Delta g; \qquad \frac{a}{b} \mathcal{W}_a\left(\frac{bg}{a}\right) = \mathcal{W}_b g = \frac{1}{b} \mathcal{W}_\Delta (bg), \tag{34}$$

$$\mathcal{W}'_{ab} g = \frac{a}{b} \mathcal{W}'_b g = \frac{a}{b}\left\{ \mathcal{W}'_\Delta (bg) + \left[b \frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right] \mathcal{V}_\Delta g \right\}, \tag{35}$$

$$\mathcal{L}_{ab}^{\pm} g = \frac{a}{b} \mathcal{L}_b^{\pm} g = \frac{a}{b}\left\{ \mathcal{L}_\Delta (bg) + \left[b \frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right] \gamma^{\pm} W_\Delta (bg) \right\}. \tag{36}$$

It is taken into account that b and its derivatives are continuous in $\mathbb{R}^3$ and

$$\mathcal{L}_\Delta(bg) := \mathcal{L}_\Delta^{+}(bg) = \mathcal{L}_\Delta^{-}(bg)$$

by the Liapunov-Tauber theorem. Hence,

$$\Delta(bV_b g) = 0, \ \Delta(bW_b g) = 0 \quad \text{in} \quad \Omega, \quad \forall g \in H^s(\partial\Omega) \quad (\forall s \in \mathbb{R}), \tag{37}$$

$$\Delta(b\mathcal{P} g) = g \quad \text{in}\ \Omega, \quad \forall g \in \widetilde{H}^s(\Omega) \quad (\forall s \in \mathbb{R}). \tag{38}$$

Theorem 1. For $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$, there hold the following relations on $\partial\Omega$,

$$\gamma^{\pm} V_b g_1 = \mathcal{V}_b g_1, \tag{39}$$

$$\gamma^{\pm} W_b g_2 = \mp\frac{1}{2} g_2 + \mathcal{W}_b g_2, \tag{40}$$

$$T_a^{\pm} V_b g_1 = \pm\frac{1}{2}\frac{a}{b} g_1 + \mathcal{W}'_{ab} g_1. \tag{41}$$

For $a = b$, the jump relations (39)-(41) are stated and proved in[5] Theorem 3.3 , and[4] Theorem A.3. For $a \neq b$ the proof follows from relations (20)-(31). The mapping properties of the volume and surface potentials are summarized in Appendix A, see also[11] Theorem A.6.

4 | THE TWO-OPERATOR THIRD GREEN IDENTITY AND INTEGRAL RELATIONS

In this section applying some limiting procedures (cf.[23],[19] S.3.8), we obtain the parametrix based third Green identities.

Theorem 2. (i) If $u \in H^1(\Omega)$, then the following third Green identity holds,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \gamma^{+} u = \mathcal{P}_b \check{A} u \quad \text{in} \quad \Omega, \tag{42}$$

where the operator $\check{A}$ is defined in (4), and for $u \in C^1(\overline{\Omega})$,

$$\mathcal{P}_b\check{A}u(y) := \langle \check{A}u, P_b(.,y)\rangle_\Omega = -\mathcal{E}_a(u, P_b(.,y)) = -\int_\Omega a(x)\nabla u(x).\nabla_x P_b(x,y)dx \tag{43}$$

(ii) If $Au = r_\Omega\tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, then recalling the definition of $T_a^+(\tilde{f},u)$, in (5), we arrive at the generalised two-operator third Green identity in the following form,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+(\tilde{f},u) + W_b\gamma^+ u = \mathcal{P}_b\tilde{f} \quad \text{in} \quad \Omega, \tag{44}$$

where it was taken into account that

$$\langle T_a^+(\tilde{f},u), P_b(x,y)\rangle_{\partial\Omega} = -V_b T_a^+(\tilde{f},u)\,, \quad \langle \tilde{f}, P_b(x,y)\rangle_\Omega = \mathcal{P}_b\tilde{f}, \tag{45}$$

and

$$\mathcal{Z}_b u = -\int_\Omega [a(x)-b(x)]\nabla_x P_b(x,y)\cdot\nabla u(x)dx = \frac{1}{b(y)}\sum_{j=1}^{3}\partial_j\mathcal{P}_\Delta\left[(a-b)\partial_j u\right] \quad \text{in} \quad \Omega. \tag{46}$$

Proof. (i) Let first $u \in D(\overline{\Omega})$. Let $y \in \Omega$, $B_\epsilon(y) \subset \Omega$ be a ball centered at y with sufficiently small radius ϵ, and $\Omega_\epsilon := \Omega\setminus\overline{B_\epsilon}(y)$. For the fixed y, evidently, $P_b(.,y) \in D(\overline{\Omega_\epsilon}) \subset H^{1,0}(A;\Omega_\epsilon)$ and has the coinciding classical and canonical co-normal derivatives on $\partial\Omega_\epsilon$. Then from (18) and the first Green identity (13) employed for Ω_ϵ with $v = P_b(.,y)$ we obtain

$$-\int_{\partial B_\epsilon(y)} T_x^+ P_b(x,y)\gamma^+u(x)ds_x - \int_{\partial\Omega} T_x P_b(x,y)\gamma^+u(x)ds_x = -\int_{\Omega_\epsilon} b(x)\nabla u(x).\nabla_x P_b(x,y)dx,$$

which we rewrite as

$$\begin{aligned}-\int_{\partial B_\epsilon(y)} T_x^+ P_b(x,y)\gamma^+u(x)ds_x - \int_{\partial\Omega} T_x P_b(x,y)\gamma^+u(x)ds_x - \int_{\Omega_\epsilon}[a(x)-b(x)]\,\nabla u(x)\nabla_x P_b(x,y)dx \\ = -\int_{\Omega_\epsilon} a(x)\nabla u(x).\nabla_x P_b(x,y)dx.\end{aligned} \tag{47}$$

Taking the limit as $\epsilon \to 0$, equation (47) reduces to the third Green identity (42)–(43) for any $u \in \mathcal{D}(\overline{\Omega})$. Taking into account the density of $\mathcal{D}(\overline{\Omega})$ in $H^1(\Omega)$, and the mapping properties of the integral potentials, see Appendix, we obtain that (42)–(43) hold true also for any $u \in H^1(\Omega)$.

(ii) Let $\{\tilde{f}_k\} \in \mathcal{D}(\Omega)$ be a sequence of covering to $\tilde{f}$ in $\widetilde{H}^{-1}(\Omega)$ as $k \to \infty$. Then, according to[13] Theorem B.1 there exists a sequence $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ converging to u in $H^1(\Omega)$ such that $Au_k = r_\Omega\tilde{f}_k$ and $T_a^+(u_k) = T_a^+(\tilde{f}_k,u_k)$ converge to $T_a^+(\tilde{f},u)$ in $H^{-\frac{1}{2}}(\partial\Omega)$. For such u_k by (43) and (5), we have

$$\begin{aligned}\mathcal{P}_b\check{A}u_k(y) &= \frac{1}{b(y)}\nabla_y.\int_\Omega a(x)P_\Delta(x,y)\nabla u_k(x)dx = -\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} a(x)\nabla u_k(x)P_\Delta(x,y)dx = -\lim_{\epsilon\to 0}\mathcal{E}_{\Omega_\epsilon}(u_k, P_b(.,y)) \\ &= -\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon}\tilde{f}_k P_b(x,y)dx - \int_{\partial B_\epsilon(y)} P_b(x,y)T_a^+u_k(x)dS(x) - \int_{\partial\Omega} P_b(x,y)T_a^+u_k(x)dS(x)\Big] = \mathcal{P}_b\tilde{f}_k + V_bT_a^+u_k(y).\end{aligned} \tag{48}$$

Taking the limits as $k \to \infty$, we obtain $\mathcal{P}_b\check{A}u(y) = \mathcal{P}_b\tilde{f} + V_bT_a^+(\tilde{f},u)$, which substitution to (42) gives (44). □

Using the Gauss divergence theorem, we can rewrite $\mathcal{Z}_b u(y)$ in the form that does not involve derivatives of u,

$$\mathcal{Z}_b u(y) := \left[\frac{a(y)}{b(y)} - 1\right]u(y) + \widehat{\mathcal{Z}}_b u(y), \tag{49}$$

$$\widehat{\mathcal{Z}}_b u(y) := \frac{a(y)}{b(y)}W_a\gamma^+u(y) - W_b\gamma^+u(y) + \frac{a(y)}{b(y)}\mathcal{R}_a u(y) - \mathcal{R}_b u(y), \tag{50}$$

which allows to call $\mathcal{Z}_b$ integral operator in spite of its integro-differential representation (46). Note that the operator $\mathcal{Z}_b$ does not vanish unless operators A and B are equal. Substituting equation (49) and (50) into equations (42) and (44), we obtain Eqs. (4.1) and (4.3) in[13] (and also for $s = 1$ in[14]) respectively.

For some functions $\tilde{f}$, Ψ, Φ let us consider a more general "indirect" integral relation, associated with (44).

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\Psi + W_b\Phi = \mathcal{P}_b\tilde{f} \quad \text{in} \quad \Omega. \tag{51}$$

For the case $a = b$, the following lemma is stated and proved in[13] Lemma 4.2 (also in[14] Lemmas 4.2 and 4.4 for $s = 1$), and in[18] Lemma 1 for $a \neq b$.

Lemma 1. Let $u \in H^1(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial\Omega), \Phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, satisfy (51). Then

$$Au = r_\Omega \tilde{f} \quad \text{in} \quad \Omega, \tag{52}$$

$$r_\Omega V_b(\Psi - T_a^+(\tilde{f}, u)) - r_\Omega W_b(\Phi - \gamma^+ u) = 0 \quad \text{in} \quad \Omega, \tag{53}$$

$$\gamma^+ u + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_b\Psi - \frac{1}{2}\Phi + \mathcal{W}_b\Phi = \gamma^+\mathcal{P}_b\tilde{f} \qquad \text{on} \quad \partial\Omega, \tag{54}$$

$$T_a^+(\tilde{f}, u) + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \frac{a}{2b}\Psi - \mathcal{W}'_{ab}\Psi + \mathcal{L}_{ab}^+\Phi = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) \quad \text{on} \quad \partial\Omega, \tag{55}$$

where

$$\mathcal{R}_*^b\tilde{f}(y) := -\sum_{j=1}^{3}\partial_j[(\partial_j b)\mathcal{P}_b\tilde{f}]. \tag{56}$$

Proof. Subtracting (51) from identity (42), we obtain

$$V_b\Psi(y) - W_b(\Phi - \gamma^+ u)(y) = \mathcal{P}_b[\check{A}u(y) - \tilde{f}](y), \quad y \in \Omega. \tag{57}$$

Multiplying equality (57) by $b(y)$, applying the Laplace operator Δ and taking into account Eqs. (37) and (38), we get $r_\Omega\tilde{f} = r_\Omega(\check{A}u) = Au$ in Ω. This means $\tilde{f}$ is an extension of the distribution $Au \in H^{-1}(\Omega)$ to $\widetilde{H}^{-1}(\Omega)$, and u satisfies (52). Then (5) implies

$$\mathcal{P}_b[\check{A}u - \tilde{f}](y) = \langle \check{A}u - \tilde{f}, P_b(., y)\rangle_\Omega = -\langle T_a^+(\tilde{f}, u), P_b(.y)\rangle_{\partial\Omega} = V_b T_a^+(\tilde{f}, u), \quad y \in \Omega. \tag{58}$$

Substituting (58) into (57) leads to (53). Equation (54) follows from (51) and jump relations in (39) and (40) in Theorem 1.

To prove (55), let us first remark that for $u \in H^1(\Omega)$, we have $H^1(\Omega; A) = H^1(\Omega; \Delta) = H^1(\Omega; B)$ and

$$B\mathcal{P}_b\tilde{f} = \tilde{f} + \mathcal{R}_*^b\tilde{f} \;\text{ in } \Omega, \tag{59}$$

due to (52), which implies $B(\mathcal{P}_b\tilde{f} - u) = \mathcal{R}_*^b\tilde{f}$ in Ω, with $\mathcal{R}_*^b\tilde{f}$ given by (56) and thus $\mathcal{R}_*^b\tilde{f} \in L_2(\Omega)$. Then $B(\mathcal{P}_b\tilde{f} - u)$ can be canonically extended (by zero) to $\widetilde{B}(\mathcal{P}_b\tilde{f} - u) = \mathring{E}\mathcal{R}_*^b\tilde{f} \in \widetilde{H}^0(\Omega) \subset \widetilde{H}^{-1}(\Omega)$. Thus there exists a canonical co-normal derivative $T_b^+(\mathcal{P}_b\tilde{f} - u)$ written as (see, e.g.,[13] Eq. (4.14),[14] Eq. (4.23).)

$$T_b^+(\mathcal{P}_b\tilde{f} - u) = T_b^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - T_b^+(\tilde{f}, u), \tag{60}$$

and hence

$$T_a^+\big(\mathcal{P}_b\tilde{f} - u\big) = \frac{a}{b}T_b^+\big(\mathcal{P}_b\tilde{f} - u\big) = \frac{a}{b}\big[T_b^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) - T_b^+(\tilde{f}, u)\big] = T_a^+\big(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}\big) - T_a^+(\tilde{f}, u) \tag{61}$$

From (51) it follows that $\mathcal{P}_b\tilde{f} - u = \mathcal{Z}_b u + \mathcal{R}_b u - V_b\Psi + W_b\Phi$ in Ω. Substituting this on the left-hand side of (60) and taking into account (36) and the jump relation (41), we arrive at (55). □

Remark 1. If $\tilde{f} \in \widetilde{H}^{-\frac{1}{2}}(\Omega) \subset \widetilde{H}^{-1}(\Omega)$, then $\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f} \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$ as well, which implies $\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f} = \tilde{A}\mathcal{P}_b\tilde{f}$ and

$$T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) = T_a^+(\tilde{B}\mathcal{P}_b\tilde{f}, \mathcal{P}_b\tilde{f}) = T_a^+\mathcal{P}_b\tilde{f}. \tag{62}$$

Furthermore, if the hypotheses of Lemma 1 are satisfied, then (52) implies $u \in H^{1,-\frac{1}{2}}(\Omega; A)$ and $T_a^+(\tilde{f}, u) = T_a^+(\tilde{A}u, u) = T_a^+ u$. Henceforth (55), takes the familiar form, cf.[11] equation (3.23),

$$T_a^+ u + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \frac{a}{2b}\Psi - \mathcal{W}'_{ab}\Psi + \mathcal{L}_{ab}^+\Phi = T_a^+\mathcal{P}_b\tilde{f} \quad \text{on} \quad \partial\Omega.$$

Remark 2. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and a sequence $\{\phi_i\} \in \widetilde{H}^{-1}(\Omega)$ converge to $\tilde{f}$ in $\widetilde{H}^{-1}(\Omega)$. By the continuity of operators[13] C.1 and C.2, estimate (6) and relation (62) for ϕ_i, we obtain that

$$T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b\tilde{f}, \mathcal{P}_b\tilde{f}) = \lim_{i\to\infty} T_a^+(\phi_i + \mathring{E}\mathcal{R}_*^b\phi_i, \mathcal{P}_b\phi_i) = \lim_{i\to\infty} T_a^+\mathcal{P}_b\phi_i.$$

in $H^{-\frac{1}{2}}(\partial\Omega)$, cf. also[13] Theorem B.1.

Lemma 1 and the third Green identity (44) imply, the following assertion.

Corollary 1. If $u \in H^1(\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ are such that $Au = r_\Omega \tilde{f}$ in Ω, then

$$\frac{1}{2}\gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b T_a^+(\tilde{f}, u) + \mathcal{W}_b \gamma^+ u = \gamma^+ \mathcal{P}_b \tilde{f} \quad \text{on} \quad \partial\Omega, \tag{63}$$

$$\left(1 - \frac{a}{2b}\right) T_a^+(\tilde{f}, u) + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} T_a^+(\tilde{f}, u) + \mathcal{L}_{ab}^+ \gamma^+ u = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) \quad \text{on} \quad \partial\Omega. \tag{64}$$

Lemma 2.

(i) If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_\Omega V_b \Psi^* = 0$ in Ω, then $\Psi = 0$.

(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $r_\Omega W_b \Phi^* = 0$ in Ω, then $\Phi = 0$.

(iii) Let $\partial\Omega = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are nonintersecting simply connected sub-manifolds of $\partial\Omega$ with infinitely smooth boundaries and S_1 is nonempty. Let $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(S_1), \Phi^* \in \widetilde{H}^{\frac{1}{2}}(S_2)$. If $r_\Omega V_b \Psi^* - r_\Omega W_b \Phi^* = 0$, in Ω, then $\Psi^* = 0$ and $\Phi^* = 0$ on $\partial\Omega$.

Proof. For the case $a = b$, items (i) and (ii) are proved in [13] Lemma 4.6. Due to relations in (33), they hold true for $a \neq b$ as well. From [11] Lemma 3.2 (iii) follows the proof of item (iii). □

Theorem 3. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE $Au = r_\Omega \tilde{f}$ in Ω if and only if it is a solution of BDIDE (44).

Proof. If $u \in H^1(\Omega)$ solves PDE $Au = r_\Omega \tilde{f}$ in Ω, then it satisfies (44). On the other hand, if u solves BDIDE (44), then using Lemma 1 for $\Psi = T_a^+(\tilde{f}, u),\ \Phi = \gamma^+ u$ completes the proof. □

5 | THE DIRICHLET PROBLEM AND TWO-OPERATOR BDIE SYSTEMS

In this section, we shall derive and investigate the two-operator BDIE systems for the following Dirichlet problem: *Find a function $u \in H^1(\Omega)$ satisfying equations*

$$Au = f \qquad \text{in} \quad \Omega, \tag{65}$$

$$\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial\Omega, \tag{66}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in H^{-1}(\Omega)$.

Equation (65) is understood in the distributional sense (3) and the Dirichlet boundary condition (66) in the trace sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

Theorem 4. The Dirichlet problem (65)-(66) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (\mathcal{A}^D)^{-1}(f, \varphi_0)^T$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{\frac{1}{2}}(\partial\Omega) \times H^{-1}(\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \times H^{-1}(\Omega)$, of the Dirichlet problem (65)-(66), is continuous.

Following Mikhailov,[13] for $u \in H^1(\Omega)$, we shall reduce the Dirichlet problem (65)-(66) with $f \in H^{-1}(\Omega)$ in to two different *segregated two-operator* BDIE systems.

Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ be an extension of $f \in H^{-1}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists, see, Lemma 2.15 and Theorem 2.16 in Mikhailov[13]. We represent in (44), (63) and (64) the generalized co-normal derivative and the trace of the function u as

$$T^+(\tilde{f}, u) = \psi, \quad \gamma^+ u = \varphi_0$$

respectively, and will regard the new unknown function $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ as formally segregated of u. Thus we will look for the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

5.1 | BDIE system (D1)

To reduce BVP (65)-(66) to one of BDIE systems we will use equation (44) in Ω and equation (63) on $\partial\Omega$. Then we arrive at the system of BDIEs (D1),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi = \mathcal{F}_1^{D1} \qquad \text{in} \quad \Omega, \tag{67}$$

$$\gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b \psi = \mathcal{F}_2^{D1} \qquad \text{on} \quad \partial\Omega, \tag{68}$$

where

$$\mathcal{F}^{D1} := \begin{bmatrix} \mathcal{F}_1^{D1} \\ \mathcal{F}_2^{D1} \end{bmatrix} = \begin{bmatrix} F_0^D \\ \gamma^+ F_0^D - \varphi_0 \end{bmatrix} \quad \text{and} \quad F_0^D := \mathcal{P}_b \tilde{f} - W_b \varphi_0. \tag{69}$$

For $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusions $F_0^D \in H^1(\Omega)$ if $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ an due to the mapping properties of operators involved in (69), we have the inclusion $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

5.2 | BDIE system (D2)

To obtain a segregated BDIE system of *the second kind*, we will use equation (44) in Ω and equation (64) on $\partial\Omega$. Then we arrive at the system, (D2), of BDIEs,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi = \mathcal{P}_b \tilde{f} - W_b \varphi_0 \quad \text{in} \quad \Omega, \tag{70}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab}\psi = T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \mathcal{L}_{ab}^+ \varphi_0 \quad \text{on} \quad \partial\Omega, \tag{71}$$

where

$$\mathcal{F}^{D2} := \begin{bmatrix} \mathcal{F}_1^{D2} \\ \mathcal{F}_2^{D2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} - W_b \varphi_0 \\ T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \mathcal{L}_{ab}^+ \varphi_0 \end{bmatrix}. \tag{72}$$

Due to the mapping properties of operators involved in (72), we have the inclusion $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

6 | EQUIVALENCE AND INVERTIBILITY OF BOUNDARY DOMAIN INTEGRAL EQUATION SYSTEMS

Theorem 5. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in H^{-1}(\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ is such that $r_\Omega \tilde{f} = f$. Then

(i) If $u \in H^1(\Omega)$ solves the BVP (65)-(66), then the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Omega)$, where

$$\psi = T_a^+(\tilde{f}, u), \qquad \text{on } \partial\Omega, \tag{73}$$

solves the BDIE systems (D1) and (D2).

(ii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves one of the BDIE systems, (D1) or (D2), then this solution is unique and solves the other system, while u solves the Dirichlet BVP, and ψ satisfies (73).

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to BVP (65)–(66). Due to Theorem 4 it is unique. Setting ψ by (73) evidently implies, $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. From Theorem 3 and relations (63)–(64) follows that the couple (u, ψ) satisfies the BDIE systems (D1) and (D2), with the right-hand sides (69) and (72) respectively, which completes the proof of item (i).

(ii) Let now a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (67)–(68). Taking trace of equation (67) on $\partial\Omega$ and subtracting equation (68) from it we obtain

$$\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial\Omega, \tag{74}$$

i.e. u satisfies the Dirichlet condition (66).

Equation (67) and Lemma 1 with $\Psi = \psi$, $\Phi = \varphi_0$ imply that u is a solution of PDE (65) and $V_b\Psi^* - W_b\Phi^* = 0$, in Ω, where $\Psi^* = \psi - T_a^+(\tilde{f}, u)$ and $\Phi^* = \varphi_0 - \gamma^+ u$. Due to equation (74), $\Phi^* = 0$. Then Lemma (2)(i) implies $\Psi^* = 0$, which proves condition (73). Thus u obtained from the solution of BDIE system (D1) solves the Dirichlet problem and hence, by item (i) of the theorem, (u, ψ) solve also BDIE system (D2).

Due to (69), the BDIE system (67)-(68) with zero right-hand side can be considered as obtained for $\tilde{f} = 0$, $\varphi_0 = 0$, implying that its solution is given by a solution of the homogeneous problem (65)-(66), which is zero by Theorem 4. This implies uniqueness of the solution of the the inhomogeneous BDIE system (67)-(68). Similar arguments work if we suppose that instead of the BDIE system (D1), the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (70)-(71). $\square$

BDIE systems (D1) and (D2) can be written as

$$\mathfrak{D}^1\mathcal{U}^D = \mathcal{F}^{D1} \quad \text{and} \quad \mathfrak{D}^2\mathcal{U}^D = \mathcal{F}^{D2},$$

respectively.

Here $\mathcal{U}^D := (u, \psi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$,

$$\mathfrak{D}^1 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b \\ \gamma^+\mathcal{Z}_b + \gamma^+\mathcal{R}_b & \mathcal{V}_b \end{bmatrix}, \tag{75}$$

$$\mathfrak{D}^2 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b \\ T_a^+\mathcal{Z}_b + T_a^+\mathcal{R}_b & \left(1 - \frac{a}{2b}\right) I - \mathcal{W}'_{ab} \end{bmatrix}, \tag{76}$$

while $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ are given by (69) and (72) respectively.

Due to the mapping properties of the operators participating in the definitions of the operators $\mathfrak{D}^1$ and $\mathfrak{D}^2$ as well as the right-hand sides $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ (see, e.g.,[5,4], we have $\mathcal{F}^{D1} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, while the operators

$$\mathfrak{D}^1 \ : \ H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{77}$$

$$\mathfrak{D}^2 \ : \ H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \tag{78}$$

are continuous. Due to Theorem 5 (ii), operators (77) and (78) are injective.

Lemma 3. For any couple $(\mathcal{F}_1, \mathcal{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\mathcal{F}_1 = \mathcal{P}_b\tilde{f}_{**} - W_b\Phi_* \tag{79}$$

$$\mathcal{F}_2 = T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b\tilde{f}_{**}, \mathcal{P}_b\tilde{f}_{**}) - \mathcal{L}_{ab}^+\Phi_* \tag{80}$$

Moreover, $(\tilde{f}_{**}, \Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$ with $\mathcal{C}_{**} \ : \ H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ a linear continuous operator given by

$$\tilde{f}_{**} = \check{\Delta}(b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + \gamma^+\mathcal{F}_1)\partial_n b) \tag{81}$$

$$\Phi_* = \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\Big\{-b\mathcal{F}_1 + \mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big]\Big\} \tag{82}$$

where $\check{\Delta}(b\mathcal{F}_1) = \nabla \cdot \check{E}\nabla(b\mathcal{F}_1)$.

Let us first assume that there exist $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (79)-(80) and find their expression in terms of $\mathcal{F}_1$ and $\mathcal{F}_2$. Let us re write (79) as

$$\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**} = -W_b\Phi_* \quad \text{in } \Omega. \tag{83}$$

Multiplying (83) by b and applying Laplacian to it, we obtain,

$$\Delta(b\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_{**}) = \Delta(b\mathcal{F}_1) - \tilde{f}_{**} = -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \tag{84}$$

which means

$$\Delta(b\mathcal{F}_1) = r_\Omega\tilde{f}_{**} \quad \text{in } \Omega, \tag{85}$$

and $b\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)$ and hence $\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)$. The latter imply that the canonical co-normal derivatives $T_b^+(\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**})$ and $T_a^+(\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**})$ are well defined and can be also written in terms of their generalized co-normal derivatives

$$\begin{aligned}\frac{b}{a}T_a^+(\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}) &= T_b^+(\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}) = T_b^+(\tilde{B}(\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}), \mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}) = T_b^+(\mathring{E}\nabla \cdot (b\nabla(\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**})), \mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}) \\ &= T_b^+(\mathring{E}\Delta(b\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_{**}) - \mathring{E}\nabla \cdot ((\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**})\nabla b), \mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}) = T_b^+(-\mathring{E}\nabla \cdot (\mathcal{F}_1\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_{**}, \mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**})\end{aligned}$$

and therefore,

$$T_a^+(\mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}) = T_a^+(-\mathring{E}\nabla \cdot (\mathcal{F}_1\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_{**},\ \mathcal{F}_1 - \mathcal{P}_b\tilde{f}_{**}) \tag{86}$$

where (59) and (85) were taken into account. Applying the co-normal derivative operator T_a^+ to both sides of equation (83), substituting their (86), taking into account (8), we obtain,

$$T_a^+(\tilde{f}_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1\nabla b),\ \mathcal{F}_1) - T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b\tilde{f}_{**}, \mathcal{P}_b\tilde{f}_{**}) = -\mathcal{L}_{ab}^+\Phi_*, \text{ on } \partial\Omega. \tag{87}$$

Subtracting this from (80), we get,

$$\mathcal{F}_2 = T_a^+\big(\tilde{f}_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big) \quad \text{on } \Omega. \tag{88}$$

Due to (85), we can represent

$$\tilde{f}_{**} = \check{\Delta}(b\mathcal{F}_1) + \tilde{f}_{1*} = \nabla\cdot\mathring{E}\nabla(b\mathcal{F}_1) - \gamma^*\Psi_{**} \tag{89}$$

where $\tilde{f}_{1*} \in H^{-1}_{\partial\Omega}$ is defined by (2) and hence, due to e.g.[15] Theorem 2.10 can be in turn represented as $\tilde{f}_{1*} = -\gamma^*\Psi_{**}$, with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$. Then (85) is satisfied and

$$\begin{aligned}
&\frac{b}{a}T_a^+\big(\tilde{f}_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big) = T_b^+\big(\tilde{f}_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1\big)\\
&= (\gamma^{-1})^*[\tilde{f}_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b) - \check{B}\mathcal{F}_1] = (\gamma^{-1})^*[\tilde{f}_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b) - \nabla\cdot\mathring{E}(b\nabla\mathcal{F}_1)]\\
&= (\gamma^{-1})^*[\nabla\cdot\mathring{E}\nabla(b\mathcal{F}_1) - \nabla\cdot\mathring{E}(b\nabla\mathcal{F}_1) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)]\\
&(\gamma^{-1})^*[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)] = -\Psi_{**} - (\gamma^+\mathcal{F}_1)\partial_n b
\end{aligned}$$

for which

$$T_a^+(\tilde{f}_{**} - \mathring{E}\Delta\cdot(\mathcal{F}_1\nabla b), \mathcal{F}_1) = \frac{a}{b}\left[-\Psi_{**} - (\gamma^+\mathcal{F}_1)\partial_n b\right] \tag{90}$$

because

$$\begin{aligned}
&\langle(\gamma^{-1})^*[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], w\rangle_{\partial\Omega} = \langle[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b) - \gamma^*\Psi_{**} - \mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], \gamma^{-1}w\rangle_\Omega\\
&= \langle[\nabla\cdot\mathring{E}(\mathcal{F}_1\nabla b), \gamma^{-1}w\rangle_{\mathbb{R}^3} - \gamma^*\Psi_{**} - \langle\mathring{E}\nabla\cdot(\mathcal{F}_1\nabla b)], \gamma^{-1}w\rangle_\Omega\\
&= -\langle[\mathring{E}(\mathcal{F}_1\nabla b), \nabla(\gamma^{-1}w)\rangle_{\mathbb{R}^3} - \gamma^*\Psi_{**} + \langle(\mathcal{F}_1\nabla b), \nabla(\gamma^{-1}w)\rangle_\Omega - \langle n\cdot\gamma^+(\mathcal{F}_1\nabla b), \gamma^+\gamma^- w\rangle_\Omega\\
&= -\langle(\gamma^+(\mathcal{F}_1)\nabla b), w\rangle_{\partial\Omega} - \Psi_{**}.
\end{aligned} \tag{91}$$

Hence (88) reduces to

$$\Psi_{**} = -\frac{b}{a}\mathcal{F}_2 - (\gamma^+\mathcal{F}_1)\partial_n b = T_b^+\mathcal{F}_1 - (\gamma^+\mathcal{F}_1)\partial_n b, \tag{92}$$

and (89) to (81).

Now (83) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{F}_\Delta \quad \text{in } \Omega, \tag{93}$$

where

$$\mathcal{F}_\Delta := -b\mathcal{F}_1 + \mathcal{P}_\Delta\tilde{f}_{**} = -b\mathcal{F}_1 + \mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big] \tag{94}$$

is harmonic function in Ω due to (84). The trace of equation (94) gives

$$-\frac{1}{2}b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+\mathcal{F}_\Delta \quad \text{on } \partial\Omega. \tag{95}$$

Since the operator $-\frac{1}{2}I + \mathcal{W}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is an isomorphism (see e.g.[19] Ch.XI, Part B, §2, Remark 8 this implies

$$\begin{aligned}
\Phi_* &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\mathcal{F}_\Delta\\
&= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\Big\{-b\mathcal{F}_1 + \mathcal{P}_\Delta\Big[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\Big(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b\Big)\Big]\Big\},
\end{aligned}$$

which is Eq.(82). Relations (81), (82) can be written as $(\tilde{f}_{**}, \Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{C}_{**} : H^1(\Omega)\times H^{\vee-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions $\tilde{f}_{**}$ and Φ_*, given by (81) and (82), satisfy equations (79) and (80). Indeed, Φ_* given by (82) satisfies equation (95) and thus $\gamma^+W_\Delta(a\Phi_*) = \gamma^+\mathcal{F}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and $\mathcal{F}_\Delta$ are harmonic functions, this implies (93)-(94) and by (81) also (79). Finally, (81) implies by (90) that (88) is satisfied, and adding (87) to it leads to (80).

Let us prove that the operator $\mathcal{C}_{**}$ is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (79)-(80) with $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. Then (85) implies that $r_\Omega\tilde{f}_{**} = 0$ in Ω, that is $\tilde{f}_{**} \in H^{-1}_{\partial\Omega} \subset \widetilde{H}^{-1}(\Omega)$. Hence (88) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \qquad \text{on } \partial\Omega. \tag{96}$$

By the first Green identity (7), this gives,

$$0 = \langle T_a^+(\tilde{f}_{**}, 0), \gamma^+ v\rangle_{\partial\Omega} = \langle \tilde{f}_{**}, v\rangle_{\Omega}, \quad \forall v \in H^1(\Omega), \tag{97}$$

which implies $\tilde{f}_{**} = 0$ in $\mathbb{R}^3$. Finally, (82) gives $\Phi_* = 0$. Hence any solution of non-homogeneous linear system (79) − (80) has only one solution, which implies the uniqueness of the operator C_{**}. The following assertion is[3] Lemma 19 generalized to a wider space.

Lemma 4. For any couple $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\tilde{\mathcal{F}}_1 = \mathcal{P}_b \tilde{f}_{**} - W_b \Phi_* \tag{98}$$

$$\tilde{\mathcal{F}}_2 = \gamma^+(\mathcal{P}_b \tilde{f}_{**} - W_b \Phi_*) \tag{99}$$

Moreover, $(\tilde{f}_{**}, \Phi_*) = \tilde{C}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ with $\tilde{C}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ a linear continuous operator is given by

$$\tilde{f}_{**} = \check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2 \partial_n b) \tag{100}$$

$$\Phi_* = \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\Big(-b\tilde{\mathcal{F}}_2 + \gamma^+ \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2 \partial_n b)]\Big) \tag{101}$$

where $\check{\Delta}(b\tilde{\mathcal{F}}_1) = \nabla \cdot \check{E}\nabla(b\tilde{\mathcal{F}}_1)$.

Proof. Let us first assume that there exist $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (98)-(99) and find their expression in terms of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$. Let us re write (98) as

$$\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**} = -W_b \Phi_* \quad \text{in } \Omega. \tag{102}$$

Multiplying (102) by b and applying Laplacian to it, we obtain,

$$\Delta(b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) = \Delta(b\tilde{\mathcal{F}}_1) - \tilde{f}_{**} = -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \tag{103}$$

which means

$$\Delta(b\tilde{\mathcal{F}}_1) = r_\Omega \tilde{f}_{**} \quad \text{in } \Omega, \tag{104}$$

and $b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta \tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)$, while $\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)$. The latter imply that the canonical co-normal derivatives $T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$ and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$ are well defined and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**}) = \frac{b}{a} T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b \tilde{f}_{**})$.

Due to (104) and using $\tilde{f}_{1*} = -\gamma^* \Psi_{**}$ with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$ as in (92), we can represent

$$\tilde{f}_{**} = \check{\Delta}(b\tilde{\mathcal{F}}_1) + \tilde{f}_{1*} = \nabla \cdot \mathring{E}\nabla(b\tilde{\mathcal{F}}_1) - \gamma^* \Psi_{**} \tag{105}$$

where $\tilde{f}_{1*} \in H^{-1}_{\partial\Omega}$. Then (104) is satisfied. Replacing $\mathcal{F}_2$ by $T_a^+(\tilde{\mathcal{F}}_1, u)$ in Lemma 3, Eq. (92) yields,

$$\Psi_{**} = -\frac{b}{a} T_a^+ \tilde{\mathcal{F}}_1 - (\gamma^+ \tilde{\mathcal{F}}_1)\partial_n b = -T_b^+ \tilde{\mathcal{F}}_1 - \tilde{\mathcal{F}}_2 \partial_n b \tag{106}$$

and (105) reduces to (100).

Now (102) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{Q}_\Delta \quad \text{in } \Omega, \tag{107}$$

where

$$\mathcal{Q}_\Delta := -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta \tilde{f}_{**} = -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+ \tilde{\mathcal{F}}_1 + (\gamma^+ \mathcal{F}_1)\partial_n b)] \tag{108}$$

is harmonic function in Ω due to (103). The trace of equation (108) gives

$$-\frac{1}{2} b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+ \mathcal{Q}_\Delta \quad \text{on } \partial\Omega. \tag{109}$$

By similar argument as in Lemma 3 the operator $-\frac{1}{2}I + \mathcal{W}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is an isomorphism this implies

$$\begin{aligned}
\Phi_* &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\mathcal{Q}_\Delta \\
&= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\{-b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + (\gamma^+\mathcal{F}_1)\partial_n b)]\} \\
&= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\big(-b\tilde{\mathcal{F}}_2 + \gamma^+\mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + (\gamma^+\mathcal{F}_1)\partial_n b)]\big)
\end{aligned}$$

which is Eq.(101).

Relations (100), (101) can be written as $(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$, where $\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions $\tilde{f}_{**}$ and Φ_*, given by (100) and (101), satisfy equations (98) and (99). Indeed, Φ_* given by (101) satisfies equation (109) and thus $\gamma^+W_\Delta(a\Phi_*) = \gamma^+\mathcal{Q}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and $\mathcal{Q}_\Delta$ are harmonic functions, this implies (107)-(108) and by (100) also (98) while (99) follows from Esq.(100) and (107).

Let us prove that the operator $\tilde{\mathcal{C}}_{**}$ is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (98)-(99) with $\tilde{\mathcal{F}}_1 = 0$ and $\tilde{\mathcal{F}}_2 = 0$. Then (104) implies that $r_\Omega\tilde{f}_{**} = 0$ in Ω, that is $\tilde{f}_{**} \in H^{-1}_{\partial\Omega} \subset \widetilde{H}^{-1}(\Omega)$. in Ω, that is $\tilde{f}_{**} \in H^{-1}_{\partial\Omega} \subset \widetilde{H}^{-1}(\Omega)$. Hence (88) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \qquad \text{on } \partial\Omega. \tag{110}$$

By the first Green identity (7), this gives relation (97), which implies $\tilde{f}_{**} = 0$ in $\mathbb{R}^2$. Finally, (101) gives $\Phi_* = 0$. Hence any solution of non-homogeneous linear system (98) – (99) has only one solution, which implies the uniqueness of the operator $\tilde{\mathcal{C}}_{**}$. □

Theorem 6. Operators (77) and (78) are continuous and continuously invertible.

Proof. The continuity of operators (77) and (78) is proved above. To prove the invertibility of operator (77), let us consider the BDIE system (D1) with arbitrary right-hand side

$$\mathcal{F}_*^{D1} = (\mathcal{F}_{*1}^{D1}, \mathcal{F}_{*2}^{D1})^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Take $\tilde{\mathcal{F}}_1 = \mathcal{F}_{*1}^{D1}$ and $\Phi_* = \gamma^+\mathcal{F}_{*1}^{D1} - \mathcal{F}_{*2}^{D1}$ in Lemma 4, to obtain the representation of $\mathcal{F}_*^{D1}$ as:

$$\mathcal{F}_{*1}^{D1} = \tilde{\mathcal{F}}_1 \qquad \mathcal{F}_{*2}^{D1} = \gamma^+\tilde{\mathcal{F}}_1 - \Phi_*$$

where the couple

$$(\tilde{f}_*, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{111}$$

is unique and the operator

$$\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{112}$$

is linear and continuous. Applying Theorem 5 with $\tilde{f} = \tilde{f}_*$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D1) is uniquely solvable and its solution is:

$\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega\tilde{f}, \varphi_0)^T$, $\quad \mathcal{U}_2 = \gamma^+\mathcal{U}_1 - \varphi_0$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (65)–(66), is continuous. Representation (111) and continuity of the operator (112) imply invertibility of (77).

To prove the invertibility of operator (78), let us consider the BDIE system (D2) with arbitrary right-hand side

$$\mathcal{F}_*^{D2} = (\mathcal{F}_{*1}^{D2}, \mathcal{F}_{*2}^{D2})^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega).$$

Take $\mathcal{F}_1 = \mathcal{F}_{*1}^{D2}$ and $\mathcal{F}_2 = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_{*2}^{D2}$ in Lemma 3 which is the version of[13] Lemma 6.6, to represent $\mathcal{F}_*^{D2}$ as

$$\mathcal{F}_{*1}^{D2} = \mathcal{F}_1 \qquad \mathcal{F}_{*2}^{D2} = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_2$$

and the couple

$$(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\mathcal{F}_1, \mathcal{F}_2) \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is unique and the operator

$$\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{113}$$

is linear and continuous. Applying Theorem 5 with $\tilde{f} = \tilde{f}_{**}$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D2) is uniquely solvable and its solution is: $\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega\tilde{f}, \varphi_0)^\top$, $\quad \mathcal{U}_2 = T_a^+(r_\Omega\tilde{f}, \mathcal{U}_1)$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to$

$H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (65)–(66), is continuous. Representation (111) and continuity of the operator (113) imply invertibility of (78). □

7 | TWO-OPERATOR BDIE SYSTEMS FOR NEUMANN PROBLEM

In this section we shall derive and investigate the *two-operator* BDIE systems for the following Neumann problem: *Find a function $u \in H^1(\Omega)$ satisfying equations*

$$Au = r_\Omega \tilde{f} \quad \text{in} \quad \Omega, \tag{114}$$
$$T_a^+(\tilde{f}, u) = \psi_0 \quad \text{on} \quad \partial\Omega. \tag{115}$$

where $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$.

Equation (114) is understood in the distributional sense (3) and the Neumann boundary condition (115) in the weak sense (7). The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

Theorem 7.

(i) The homogeneous Neumann problem (114)-(115) admits only linearly independent solution $u^0 = 1$ in $H^1(\Omega)$.

(ii) The non-homogeneous Neumann problem (114)-(115) is solvable if and only if the following solvability condition is satisfied.

$$\langle \tilde{f}, u^0 \rangle_\Omega - \langle \psi_0, \gamma^+ u^0 \rangle_{\partial\Omega} = 0 \tag{116}$$

We explore different possibilities of reducing the Neumann problem (114)–(115) with $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to two different *segregated* Boundary-Domain Integral Equation (BDIE) systems. Corresponding formulations for the mixed problem for $u \in H^{1,0}(\Omega, \Delta)$ with $f \in L_2(\Omega)$ were introduced and analysed in[11,12,5,16,4]. Let us represent in (44), (63) and (64) the generalised co-normal derivative and the trace of the function u as

$$T_a^+(\tilde{f}, u) = \psi_0, \quad \gamma^+ u = \varphi,$$

and will regard the new unknown function $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ as formally segregated of u. Thus we will look for the couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

7.1 | BDIE system (N1)

To reduce BVP (114)-(115) to a BDIE system in this section we will use equation (44) in Ω and equation (64) on $\partial\Omega$. Then we arrive at the following system, (N1), of two boundary-domain integral equations for the couple of unknowns,(u, φ),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1^{N1} \quad \text{in} \quad \Omega, \tag{117}$$
$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ \varphi = \mathcal{F}_2^{N1} \quad \text{on} \quad \partial\Omega, \tag{118}$$

where

$$\mathcal{F}^{N1} := \begin{bmatrix} \mathcal{F}_1^{N1} \\ \mathcal{F}_2^{N1} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} + V_b \psi_0 \\ T_a^+(\tilde{f} + \mathring{E}\mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \psi_0 + \frac{a}{2b}\psi_0 + \mathcal{W}'_{ab}\psi_0 \end{bmatrix}. \tag{119}$$

Due to the mapping properties of operators involved in (119) we have $F^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

7.2 | BDIE system (N2)

To obtain a segregated BDIE system of *the second kind*, we will use equation (44) in Ω and equation (63) on $\partial\Omega$. Then we arrive at the following system, (D2), of boundary-domain integral equation systems,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{P}_b \tilde{f} + V_b \psi_0 \quad \text{in} \quad \Omega, \tag{120}$$
$$\frac{1}{2}\varphi + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u + \mathcal{W}_b \varphi = \gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0 \quad \text{on} \quad \partial\Omega, \tag{121}$$

where

$$\mathcal{F}^{N2} := \begin{bmatrix} \mathcal{F}_1^{N2} \\ \mathcal{F}_2^{N2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b\tilde{f} + V_b\psi_0 \\ \gamma^+\mathcal{P}_b\tilde{f} + \mathcal{V}_b\psi_0 \end{bmatrix}. \tag{122}$$

Due to the mapping properties of operators involved in (122), we have the inclusion $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

8 | EQUIVALENCE OF BOUNDARY-DOMAIN INTEGRAL EQUATION SYSTEMS AND THE NEUMANN PROBLEM

Theorem 8. Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$.

(i) If a function $u \in H^1(\Omega)$ solves the BVP (114)-(115) , then the couple (u, φ), with $\varphi = \gamma^+ u$ solves the BDIE systems (N1) and (N2).

(ii) Vice versa, if a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves one of the BDIE systems, (N1) or (N2), then the couple solves the other one BDIE system and u solves the Neumann problem (114)-(115) and $\gamma^+ u = \varphi$.

(iii) The homogeneous BDIE systems (N1) and (N2) have unique linearly independent solution $\mathcal{U}_0 = (u^0, \varphi^0)^\top$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Condition (116) is necessary and sufficient for solvability of the nonhomogeneous BDIE systems (N1) and (N2) in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to the Neumann BVP (114)–(115). It immediately follows from Theorem 3 and relations (63)–(64) that the couple (u, φ) with $\varphi = \gamma^+ u$ satisfies the BDIE systems (N1) and (N2), which proves item (i).

(ii) Let now a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N1). Lemma 1 for equation (117) implies that u is a solution of equation (1), and equations (53)-(55) hold for $\Psi = \psi_0$ and $\Phi = \varphi$. Subtracting (55) from (118) gives $T_a^+(\tilde{f}, u) = \psi_0$ on $\partial\Omega$. Further, from (53) we derive $W_b(\gamma^+ u - \varphi) = 0$ in Ω^+, where $\gamma^+ u = \varphi$ on $\partial\Omega$ by Lemma 2 completing item (ii) for BDIE system (N1).

Let now couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N2). Further, taking the trace of (120) on $\partial\Omega$ and comparing the results with (121), we easily derive that $\gamma^+ u = \varphi$ on $\partial\Omega$. Lemma 1 for equation (120) implies that u is a solution of (1), while equations (53)-(55) hold for $\Psi = \psi_0$ and $\Phi = \varphi$. Further, from (53) we derive

$$V_b(\psi_0 - T_a(\tilde{f}, u)) = 0 \text{ in } \Omega^+,$$

whence $T_a^+ u = \psi_0$ on $\partial\Omega$ by Lemma 2, i.e., u solves Neumann problem (114)-(115) which completes the proof of item (ii) for BDIE system (N2). (iii) Theorem 7 along with items (i) and (ii) imply the claims of item (iii) for BDIE system (N2) and (N1). □

9 | PROPERTIES OF BDIE SYSTEM OPERATORS FOR THE NEUMANN PROBLEM

BDIE systems (N1) and (N2) can be written respectively, as

$$\mathfrak{R}^1 \mathcal{U}^N = \mathcal{F}^{N1}, \qquad \mathfrak{R}^2 \mathcal{U}^N = \mathcal{F}^{N2},$$

where $\mathcal{U}^N = (u, \varphi)^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D)$,

$$\mathfrak{R}^1 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ T_a^+\mathcal{Z}_b + T_a^+\mathcal{R}_b & \mathcal{L}_{ab}^+ \end{bmatrix}, \qquad \mathfrak{R}^2 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ \gamma^+\mathcal{Z}_b + \gamma^+\mathcal{R}_b & \frac{1}{2} + \mathcal{W}_b \end{bmatrix}.$$

Due to the mapping properties of potentials in (119) and (122), the right-hand sides of BDIE systems (N1)and (N2) are such that $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Theorem 9. The operators

$$\mathfrak{R}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{123}$$

$$\mathfrak{R}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{124}$$

are continuous. They have one-dimensional null spaces, $\ker \mathfrak{R}^1 = \ker \mathfrak{R}^2$, in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, spanned over the element $(u^0, \varphi^0) = (1, 1)$.

Proof. The mapping properties of the potentials imply continuity of the operators (123) and (124). The claims that $\ker \mathfrak{R}^1$ and $\ker \mathfrak{R}^2$ are one-dimensional and the couple $(u^0, \varphi^0) = (1, 1)$ belong to $\ker \mathfrak{R}^1 = \ker \mathfrak{R}^2$ directly follows from Theorem 8(iii). $\square$

To describe in more details the range of operators (123) and (124), i.e., to give more information about the co-kernels of these operators, we will need several auxiliary assertions. First of all, let us remark that for any $v \in H^{s-\frac{3}{2}}(\partial\Omega)$, $s < \frac{3}{2}$, the single layer potential can be defined as follows:

$$V_b v(y) := -\langle \gamma P_b(., y), v\rangle_{\partial\Omega} = -\langle P_b(., y), \gamma^* v\rangle_{\mathbb{R}^3} = -\mathbf{P}_b \gamma^* v(y), \quad y \in \mathbb{R}^3 \setminus \partial\Omega. \tag{125}$$

where $\gamma^* : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-2}_{\partial\Omega}$, $s < \frac{3}{2}$, is the operator adjoined to the trace operator $\gamma : H^{2-s}(\mathbb{R}^3) \to H^{\frac{3}{2}-s}(\partial\Omega)$, and the space $H^s_{\partial\Omega}$ is defined by (2).

Lemma 5. Let $\tilde f \in \widetilde H^{s-2}(\Omega)$, $s > \frac{1}{2}$. If

$$r_\Omega \mathbf{P}_b \tilde f = 0 \quad \text{in } \Omega, \tag{126}$$

then $\tilde f = 0$ in $\mathbb{R}^3$.

Proof. Multiplying (126) by b, taking into account the first relation in (32) and applying the Laplace operator, we obtain $r_\Omega \tilde f = 0$, which means $\tilde f \in H^{s-2}_{\partial\Omega}$. If $s \geq \frac{3}{2}$, then $\tilde f = 0$ by[15] Theorem 2.10. If $\frac{1}{2} < s < \frac{3}{2}$, then by the same theorem there exists $v \in H^{s-\frac{3}{2}}(\partial\Omega)$ such that $\tilde f = \gamma^* v$. This gives $\mathbf{P}_b \tilde f = \mathbf{P}_b \gamma^* v = -V_b v$ in $\mathbb{R}^3$. Then (126) reduces to $V_b v = 0$ in Ω, which implies $v = 0$ on $\partial\Omega$ (see e.g., Lemma 2(i) for $s = 1$, which can be generalized to $\frac{1}{2} < s < \frac{3}{2}$) and thus $\tilde f = 0$ in $\mathbb{R}^3$. $\square$

Theorem 10. Let $\frac{1}{2} < s < \frac{3}{2}$. The operator

$$\mathbf{P}_b : \widetilde H^{s-2}(\Omega) \to H^s(\Omega) \tag{127}$$

and its inverse

$$(\mathbf{P}_b)^{-1} : H^s(\Omega) \to \widetilde H^{s-2}(\Omega) \tag{128}$$

are continuous and

$$(\mathbf{P_b})^{-1} g = \left[\Delta \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+\right](bg) \qquad \text{in} \quad \mathbb{R}^3, \quad \forall g \in H^s(\Omega). \tag{129}$$

Proof. The continuity of equation (127) follows from[5] Theorem 3.8. By Lemma 5 operator (127) is injective. Let us prove its surjectivity. To this end, for arbitrary $g \in H^s(\Omega)$ let us consider the following equation with respect to $\tilde f \in \tilde H^{s-2}(\Omega)$,

$$\mathbf{P}_\Delta \tilde f = g \quad \text{in} \quad \Omega. \tag{130}$$

Let $g_1 \in H^s(\Omega)$ be the (unique) solution of the following Dirichlet problem:

$$\Delta g_1 = 0 \text{ in } \Omega, \quad \gamma^+ g_1 = \gamma^+ g,$$

which due to[21] or[15] Lemma 2.6 can be particularly presented as $g_1 = V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+ g$. Let $g_0 := g - g_1$. Then $\in H^s(\Omega)$ and $\gamma^+ g_0 = 0$ and thus g_0 can be uniquely extended to $\mathring{E} g_0 \in \widetilde H^s(\Omega)$, where $\mathring{E}$ is the operator of extension by zero outside Ω . Thus by (125), equation (130) takes form

$$r_\Omega \mathbf{P}_\Delta[\tilde f + \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g] = g_0 \quad \text{in } \Omega. \tag{131}$$

Any solution $\tilde f \in \widetilde H^{s-2}(\Omega)$ of the corresponding equation on $\mathbb{R}^3$

$$\mathbf{P}_\Delta[\tilde f + \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g] = \mathring{E} g_0 \quad \text{in } \mathbb{R}^3, \tag{132}$$

solves (131). If $\tilde f$ solves (132) then acting with the Laplace operator on (132), we obtain

$$\tilde f = \tilde Q g := \Delta \mathring{E} g_0 - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g = \Delta \mathring{E}(g - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+ g) - \gamma^* \mathcal{V}_\Delta^{-1}\gamma^+ g \tag{133}$$

in $\mathbb{R}^3$. On the other hand, substituting $\tilde f$ given by (133) to (132) and taking into account that $\mathbf{P}_\Delta \Delta \tilde h = \tilde h$ for any $\tilde h \in \widetilde H^s(\Omega)$, $s \in \mathbb{R}$, we obtain that $\tilde Q g$ is indeed a solution of equation (132) and thus (131). By Lemma 5 the solution of (132) is unique, which means that the operator $\tilde Q$ is inverse to operator (127), i.e., $\tilde Q = (r_\Omega \mathbf{P}_b)^{-1}$. Since Δ is a continuous operator from $\widetilde H^s(\Omega)$ to $\widetilde H^{s-2}(\Omega)$, equation (81) implies that operator $(r_\Omega \mathbf{P})_b^{-1} = \tilde Q : H^s(\Omega) \to \widetilde H^{s-2}(\Omega)$ is continuous. The relations $\mathbf{P}_b = \frac{1}{b}\mathbf{P}_\Delta$ and $b(x) > 0$ then imply invertibility of operator (127) and anstatz (129). $\square$

Theorem 11. The co-kernel of operator (123) is spanned over the functional

$$g^{*1} := ((\gamma^+)^*\partial_n b, 1)^\top \tag{134}$$

in $\widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, i.e., $g^{*1}(\mathcal{F}_1, \mathcal{F}_2) = \langle(\gamma^+\mathcal{F}_1)\partial_n b + \mathcal{F}_2, \gamma^+ u^0\rangle_{\partial\Omega}$, where $u^0 = 1$.

Proof. The proof follows from the proof of[13] Theorem 6.7 and Lemma 3. Indeed, let us consider the equation $\mathfrak{R}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, i. e. system the system (N1)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \tag{135}$$

$$T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u + \mathcal{L}_{ab}^+\varphi = \mathcal{F}_2 \quad \text{on} \quad \partial\Omega, \tag{136}$$

with arbitrary right-hand side $(\mathcal{F}_1, \mathcal{F}_2)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. By Lemma 3 the right-hand side of the system has the form (79)-(80), i.e., system (135)-(136) reduces to

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b(\varphi + \Phi_*) = \mathcal{P}_b \tilde{f}_{**} \quad \text{in} \quad \Omega, \tag{137}$$

$$T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u + \mathcal{L}_{ab}^+(\varphi + \Phi_*) = T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b\tilde{f}_{**}, \mathcal{P}_b\tilde{f}_{**}) \quad \text{on} \quad \partial\Omega, \tag{138}$$

where the couple $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is given by (79)-(80). Up to the notations (137)-(138) is the same as in (119) with $\psi_0 = 0$. Then Theorems 8(iii) and 10 imply that the BDIE system (137)-(138) and hence (135)-(136) is solvable if and only if

$$\begin{aligned}\langle\tilde{f}_{**}, u^0\rangle_\Omega &= \langle(\check{\Delta} b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b), u^0\rangle_\Omega = \langle(\nabla\cdot\check{E}\nabla(b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b), u^0\rangle_{\mathbb{R}^3} \\ &= \langle(\nabla\cdot\check{E}\nabla(b\mathcal{F}_1, \nabla u^0\rangle_{\mathbb{R}^3} + \langle(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b), \gamma^+u^0\rangle_{\partial\Omega} = \langle(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n b), \gamma^+u^0\rangle_{\partial\Omega} = 0\end{aligned} \tag{139}$$

where we took into account that $\nabla u^0 = 0$ in $\mathbb{R}^3$. Thus the functional g^{*1} defined by (134) generates the necessary and sufficient solvability condition of equation $\mathfrak{R}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$. Hence g^{*1} is basis of the cokernel of $\mathfrak{R}^1$. □

Theorem 12. The co-kernel of operator (124) is spanned over

$$g^{*2} := \begin{pmatrix} -b\gamma^{+*}(\frac{1}{2} + \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 \\ -b(\frac{1}{2} - \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 \end{pmatrix} \tag{140}$$

in $\widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, i.e.,

$$g^{*2}(\mathcal{F}_1, \mathcal{F}_2) = \Big\langle -b\gamma^{+*}\Big(\frac{1}{2} + \mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_1\Big\rangle_\Omega + \Big\langle -b\Big(\frac{1}{2} - \mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_2\Big\rangle_{\partial\Omega}$$

where $u^0 = 1$.

Proof. The proof follows from the proof of[13] Theorem 6.8 and Lemma 3. Indeed, let us consider the equation $\mathfrak{R}^2\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, i. e. system the system (N2)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \tag{141}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u + \mathcal{W}_b\varphi = \mathcal{F}_2 \quad \text{on} \quad \partial\Omega, \tag{142}$$

with arbitrary $(\mathcal{F}_1, \mathcal{F}_2)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Introducing the new variable , $\varphi' = \varphi - (\mathcal{F}_2 - \gamma^+\mathcal{F}_1)$, BDIE system (141)-(142) takes the form

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b\varphi = \mathcal{F}'_1 \quad \text{in} \quad \Omega, \tag{143}$$

$$\frac{1}{2}\varphi' + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u + \mathcal{W}_b\varphi' = \mathcal{F}'_2 \quad \text{on} \quad \partial\Omega, \tag{144}$$

where

$$\mathcal{F}'_1 = \mathcal{F}_1 - W_b(\mathcal{F}_2 - \gamma^+\mathcal{F}_1) \in H^1(\Omega).$$

Let us recall that $\mathcal{P}_b = r_\Omega\mathbf{P}_b : \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega)$ and then by Theorem 10, the operator $\mathcal{P}_b^{-1} = (\mathbf{P}_b)^{-1} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is continuous for $\frac{1}{2} < s < \frac{3}{2}$. Hence we always represent $\mathcal{F}_1 = \mathcal{P}_b\tilde{f}_*$, with

$$\tilde{f}_* = [\Delta\mathring{E}(I - r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^*\mathcal{V}_\Delta^{-1}\gamma^+](b\mathcal{F}'_1) \in \widetilde{H}^{-1}(\Omega).$$

For $\mathcal{F}_1' = \mathcal{P}_b\tilde{f}_*$, the right-hand side of BDIE system (143)-(144) is the same as in (122) with $f = \tilde{f}_*$ and $\psi_0 = 0$. Then Theorems 8(iii) implies that the BDIE system (143)-(144) and hence (141)-(142) is solvable if and only if

$$\begin{aligned}\langle \tilde{f}_*, u^0\rangle_\Omega &= \langle [\Delta \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+) - \gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+](b\mathcal{F}_1'), u^0\rangle_{\mathbb{R}^3}\\ &= \langle \mathring{E}(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1}\gamma^+)(b\mathcal{F}_1'), \Delta u^0\rangle_{\mathbb{R}^3} - \langle (\gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+)(b\mathcal{F}_1'), u^0\rangle_{\mathbb{R}^3}\\ &= -\langle \gamma^+(b\mathcal{F}_1'), \mathcal{V}_\Delta^{-1}\gamma^+u^0\rangle_{\partial\Omega} = -\Big\langle \frac{1}{2}[\gamma^+(b\mathcal{F}_1) + (b\mathcal{F}_2)] - \mathcal{W}_\Delta[b(\mathcal{F}_2 - \gamma^+\mathcal{F}_1)], \mathcal{V}_\Delta^{-1}\gamma^+u^0\Big\rangle_{\partial\Omega}\\ &= \Big\langle -b\gamma^{+*}\Big(\frac{1}{2} + \mathcal{W}_\Delta'\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_1\Big\rangle_\Omega + \Big\langle -b\Big(\frac{1}{2} + \mathcal{W}_\Delta'\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \mathcal{F}_2\Big\rangle_{\partial\Omega} = 0.\end{aligned} \tag{145}$$

Thus the functional g^{*2} defined by (140) generates the necessary and sufficient solvability condition of equation $\mathfrak{R}^2\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$. Hence g^{*2} is basis of the cokernel of $\mathfrak{R}^2$. □

9.1 | Perterbed segregated BDIE systems for Neumann problem

Theorem 8 implies, that even when the solvability condition (116) is satisfied, the solutions of both BDIE systems, (N1) and (N2), are not unique. By Theorem 9, in turn, the BDIE left-hand side operators, $\mathfrak{R}^1$ and $\mathfrak{R}^2$, have non-zero kernels and thus are not invertible. To find a solution (u, φ) from uniquely solvable BDIE system with continuously invertible left-hand side operators, let us consider, following [24], some BDIE systems obtained form (N1) and (N2) by finite-dimensional operator perturbations, cf. [13] for the three-dimensional case. Below we use the notations $\mathcal{U} = (u, \varphi)^\top$ and $|\partial\Omega| := \int_{\partial\Omega} dS$.

9.1.1 | Perturbation of BDIE system (N1)

Let us introduce the perturbed counterparts of the BDIE system (N1),

$$\hat{\mathfrak{R}}^1\mathcal{U}^N = \mathcal{F}^{N1}, \tag{146}$$

where

$$\hat{\mathfrak{R}}^1 := \hat{\mathfrak{R}}^1 + \mathring{\mathfrak{R}}^1 \text{ and } \mathring{\mathfrak{R}}^1\mathcal{U}^N(y) := g^0(\mathcal{U}^N)\mathcal{G}^1(y) = \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)dS\begin{pmatrix}0\\1\end{pmatrix},$$

that is,

$$g^0(\mathcal{U}^N) := \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(x)dS, \quad \mathcal{G}^1(y) := \begin{pmatrix}0\\1\end{pmatrix}$$

For the functional g^{*1} given by (134) in Theorem 11, $g^{*1}(\mathcal{G}^1) = |\partial\Omega|$, while $g^0(\mathcal{U}^0) = 1$. Hence [13] Theorem D.1, [14] Theorem 6.14 imply the following assertion.

Theorem 13. (i) The operator $\hat{\mathfrak{R}}^1 : H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertable.

(ii) If condition $g^{*1}(\mathcal{F}^{N1}) = 0$ or condition (116) for $\mathcal{F}^{N1}$ in form (123) is satisfied, then the unique solution of perturbed BDIDE system (146) gives a solution of original BDIE system (N1) such that

$$g^0(\mathcal{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+u\,dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi\,dS = 0.$$

9.1.2 | Perturbation of BDIE system (N2)

Let us introduce the perturbed counterparts of the BDIE system (N2)

$$\hat{\mathfrak{R}}^2\mathcal{U} = \mathcal{F}^{N2}, \tag{147}$$

where

$$\hat{\mathfrak{R}}^2 := \mathfrak{R}^2 + \mathring{\mathfrak{R}}^2 \quad \text{and} \quad \mathring{\mathfrak{R}}^2\mathcal{U}(y) := g^0(\mathcal{U})\mathcal{G}^2(y) = \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds\begin{pmatrix}b^{-1}(y)\\ \gamma^+b^{-1}(y)\end{pmatrix},$$

that is,

$$g^0(\mathcal{U}) := \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds, \quad \mathcal{G}^2(y) := \begin{pmatrix}b^{-1}(y)u^0(y)\\ \gamma^+[b^{-1}u^0](y)\end{pmatrix}.$$

For the functional g^{*2} given by (140) in Theorem 12, since the operator $\mathcal{V}_\Delta^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned} g^{*2}(\mathcal{G}^2) &= \Big\langle -b\gamma^{+*}\Big(\frac{1}{2}+\mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, b^{-1}u^0\Big\rangle_\Omega + \Big\langle -b\Big(\frac{1}{2}-\mathcal{W}'_\Delta\Big)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+\Big(b^{-1}u^0\Big)\Big\rangle_{\partial\Omega} \\ &= -\langle(\frac{1}{2}+\mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 + (\frac{1}{2}-\mathcal{W}')\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} = -\langle\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \\ &\le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = -C|\partial\Omega|^2 < 0. \end{aligned} \tag{148}$$

Due to (148) and $g^0(\mathcal{U}^0) = 1$,[13] Theorem D.1,[14] Theorem 6.14 imply the following assertion.

Theorem 14. (i) The operator $\hat{\mathfrak{R}}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertable.

(ii) If condition $g^{*2}(\mathcal{F}^2) = 0$ or condition (116) for $\mathcal{F}^{N2}$ in form (124) is satisfied, then the unique solution of perturbed BDIDE system (147) gives a solution of original BDIE system (N2) such that

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+u\,dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi\,dS = 0.$$

10 | THE TWO-OPERATOR BOUNDARY-DOMAIN INTEGRAL EQUATION SYSTEMS FOR THE MIXED BOUNDARY VALUE PROBLEM

We shall derive and investigate BDIEs for the following mixed BVP: *Find a function $u \in H^1(\Omega)$ satisfying conditions*

$$Au = r_\Omega\tilde{f} \qquad \text{in } \Omega, \tag{149}$$

$$\gamma^+u = \varphi_0 \qquad \text{on} \quad \partial\Omega_D, \tag{150}$$

$$T^+(\tilde{f}, u) = \psi_0 \qquad \text{on} \quad \partial\Omega_N, \tag{151}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ are given functions. Equation (149) is understood in distributional sense, Eq.(150) is understood in trace sense and Eq.(151) is understood in functional sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

Theorem 15. The homogeneous version of BVP (149) – (151), i.e., with $(f, \varphi_0, \psi_0) = (0, 0, 0)$ has only the trivial solution. Hence the nonhomogeneous problem (149)–(151) may posses at most one solution.

Proof. The proof follows from Green's formula (7) with $v = u$ as a solution of the homogeneous mixed BVP (cf.[5] Theorem 2.1).

Theorem 16. The mixed problem (149)–(151) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (\mathcal{A}^M)^{-1}(\tilde{f}, \varphi_0, \psi_0)^\top$, where the inverse operator, $(\mathcal{A}^M)^{-1} : H^{-\frac{1}{2}}(\partial\Omega_N) \times H^{\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{-1}(\Omega) \to H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^M : H^1(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega_N) \times H^{\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{-1}(\Omega)$, of the mixed problem (149)–(151), is continuous.

11 | TWO-OPERATOR BOUNDARY-DOMAIN INTEGRAL EQUATIONS

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ from $\partial\Omega_D$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ from $\partial\Omega_N$ to $\partial\Omega$, respectively. Let us also denote

$$\tilde{F}_0 := \mathcal{P}_b\tilde{f} + V_b\Psi_0 - W_b\Phi_0 \qquad \text{in} \qquad \Omega.$$

Due to the mapping properties of the Newtonian (volume) and layer potentials (cf. Theorems 3.1 and 3.10 in[5]), we have the inclusion $\widetilde{F}_0 \in H^1(\Omega)$, for $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. We shall use the following notations for product spaces.

$$\begin{aligned}
\mathbb{X} &:= H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N),\\
\mathbb{Y}^{11} &:= H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N),\\
\mathbb{Y}^{22} &:= H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N),\\
\mathbb{Y}^{12} &:= H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega),\\
\mathbb{Y}^{21} &:= H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega).
\end{aligned}$$

To reduce BVP(149)–(151) to one or another two-operator BDIE system, we shall use equation (44) in Ω, and restrictions of Eq.(63) or (64) to appropriate parts of the boundary. We shall always substitute $\Phi_0+\varphi$ for γ^+u and $\Psi_0+\psi$ for $T_a^+(\tilde{f},u)$, cf.[5,11,12], where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ are considered as known, while ψ belongs to $\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and φ to $\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ due to the boundary conditions (150)- (151) and are to be found along with $u \in H^1(\Omega)$. This will lead us to *segregated* BDIE systems with respect to the unknown triple

$$\mathcal{U} := [u, \psi, \varphi]^\top \in \mathbf{X}.$$

11.1 | BDIE system (M11)

Let us use Eq. (44) in Ω, the restriction of Eq. (63) on $\partial\Omega_D$ and the restriction of Eq. (64) on $\partial\Omega_N$. Then we arrive at the following two-operator segregated system of BDIEs:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = \tilde{F}_0 \quad \text{in} \quad \Omega, \tag{152}$$

$$\gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_b\psi + \mathcal{W}_b\varphi = \gamma^+\tilde{F}_0 - \varphi_0 \quad \text{on} \quad \partial\Omega_D, \tag{153}$$

$$T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}^+_{ab}\varphi = T_a^+\tilde{F}_0 - \psi_0 \quad \text{on} \quad \partial\Omega_N\ , \tag{154}$$

which we call BDIE system M11, where M stands for the mixed problem and 11 hints that the integral equations on the Dirichlet and Neumann parts of the boundary are of the first kind. System (152)-(154) can be written in the form

$$\mathcal{A}^{11}\mathcal{U} = \mathcal{F}^{11},$$

where

$$\mathcal{F}^{11} := [\tilde{F}_0, r_{_{\partial\Omega_D}}\gamma^+\tilde{F}_0 - \varphi_0,\ r_{_{\partial\Omega_N}}T_a^+\widetilde{F}_0 - \psi_0]^\top, \tag{155}$$

$$\mathcal{A}^{11} := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ r_{_{\partial\Omega_D}}\gamma^+[\mathcal{Z}_b + \mathcal{R}_b] & -r_{_{\partial\Omega_D}}\mathcal{V}_b & r_{_{\partial\Omega_D}}\mathcal{W}_b \\ r_{_{\partial\Omega_N}}T_a^+[\mathcal{Z}_b + \mathcal{R}_b] & -r_{_{\partial\Omega_N}}\mathcal{W}'_{ab} & r_{_{\partial\Omega_N}}\mathcal{L}^+_{ab} \end{bmatrix}. \tag{156}$$

11.2 | BDIE system (M12)

To obtain another system, we use Eq.(44) in Ω and Eq.(63) on the whole boundary $\partial\Omega$, and arrive at the two-operator segregated BDIE system M12:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = \tilde{F}_0 \quad \text{in} \quad \Omega, \tag{157}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_b\psi + \mathcal{W}_b\varphi = \gamma^+\tilde{F}_0 - \Phi_0 \quad \text{on} \quad \partial\Omega\ . \tag{158}$$

System (157)-(158) can be written in the form

$$\mathcal{A}^{12}\mathcal{U} = \mathcal{F}^{12},$$

where

$$\mathcal{F}^{12} := [\tilde{F}_0,\ \gamma^+\tilde{F}_0 - \Phi_0]^\top, \tag{159}$$

$$\mathcal{A}^{12} := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ \gamma^+[\mathcal{Z}_b + \mathcal{R}_b] & -\mathcal{V}_b & \frac{1}{2}I + \mathcal{W}_b \end{bmatrix}. \tag{160}$$

11.3 | BDIE system (M21)

To obtain one more system, we use Eq.(44) in Ω and Eq.(64) on $\partial\Omega$ and arrive at the two-operator segregated BDIE system M21:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = \tilde{F}_0 \quad \text{in} \quad \Omega, \tag{161}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}_{ab}^+\varphi = T_a^+\tilde{F}_0 - \Psi_0 \quad \text{on} \quad \partial\Omega. \tag{162}$$

System (161)-(162) can be written in the form

$$\mathcal{A}^{21}\mathcal{U} = \mathcal{F}^{21},$$

where

$$\mathcal{F}^{21} := [\tilde{F}_0,\ T_a^+\tilde{F}_0 - \Psi_0]^\top, \tag{163}$$

$$\mathcal{A}^{21} := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ T_a^+[\mathcal{Z}_b + \mathcal{R}_b] & (1 - \frac{a}{2b})I - \mathcal{W}'_{ab} & \mathcal{L}_{ab}^+ \end{bmatrix}. \tag{164}$$

11.4 | BDIE system (M22)

To reduce BVP (149)-(151) to a BDIE system of almost the second kind (up to the spaces), we use Eq.(44) in Ω, the restriction of Eq.(64) to $\partial\Omega_D$, and the restriction of Eq.(63) to $\partial\Omega_N$. Then we arrive at the following two-operator segregated BDIE system M22:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = \tilde{F}_0 \quad \text{in} \quad \Omega, \tag{165}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}_{ab}^+\varphi = T_a^+\tilde{F}_0 - \Psi_0 \quad \text{on} \quad \partial\Omega_D, \tag{166}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{Z}_b u + \gamma^+\mathcal{R}_b u - \mathcal{V}_a\psi + \mathcal{W}_a\varphi = \gamma^+\tilde{F}_0 - \Phi_0 \quad \text{on} \quad \partial\Omega_N. \tag{167}$$

System (165)-(167) can be rewritten in the form

$$\mathcal{A}^{22}\mathcal{U} = \mathcal{F}^{22},$$

where

$$\mathcal{F}^{22} := [\tilde{F}_0,\ r_{\partial\Omega_D}(T_a^+\tilde{F}_0 - \Psi_0),\ r_{\partial\Omega_N}(\gamma^+\tilde{F}_0 - \Phi_0)]^\top, \tag{168}$$

$$\mathcal{A}^{22} := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ r_{\partial\Omega_D} T_a^+[\mathcal{Z}_b + \mathcal{R}_b] & (1 - \frac{a}{2b})I - r_{\partial\Omega_D}\mathcal{W}'_{ab} & r_{\partial\Omega_D}\mathcal{L}_{ab}^+ \\ r_{\partial\Omega_N}\gamma^+[\mathcal{Z}_b + \mathcal{R}_b] & -r_{\partial\Omega_N}\mathcal{V}_b & \frac{1}{2}I + r_{\partial\Omega_N}\mathcal{W}_b \end{bmatrix}. \tag{169}$$

12 | EQUIVALENCE AND INVERTIBILITY

Now let us prove the equivalence of BVP (149)-(151) with the BDIE systems M11, M12, M21 and M22.

Theorem 17. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, respectively.

(i) If some $u \in H^1(\Omega)$ solves the mixed BVP (149)-(151) in Ω, then the solution is unique and the triplet $(u, \psi, \varphi) \in \mathbb{X}$, where

$$\psi = T_a^+(\tilde{f}, u) - \Psi_0, \qquad \varphi = \gamma^+ u - \Phi_0 \qquad \text{on} \quad \partial\Omega, \tag{170}$$

solves the BDIE systems M11, M12, M21 and M22.

(ii) Vise versa, if a triplet $(u, \psi, \varphi) \in \mathbb{X}$ solves BDIE system M11 or M12 or M21 or M22, then the solution is unique, the function u solves BVP (149)-(151), and relations in (170) hold.

Proof. Let $u \in H^1(\Omega)$ be a solution to BVP (149)-(151). Then due to Theorem 15 it is unique. Set $\psi := T_a^+(\tilde{f}, u) - \Psi_0$, $\varphi := \gamma^+ u - \Phi_0$. Evidently, $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Then from Theorem 3 and relations (152)-(154) follows that the triplet (u, ψ, φ) satisfies the BDIE systems M11, M12, M21 and M22 with the right-hand sides (155), (159), (163) and (168) respectively, which completes the proof of item (i).

We give below proofs of item (ii) for the four BDIE systems M11, M12, M21 and M22 one by one.

BDIE system M11.

Let a triplet $(u, \psi, \varphi) \in H^1(\Omega) \times \mathbb{X}$ solves BDIE system (152)-(154). Let us consider the trace of Eq.(152) on $\partial\Omega_D$, taking into account the jump properties (see, Theorem 1), and subtract Eq.(153) to obtain

$$\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial\Omega_D, \tag{171}$$

i.e., u satisfies the Dirichlet condition (150). Taking the co-normal derivative T_a^+ of Eq. (152) on $\partial\Omega_N$, again with account of the jump properties, and subtracting Eq. (154), we obtain

$$T_a^+(\tilde{f}, u) = \psi_0, \qquad \text{on} \quad \partial\Omega_N, \tag{172}$$

i.e. u satisfies the Neumann condition (151). Taking into account that $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, equations (171) and (172) imply that the first equation of (170) is satisfied on $\partial\Omega_N$ and the second equation of (170) is satisfied on $\partial\Omega_D$.

Eq.(152) and Lemma 1 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is a solution to (52) and due to (53)

$$r_\Omega V_b \Psi^* - r_\Omega W_b \Phi^* = 0, \qquad \text{in} \quad \Omega,$$

where $\Psi^* = \Psi_0 + \psi - T_a^+(\tilde{f}, u)$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$. Since first equation in (170) on $\partial\Omega_N$ and the second equation in (170) on $\partial\Omega_D$, already proved, we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$, $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Then Lemma 2 (iii) with $S_1 = \partial\Omega_D$, $S_2 = \partial\Omega_N$, implies $\Psi = \Phi = 0$, which completes the the proof of conditions (170).

BDIE system M12.

Let the triplet $(u, \psi, \varphi) \in \mathbb{X}$ solve BDIE system (157)-(158). Let us consider the trace of equation (157) on $\partial\Omega$, taking into account the jump properties, and subtract it from (158) to obtain,

$$\gamma^+ u = \Phi_0 + \varphi \quad \text{on} \quad \partial\Omega. \tag{173}$$

This means that the second equation in (170) holds. Since $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$ we see that the Dirichlet condition (150) is satisfied.

Equation (157) and Lemma 1 with $\Psi = \psi + \Psi_0$, $\quad \Phi = \varphi + \Phi_0$ imply that u is a solution to Eq. (52) and

$$r_\Omega V_b(\Psi_0 + \psi - T_a^+(\tilde{f}, u)) - r_\Omega W_b(\Phi_0 + \varphi - \gamma^+ u) = 0 \qquad \text{in} \quad \Omega. \tag{174}$$

Due to (173), the second term in (174) vanishes, and by Lemma 2 (i) we obtain

$$\Psi_0 + \psi - T_a^+(\tilde{f}, u) = 0 \quad \text{on} \quad \partial\Omega, \tag{175}$$

i.e., the first equation in (170) is satisfied as well. Since $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$ equation (175) implies that u satisfies the Neumann boundary condition (151).

BDIE system M21.

Let now a triplet $(u, \psi, \varphi) \in \mathbb{X}$ solve BDIE system (161)-(162). Taking the co-normal derivative of Eq.(161) on $\partial\Omega$ and subtracting it from equation (162), we obtain

$$\psi + \Psi_0 - T_a^+(\tilde{f}, u) = 0 \qquad \text{on} \quad \partial\Omega. \tag{176}$$

which proves the first equation in (170). Since $\psi = 0$ on $\partial\Omega_N$ and $\Psi_0 = \psi_0$ on $\partial\Omega_N$, we see that u satisfies the Neumann condition (151). Equation (161) and Lemma (1) with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is a solution to equation (52) and

$$r_\Omega V_b(\Psi_0 + \psi - T_a^+(\tilde{f}, u)) - r_\Omega W_b(\Phi_0 + \varphi - \gamma^+ u) = 0 \qquad \text{in} \quad \Omega. \tag{177}$$

Due to Eq.(176) the first term vanishes in (177), and by Lemma 2 (ii) we obtain,

$$\Phi_0 + \varphi - \gamma^+ u = 0 \qquad \text{on} \quad \partial\Omega,$$

which means the second condition in (170) holds as well. Taking into account $\varphi = 0$ on $\partial\Omega_D$ and $\Phi_0 = \varphi$ on $\partial\Omega_D$, we conclude that u satisfies the Dirichlet condition (150).

BDIE system M22.

Let now a triplet $(u, \psi, \varphi) \in \mathbb{X}$ solve BDIE system (165)-(167). Taking the co-normal derivative of Eq. (170) on $\partial\Omega_D$ and subtracting it from Eq. (166), we obtain

$$\psi = T_a^+(\tilde{f}, u) - \Psi_0 \qquad \text{on} \quad \partial\Omega_D. \tag{178}$$

Further, take the trace of Eq. (165) on $\partial\Omega_N$ and subtract it from Eq. (167). We get

$$\varphi = \gamma^+ u - \Phi_0 \qquad \text{on} \quad \partial\Omega_N. \tag{179}$$

Equations (178) and (179) imply that the first equation (170) is satisfied on $\partial\Omega_D$ and the second equation in (170) is satisfied on $\partial\Omega_N$. Eq. (165) and Lemma 1 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is a solution to Eq. (52) and $r_\Omega V_b \Psi^* - r_\Omega W_b \Psi^* = 0$ in Ω, where $\Psi^* = \Psi_0 + \psi - T_a^+(\tilde{f}, u)$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$. Due to (170) and (179), we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_N)$, $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_D)$. Lemma 2 (iii) with $S_1 = \partial\Omega_N$ and $S_2 = \partial\Omega_D$ implies $\Psi^* = \Phi^* = 0$ which completes the proof of conditions (170) on the whole boundary $\partial\Omega$. Taking into account that $\varphi = 0$ on $\partial\Omega_D$ and $\Phi_0 = \varphi_0$ on $\partial\Omega_D$, and $\psi = 0$ on $\partial\Omega_N$ and $\Psi_0 = \psi_0$ on $\partial\Omega_N$, Eq. (170) imply the boundary conditions (150) and (151).

Unique solvability of the BDIE systems M11, M12, M12 and M22 then follows from the already proved relations (170) and the unique solvability of BVP (149)-(151) stated in item (i). □

The mapping properties of operators in (156), (160), (164) and (169) described in Appendix A and Theorem 17 imply the following statement.

Corollary 2. The following operators are continuous and injective

$$\mathcal{M}^{11} : \mathbb{X} \to \mathbb{Y}^{11}, \tag{180}$$

$$\mathcal{M}^{12} : \mathbb{X} \to \mathbb{Y}^{12}, \tag{181}$$

$$\mathcal{M}^{21} : \mathbb{X} \to \mathbb{Y}^{21}, \tag{182}$$

$$\mathcal{M}^{22} : \mathbb{X} \to \mathbb{Y}^{22}. \tag{183}$$

Now we are in the position to analyse the invertibility of Now we are in the position to analyse the invertibility of the operators $\mathcal{M}^{11}$, $\mathcal{M}^{12}$, $\mathcal{M}^{21}$and $\mathcal{M}^{22}$.

Theorem 18. Operators (180)-(183) are continuously invertible.

Proof. To prove the invertibility of operator (180), let us consider BDIE system M11 with an arbitrary right-hand side $\mathcal{F}_*^{11} = \{\mathcal{F}_{*1}^{11}, \mathcal{F}_{*2}^{11}, \mathcal{F}_{*3}^{11}\}^\top \in \mathbb{X}$. Taking $S_1 = \partial\Omega_N$, $S_2 = \partial\Omega_D$ and

$$F = \mathcal{F}_{*1}^{11}, \quad \Psi = r_{\partial\Omega_N} T_a^+ \mathcal{F}_{*1}^{11} - \mathcal{F}_{*3}^{11}, \quad \Phi = r_{\partial\Omega_D} \gamma^+ \mathcal{F}_{*1}^{11} - \mathcal{F}_{*2}^{11}$$

in[5] Lemma 5.13, presented as Lemma 6 in the Appendix, we obtain that $\mathcal{F}_*^{11}$ can be represented as

$$\begin{aligned}
\mathcal{F}_{*1}^{11} &= \mathcal{P}_b \tilde{f}_* + V_b \Psi_* - W_b \Phi_* \ \text{ in } \ \Omega, \\
\mathcal{F}_{*2}^{11} &= r_{\partial\Omega_D} \left[\gamma^+ \mathcal{F}_{*1}^{11} - \Phi_*\right], \\
\mathcal{F}_{*3}^{11} &= r_{\partial\Omega_N} \left[T_a^+ \mathcal{F}_{*1}^{11} - \Psi_*\right],
\end{aligned}$$

where the triple

$$(\tilde{f}_*, \Psi_*, \Phi_*)^\top = \mathcal{C}_{\partial\Omega_N, \partial\Omega_D} \mathcal{F}_*^{11} \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{184}$$

is unique and the operator

$$\mathcal{C}_{\partial\Omega_N, \partial\Omega_D} : \mathbb{X} \to \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{185}$$

is linear and continuous.

Applying Theorem 17 with

$$f = \tilde{f}_*, \quad \Psi_0 = \Psi_*, \quad \Phi_0 = \Phi_*, \quad \psi_0 = r_{\partial\Omega_N} \Psi_0, \quad \varphi_0 = r_{\partial\Omega_D} \Phi_0, \tag{186}$$

we obtain that the system M11 is uniquely solvable and its solution is

$$\mathcal{U}_1 = (A^{DN})^{-1}(\tilde{f}_*, r_{\partial\Omega_D} \Phi_*, r_{\partial\Omega_N} \Psi_*)^\top, \qquad \mathcal{U}_2 = T_a^+ \mathcal{U}_1 - \Psi_*, \qquad \mathcal{U}_3 = \gamma^+ \mathcal{U}_1 - \Phi_* \tag{187}$$

while $r_{_{\partial\Omega_N}}\mathcal{U}_2 = 0$, $r_{_{\partial\Omega_D}}\mathcal{U}_3 = 0$. Here $(A^{DN})^{-1}$ is the continuous inverse operator to the left-hand side operator of the mixed BVP (149)-(151), $A^{DN} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega)$, cf.[5] Corollary 5.16. Representation (184), and continuity of operator (185) complete the proof for $\mathcal{M}^{11}$.

To prove invertibility of operator (183), we apply similar arguments. Let us consider the BDIE system M22 with an arbitrary right-hand side $\mathcal{F}^{22}_* = \{\mathcal{F}^{22}_{*1}, \mathcal{F}^{22}_{*2}, \mathcal{F}^{22}_{*3}\}^\top \in \mathbb{X}$. Taking now $S_1 = \partial\Omega_D$, $S_2 = \partial\Omega_N$,

$$F = \mathcal{F}^{22}_{*1}, \quad \Psi = r_{_{\partial\Omega_D}} T^+_a \mathcal{F}^{22}_{*1} - \mathcal{F}^{22}_{*2}, \quad \Phi = r_{_{\partial\Omega_N}} \gamma^+ \mathcal{F}^{22}_{*1} - \mathcal{F}^{22}_{*3}$$

in[5] Lemma 5.13, i.e., Lemma 6 in the Appendix, we obtain that $\mathcal{F}^{22}_*$ can be represented as

$$\begin{aligned} \mathcal{F}^{22}_{*1} &= \mathcal{P}_b \tilde{f}_* + V_b \Psi_* - W_b \Phi_* \ \text{ in } \ \Omega, \\ \mathcal{F}^{22}_{*2} &= r_{_{\partial\Omega_D}} \left[T^+_a \mathcal{F}^{22}_{*1} - \Psi_* \right], \\ \mathcal{F}^{22}_{*3} &= r_{_{\partial\Omega_N}} \left[\gamma^+ \mathcal{F}^{22}_{*1} - \Phi_* \right], \end{aligned}$$

where the triple

$$(\tilde{f}_*, \Psi_*, \Phi_*)^\top = \mathcal{C}_{\partial\Omega_D, \partial\Omega_N} \mathcal{F}^{22}_* \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{188}$$

is unique and the operator

$$\mathcal{C}_{\partial\Omega_N, \partial\Omega_D} : \mathbb{X} \to \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{189}$$

is linear and continuous.

Applying now Theorem 17 with the same substitutions (186), we obtain that the system M22 is uniquely solvable and its solution is given by (187). Representation (188), and continuity of operator (189) complete the proof for $\mathcal{M}^{22}$.

To prove invertibility of operator (181), let us consider the BDIE system M12 with an arbitrary right-hand side $\mathcal{F}^{12}_* = \{\mathcal{F}^{12}_{*1}, \mathcal{F}^{12}_{*2}\}^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Taking $F = \mathcal{F}^{12}_{*1}$, $\Phi = \gamma^+ \mathcal{F}^{12}_{*1} - \mathcal{F}^{12}_{*2}$ on $\partial\Omega$ in Corollary 4 in the Appendix, we obtain the representation

$$\begin{aligned} \mathcal{F}^{12}_{*1} &= \mathcal{P}_b \tilde{f}_* + V_b \Psi_* - W_b \Phi_* \ \text{ in } \ \Omega, \\ \mathcal{F}^{12}_{*2} &= \gamma^+ \mathcal{F}^{12}_{*1} - \Phi_* \ \text{ on } \ \partial\Omega, \end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Phi*} \mathcal{F}_* \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{190}$$

is unique and the operator

$$\widetilde{\mathcal{C}}_{\Phi*} : \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{191}$$

is linear and continuous.

Applying Theorem 17 with substitutions (186), we obtain that the system M12 is uniquely solvable and its solution is given by (187). Representation (190), and continuity of operator (191) complete the proof for M12.

Finally to prove invertibility of operator (182), let us consider the BDIE system M21 with an arbitrary right-hand side $\mathcal{F}^{21}_* = \{\mathcal{F}^{21}_{*1}, \mathcal{F}^{21}_{*2}\}^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. Taking $F = \mathcal{F}^{21}_{*1}$, $\Psi = T^+_a \mathcal{F}^{21}_{*1} - \mathcal{F}^{21}_{*2}$ on $\partial\Omega$ in Corollary 3 in the Appendix, we obtain that

$$\begin{aligned} \mathcal{F}^{21}_{*1} &= \mathcal{P}_b \tilde{f}_* + V_b \Psi_* - W_b \Phi_* \ \text{ in } \ \Omega, \\ \mathcal{F}^{21}_{*2} &= \ T^+_a \mathcal{F}^{21}_{*1} - \Psi_* \ \text{ on } \ \partial\Omega. \end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Psi*} \mathcal{F}_* \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{192}$$

is unique and the operator

$$\widetilde{\mathcal{C}}_{\Psi*} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{193}$$

is linear and continuous. Applying Theorem 17 with substitutions (186), we obtain that the system M21 is uniquely solvable and its solution is given by (187). Representation (192), and continuity of operator (193) complete the proof for M21. □

ACKNOWLEDGMENTS

The work on this paper of the author was supported by the Alexander von Humboldt Foundation grant 3.4-ETH/1144171. He would like also to thank the Department of Mathematics at TU Kaiserslautern for hosting his renewed research stay and the Department of Mathematics at Addis Ababa University for all other supports.

Author contributions

The paper was written by the author personally. The author read and approved the final manuscript.

Financial disclosure

None reported.

Conflict of interest

The author declares that there is no potential conflict of interests.

SUPPORTING INFORMATION

The following supporting information is available as part of the online article:

References

1. Mikhailov SE. Localized boundary-domain integral formulations for problems with variable coefficients. *Engineering Analysis with Boundary Elements.* 2002;26:681–690.

2. Mikhailov SE, Nakhova IS. Mesh-based numerical implementation of the localized boundary-domain integral-equation method to a variable-coefficient Neumann problem. *Journal of Engineering Mathematics.* 2005;51:251–259.

3. Mikhailov SE. Analysis of extended boundary-domain integral and integro-differential equations of some variable-coefficient BVP:106–125. 2005.

4. Mikhailov SE. Analysis of united boundary-domain integro-differential and integral equations for a mixed BVP with variable coefficient. *Math. Methods Appl. Sci..* 2006;29(6):715–739.

5. Chkadua O, Mikhailov SE, Natroshvili D. Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. I. Equivalence and invertibility. *J. Integral Equations Appl..* 2009;21(4):499–543.

6. Chkadua O, Mikhailov SE, Natroshvili D. Analysis of segregated boundary-domain integral equations for variable-coefficient problems with cracks. *Numer. Methods Partial Differential Equations.* 2011;27(1):121–140.

7. Mikhailov SE. Localized direct boundary-domain integro-differential formulations for scalar nonlinear boundary-value problems with variable coefficients. *J. Engrg. Math..* 2005;51(3):283–302.

8. Lions JL, Magenes E. *Non-homogeneous boundary value problems and applications [by] J. L. Lions [and] E. Magenes. Translated from the French by P. Kenneth*. Springer-Verlag Berlin, New York; 1972.

9. Grisvard P. *Elliptic Problems in Nonsmooth Domains*. Society for Industrial and Applied Mathematics; 2011.

10. McLean W. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press; 2000.

11. Ayele TG, Mikhailov SE. Analysis of two-operator boundary-domain integral equations for variable-coefficient mixed BVP. *Eurasian Math. J..* 2011;2(3):20–41.

12. Ayele TG, Mikhailov SE. Two-operator boundary-domain integral equations for a variable-coefficient BVP. In: Birkhäuser Boston, Inc., Boston, MA 2010 (pp. 29–39).

13. Mikhailov SE. *Analysis of Segregated Boundary-Domain Integral Equations for Variable-Coefficient Dirichlet and Neumann Problems with General Data.*. ArXiv:1509.03501v3[math.AP]; 2017.

14. Mikhailov SE. Analysis of segregated boundary-domain integral equations for BVPs with non-smooth coefficients on Lipschitz domains. *Boundary Value Problems.* 2018;2018(1):87.

15. Mikhailov SE. Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Anal. Appl.*. 2011;378(1):324–342.

16. Chkadua O, Mikhailov SE, Natroshvili D. Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. II. Solution regularity and asymptotics. *J. Integral Equations Appl.* 2010;22(1):19–37.

17. Mikhailov SE. Solution regularity and co-normal derivatives for elliptic systems with non-smooth coefficients on Lipschitz domains. *J. Math. Anal. Appl.*. 2013;400(1):48–67.

18. Ayele TG. Two-Operator Boundary Domain Integral Equations for Variable Coefficient Dirichlet Problem with General Data:37–52. Springer Nature Switzerland AG 2019.

19. Dautray R, Artola M, Amson JC, et al. *Mathematical Analysis and Numerical Methods for Science and Technology: Volume 4 Integral Equations and Numerical Methods*. Mathematical Analysis and Numerical Methods for Science and TechnologySpringer Berlin Heidelberg; 1999.

20. Kohr M, Cristoforis M, Mikhailov SE, Wendland WL. Integral potential method for a transmission problem with Lipschitz interface in $\mathbb{R}^3$ for the Stokes and Darcy-Forchheimer-Brinkman PDE systems. *Z. Angew. Math. Phys.*. 2016;67(5):Art. 116, 30.

21. Costabel M. Boundary Integral Operators on Lipschitz Domains: Elementary Results. *SIAM Journal on Mathematical Analysis.* 1988;19(3):613-626.

22. Hellwig G. *Partial Differential Equations: An Introduction. Stuttgart.* B.G. Teubner; 1977.

23. Miranda C. *Partial differential equations of elliptic type. Translated by Zane C. Motteler*. Springer-Verlag Berlin, New York; 2d., rev. ed. ed.1970.

24. Mikhailov SE. Finite-dimensional perturbations of linear operators and some applications to boundary integral equations. *Engineering Analysis with Boundary Elements.* 1999;23:805–813.

How to cite this article: Tsegaye G. Ayele (2020), Analysis of Two-Operator Boundary-Domain Integral Equations for Variable Coefficient BVPs with General Data, *journal Math. Methods in Appl. Sci.*, *2020;00:1–32*.

APPENDIX

A MAPPING AND JUMP PROPERTIES OF THE VOLUME AND SURFACE POTENTIALS

The mapping properties of the parametrix-based volume and surface potentials formulated in Appendix A are proved or immediately follow from[5,11,12] (see also[4]).

Theorem 19. Let Ω be a bounded open three-dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary $\partial\Omega$. The operators

$$\mathcal{P}_b \;:\; \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R} \tag{A1}$$
$$:\; H^s(\Omega) \to H^{s+2}(\Omega), \quad s > -\frac{1}{2}, \tag{A2}$$
$$:\; H^s(\Omega) \to H^{s+2,0}(\Omega; L_a), \quad s \geq 0, \tag{A3}$$
$$\mathcal{R}_b, \mathcal{R}_*^b \;:\; \widetilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{A4}$$
$$:\; H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{A5}$$
$$:\; H^s(\Omega) \to H^{s+1,0}(\Omega; L_a), \quad s \geq 1, \tag{A6}$$
$$\gamma^+\mathcal{P}_b \;:\; \widetilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{3}{2}, \tag{A7}$$
$$:\; H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, \tag{A8}$$
$$\gamma^+\mathcal{R}_b \;:\; \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, \tag{A9}$$
$$:\; H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, \tag{A10}$$
$$T_a^+\mathcal{P}_b \;:\; \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, \tag{A11}$$
$$:\; H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, \tag{A12}$$
$$T_a^+\mathcal{R}_b \;:\; \widetilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2}, \tag{A13}$$
$$:\; H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2} \tag{A14}$$

are continuous and the operators

$$\mathcal{R}_b \;:\; H^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}, \tag{A15}$$
$$:\; H^s(\Omega) \to H^{s,0}(\Omega; A), \quad s > 1, \tag{A16}$$
$$\gamma^+\mathcal{R}_b \;:\; H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, \tag{A17}$$
$$T_a^+\mathcal{R}_b \;:\; H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad s > \frac{1}{2} \tag{A18}$$

are compact for any non-empty, open sub-manifold S_1 of $\partial\Omega$ with an infinitely smooth boundary.

Proof. For $a = b$, the mapping properties are stated and proved in Theorem 3.8 in[5] and Corollary B.3 in[4]. The case $a \neq b$ then follows by taking into account the relation $T_a^+ = \frac{a}{b} T_b^+$, for (A11)-(A14) and (A18). □

Theorem 20. The following operators are continuous

$$V_b \;:\; H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega), \quad s \in \mathbb{R}, \tag{A19}$$
$$W_b \;:\; H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega), \quad s \in \mathbb{R}, \tag{A20}$$
$$V_b \;:\; H^s(\partial\Omega) \to H^{s+\frac{3}{2},0}(\Omega, A), \quad s \geq -\frac{1}{2}, \tag{A21}$$
$$W_b \;:\; H^s(\partial\Omega) \to H^{s+\frac{1}{2},0}(\Omega, A), \quad s \geq \frac{1}{2}. \tag{A22}$$

Theorem 21. The operators

$$\mathcal{Z}_b \;:\; H^s(\Omega) \to H^s(\Omega), \quad s \geq 1 \tag{A23}$$
$$\widehat{\mathcal{Z}}_b \;:\; H^s(\Omega) \to H^{s,-\frac{1}{2}}(\Omega; B), \quad s > -\frac{1}{2} \tag{A24}$$

are continuous.

Proof. The proof follows from Theorems 19 and 20. □

Theorem 22. Let $s \in \mathbb{R}$. The following pseudodifferential operators are continuous

$$\begin{aligned} \mathcal{V}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega) \\ \mathcal{W}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega) \\ \mathcal{W}'_{ab} &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega) \\ \mathcal{L}^{\pm}_{ab} &: H^s(\partial\Omega) \to H^{s-1}(\partial\Omega). \end{aligned}$$

Due to the Rellich compact embedding theorem, Theorem 22 implies the following assertion.

Theorem 23. Let $s \in \mathbb{R}$. Let S_1 and S_2 with $\partial S_1, \partial S_2 \in C^\infty$ be nonempty open submanifolds of $\partial\Omega$. The operators

$$\begin{aligned} r_{S_2}\mathcal{V}_b &: \widetilde{H}^s(\partial\Omega) \to H^s(\partial\Omega) \\ r_{S_2}\mathcal{W}_b &: \widetilde{H}^s(\partial\Omega) \to H^s(\partial\Omega) \\ r_{S_2}\mathcal{W}'_{ab} &: \widetilde{H}^s(\partial\Omega) \to H^s(\partial\Omega) \end{aligned}$$

are compact.

Theorems 21, 22, 1, 23 and the Rellich embedding theorem imply the following assertion.

Theorem 24. The operator

$$\mathcal{V}_b : \widetilde{H}^{s-1}(\partial\Omega) \to H^s(\partial\Omega)$$

is continuously invertible for all $s \in \mathbb{R}$.

B REPRESENTATION LEMMAS

To prove invertibility of the BDIE operators we need the following representation statements.

Lemma 6 ([5], Lemma 5.13)**.** Let $\partial\Omega = \bar{S}_1 \cup \bar{S}_2$, where S_1 and S_2 are nonintersecting simply connected nonempty sub-manifolds of $\partial\Omega$ with infinitely smooth boundaries. For any triplet

$$\mathcal{F}_* = (F, \Psi, \Phi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2)$$

there exists a unique triplet

$$(f_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{S_1,S_2}\mathcal{F}_* \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} &\mathcal{P}_b f_* + V_b \Psi_* - W_b \Phi_* = F \quad in \ \ \Omega, \\ &r_{S_1} \Psi_* = \Psi \ \ , \\ &r_{S_2} \Phi_* = \Phi \ \ . \end{aligned}$$

Moreover, the operator

$$\widetilde{\mathcal{C}}_{S_1,S_2} : \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

The cases when $S_1 = \emptyset$ or $S_2 = \emptyset$ need to be considered separately. The following assertion is[3] Lemma 19 generalized to a wider space.

Lemma 7. For any function $\mathcal{F}_{\Phi_*} \in H^1(\Omega)$ there exists a unique couple $(\tilde{f}_*, \Phi_*) = \mathcal{C}_\Phi \mathcal{F}_{\Phi_*} \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that

$$\mathcal{F}_{\Phi_*} = \mathcal{P}_b \tilde{f}_* - W_b \Phi_* \quad \text{in } \Omega, \tag{B25}$$

$$T^+_a \mathcal{F}_{\Phi_*} = T^+_a\big(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b), \mathcal{F}_{\Phi_*}\big) \quad \text{on } \partial\Omega. \tag{B26}$$

Moreover, $(\tilde{f}_*, \Phi_*) = \mathcal{C}_{\Phi_*}\mathcal{F}_{\Phi_*}$ and $\mathcal{C}_{\Phi_*} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear and bounded operator given by

$$\tilde{f}_* = \check{\Delta}(b\mathcal{F}_{\Phi_*}) + \gamma^*(T^+_b \mathcal{F}_{\Phi_*} - (\gamma^+\mathcal{F}_{\Phi_*})\partial_n b) \tag{B27}$$

$$\Phi_* = \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\big\{-b\mathcal{F}_{\Phi_*} + \mathcal{P}_\Delta\big[\check{\Delta}(b\mathcal{F}_{\Phi_*}) + \gamma^*\big(T^+_b\mathcal{F}_{\Phi_*} - (\gamma^+\mathcal{F}_{\Phi_*})\partial_n b\big)\big]\big\} \tag{B28}$$

where $\check{\Delta}(b\mathcal{F}_{\Phi_*}) = \nabla\cdot\check{E}\nabla(b\mathcal{F}_{\Phi_*})$.

Proof. Let us first assume that there exist $(\tilde{f}_*,\Phi_*)\in\widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equation (B27) and find their expression in terms of $\mathcal{F}_{\Phi_*}$. Let us rewrite (B27) as

$$\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_* = -W_b\Phi_* \quad \text{in } \Omega. \tag{B29}$$

Multiplying (B29) by b and applying Laplacian to it, we obtain,

$$\Delta(b\mathcal{F}_{\Phi_*} - \mathcal{P}_\Delta\tilde{f}_*) = \Delta(b\mathcal{F}_{\Phi_*}) - \tilde{f}_* = -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \tag{B30}$$

which means

$$\Delta(b\mathcal{F}_{\Phi_*}) = r_\Omega\tilde{f}_* \quad \text{in } \Omega, \tag{B31}$$

and $b\mathcal{F}_{\Phi_*} - \mathcal{P}_\Delta\tilde{f}_* \in H^{1,0}(\Omega,\Delta)$ and hence $\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_* \in H^{1,0}(\Omega,B) = H^{1,0}(\Omega,A)$. The latter imply that the canonical co-normal derivatives $T_b^+(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)$ and $T_a^+(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)$ are well defined and can be also written in terms of their generalized co-normal derivatives

$$\begin{aligned}
\frac{b}{a}T_a^+(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*) &= T_b^+(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*) = T_b^+(\tilde{B}(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*),\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)\\
&= T_b^+(\mathring{E}\nabla\cdot(b\nabla(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)),\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)\\
&= T_b^+(\mathring{E}\Delta(b\mathcal{F}_{\Phi_*} - \mathcal{P}_\Delta\tilde{f}_*) - \mathring{E}\nabla\cdot((\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)\nabla b),\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)\\
&= T_b^+(-\mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)
\end{aligned}$$

where (59) and (B31) were taken into account. Hence,

$$T_a^+(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*) = T_a^+(-\mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\ \mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*), \tag{B32}$$

and using (8) Eq.(B32) can be written as

$$T_a^+(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*) = T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b),\mathcal{F}_{\Phi_*}) - T_a^+(\tilde{f}_* + \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\mathcal{P}_b\tilde{f}_*). \tag{B33}$$

Indeed,

$$\begin{aligned}
T_a^+(\mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*) &= T_a^+(-\mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\ \mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)\\
&= T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b) - \tilde{f}_* - \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\ \mathcal{F}_{\Phi_*} - \mathcal{P}_b\tilde{f}_*)\\
&= T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b),\mathcal{F}_{\Phi_*}) - T_a^+(\tilde{f}_* + \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\mathcal{P}_b\tilde{f}_*)
\end{aligned}$$

which is (B33).

Applying the co-normal derivative operator T_a^+ to both sides of equation (B29) and substituting their (B33), we obtain,

$$T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b),\ \mathcal{F}_{\Phi_*}) - T_a^+(\tilde{f}_* + \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\mathcal{P}_b\tilde{f}_*) = -\mathcal{L}_{ab}^+\Phi_*, \quad \text{on } \partial\Omega, \tag{B34}$$

and

$$T_a^+\mathcal{F}_{\Phi_*} = T_a^+(\tilde{f}_*,\mathcal{F}_{\Phi_*}) = T_a^+\big(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Phi_*}\nabla b),\mathcal{F}_{\Phi_*}\big) \quad \text{on } \partial\Omega, \tag{B35}$$

which is (B26). Due to (B31), we can represent

$$\tilde{f}_* = \check{\Delta}(b\mathcal{F}_{\Phi_*}) + \tilde{f}_{1*} = \nabla\cdot\mathring{E}\nabla(b\mathcal{F}_{\Phi_*}) - \gamma^*\Psi_{**} \tag{B36}$$

where $\tilde{f}_{1*}\in H^{-1}_{\partial\Omega}$ defined in (2) and hence, due to e.g.[15] Theorem 2.10 can be in turn represented as $\tilde{f}_{1*} = -\gamma^*\Psi_{**}$, with some $\Psi_{**}\in H^{-\frac{1}{2}}(\partial\Omega)$. Then (B31) is satisfied and hence (B35) reduces to

$$\Psi_{**} = -\frac{b}{a}T_a^+\mathcal{F}_{\Phi_*} - (\gamma^+\mathcal{F}_{\Phi_*})\partial_n b = T_b^+\mathcal{F}_{\Phi_*} - (\gamma^+\mathcal{F}_{\Phi_*})\partial_n b, \tag{B37}$$

and (B36) to (B27).

Now Eq. (B29) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{G}_\Delta \quad \text{in } \Omega, \tag{B38}$$

where

$$\mathcal{G}_\Delta := -b\mathcal{F}_{\Phi_*} + \mathcal{P}_\Delta\tilde{f}_* = -b\mathcal{F}_{\Phi_*} + \mathcal{P}_\Delta\big[\check{\Delta}(b\mathcal{F}_{\Phi_*}) + \gamma^*\big(T_b^+\mathcal{F}_{\Phi_*} - (\gamma^+\mathcal{F}_{\Phi_*})\partial_n b\big)\big] \tag{B39}$$

is harmonic function in Ω due to (B30). The trace of equation (B38) gives

$$-\frac{1}{2}b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+\mathcal{G}_\Delta \quad \text{on } \partial\Omega. \tag{B40}$$

Since the operator $-\frac{1}{2}I + \mathcal{W}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is an isomorphism (see e.g.[19] Ch.XI, Part B, §2, Remark 8 this implies

$$\begin{aligned}\Phi_* &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\mathcal{G}_\Delta\\ &= \frac{1}{b}\Big(-\frac{1}{2}I + \mathcal{W}_\Delta\Big)^{-1}\gamma^+\big\{-b\mathcal{F}_{\Phi_*} + \mathcal{P}_\Delta\big[\check{\Delta}(b\mathcal{F}_{\Phi_*}) + \gamma^*\big(T_b^+\mathcal{F}_{\Phi_*} - (\gamma^+\mathcal{F}_{\Phi_*})\partial_n b\big)\big]\big\},\end{aligned}$$

which is Eq. (B27). Evidently $\tilde{f}_*$ and Φ_* chosen in this manner satisfy Eqs. (B25) and (B26).

Considering a couple $(F,\Psi)^\top \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ and employing Lemma 7 for $\mathcal{F}_{\Phi*} = F - V_b\Psi \in H^1(\Omega)$, we arrive at the following statement.

Corollary 3. For any couple

$$(F,\Psi)^\top = \mathcal{F}_* \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$$

there exists a unique triplet

$$(\tilde{f}_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Psi*}\mathcal{F}_* \in \widetilde{H}^{-1}(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\mathcal{P}_b\tilde{f}_* + V_b\Psi_* - W_b\Phi_* = F \quad \text{in } \Omega, \qquad \Psi_* = \Psi \quad \text{on } \partial\Omega.$$

Moreover, the operator $\widetilde{\mathcal{C}}_{\Psi*} : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is linear and continuous.

Let us first present a generalized version of Lemma 5.5 in[4] to a wider space.

Lemma 8. For any function $\mathcal{F}_{\Psi*} \in H^1(\Omega)$, there exists a couple $(\tilde{f}_*, \Psi_*) = \mathcal{C}_{\Psi_*}\mathcal{F}_{\Psi*} \in \widetilde{H}^{-1}(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ such that

$$\mathcal{F}_{\Psi*} = \mathcal{P}_b\tilde{f}_* + V_b\Psi_*, \qquad \text{in } \Omega, \tag{B41}$$

$$T_a^+\mathcal{F}_{\Psi_*} = T_a^+(\tilde{f}_*, \mathcal{F}_{\Psi_*}) = T_a^+\big(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b), \mathcal{F}_{\Psi_*}\big) \quad \text{on } \partial\Omega. \tag{B42}$$

Moreover, $(\tilde{f}_*, \Phi_*) = \mathcal{C}_{\Psi_*}\mathcal{F}_{\Psi_*}$ and $\mathcal{C}_{\Psi_*} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is a linear and bounded operator given by

$$\tilde{f}_* = \check{\Delta}(b\mathcal{F}_{\Psi_*}) + \gamma^*(T_b^+\mathcal{F}_{\Psi_*} - (\gamma^+\mathcal{F}_{\Psi_*})\partial_n b) \tag{B43}$$

$$\Psi_* = \mathcal{V}_\Delta^{-1}\gamma^+\big\{-b\mathcal{F}_{\Psi_*} + \mathcal{P}_\Delta\big\{\check{\Delta}(b\mathcal{F}_{\Psi_*}) + \gamma^*\big(T_b^+\mathcal{F}_{\Psi_*} - (\gamma^+\mathcal{F}_{\Psi_*})\partial_n b\big)\big\} \qquad \text{on } \partial\Omega, \tag{B44}$$

where $\check{\Delta}(b\mathcal{F}_{\Psi_*}) = \nabla\cdot\check{E}\nabla(b\mathcal{F}_{\Psi_*})$.

Proof. Let us first assume that there exist $(\tilde{f}_*, \Psi_*) \in \widetilde{H}^{-1}(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ satisfying Eq. (B41) and find their expression in terms of $\mathcal{F}_{\Psi_*}$. Let us rewrite (B41) as

$$\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_* = V_b\Psi_* \quad \text{in } \Omega. \tag{B45}$$

Multiplying (B45) by b and applying Laplacian to it, we obtain,

$$\Delta(b\mathcal{F}_{\Psi_*} - \mathcal{P}_\Delta\tilde{f}_*) = \Delta(b\mathcal{F}_{\Psi_*}) - \tilde{f}_* = \Delta(V_\Delta\Psi_*) = 0 \quad \text{in } \Omega, \tag{B46}$$

which means

$$\Delta(b\mathcal{F}_{\Psi_*}) = r_\Omega\tilde{f}_* \quad \text{in } \Omega, \tag{B47}$$

and $b\mathcal{F}_{\Psi_*} - \mathcal{P}_\Delta\tilde{f}_* \in H^{1,0}(\Omega,\Delta)$ and hence $\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_* \in H^{1,0}(\Omega,B) = H^{1,0}(\Omega,A)$. The latter imply that the canonical co-normal derivatives $T_b^+(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)$ and $T_a^+(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)$ are well defined and can be also written in terms of their generalized co-normal derivatives

$$\begin{aligned}\frac{b}{a}T_a^+(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*) &= T_b^+(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*) = T_b^+(\tilde{B}(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*), \mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)\\ &= T_b^+(\mathring{E}\nabla\cdot(b\nabla(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)), \mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)\\ &= T_b^+(\mathring{E}\Delta(b\mathcal{F}_{\Psi_*} - \mathcal{P}_\Delta\tilde{f}_*) - \mathring{E}\nabla\cdot((\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)\nabla b), \mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)\\ &= T_b^+(-\mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_*, \mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*)\end{aligned}$$

where (59) and (B47) were taken into account. Hence,

$$T_a^+(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*) = T_a^+(-\mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\ \mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*), \tag{B48}$$

and using (8) Eq.(B48) can be written as

$$T_a^+(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*) = T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b), \mathcal{F}_{\Psi_*}) - T_a^+(\tilde{f}_* + \mathring{E}\mathcal{R}_*^b\tilde{f}_*, \mathcal{P}_b\tilde{f}_*). \tag{B49}$$

Indeed,

$$\begin{aligned} T_a^+(\mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*) &= T_a^+(-\mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b) - \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\ \mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*) \\ &= T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b) - \tilde{f}_* - \mathring{E}\mathcal{R}_*^b\tilde{f}_*,\ \mathcal{F}_{\Psi_*} - \mathcal{P}_b\tilde{f}_*) \\ &= T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b), \mathcal{F}_{\Psi_*}) - T_a^+(\tilde{f}_* + \mathring{E}\mathcal{R}_*^b\tilde{f}_*, \mathcal{P}_b\tilde{f}_*) \end{aligned}$$

which is (B49).

Applying the co-normal derivative operator T_a^+ to both sides of equation (B45) and substituting their (B49), we obtain,

$$T_a^+(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b),\ \mathcal{F}_{\Psi_*}) - T_a^+(\tilde{f}_* + \mathring{E}\mathcal{R}_*^b\tilde{f}_*, \mathcal{P}_b\tilde{f}_*) = \frac{a}{2b}\Psi_* + \mathcal{W}'_{ab}\Psi_*, \quad \text{on } \partial\Omega, \tag{B50}$$

and

$$T_a^+\mathcal{F}_{\Psi_*} = T_a^+(\tilde{f}_*, \mathcal{F}_{\Psi_*}) = T_a^+\big(\tilde{f}_* - \mathring{E}\nabla\cdot(\mathcal{F}_{\Psi_*}\nabla b), \mathcal{F}_{\Psi_*}\big) \quad \text{on } \partial\Omega, \tag{B51}$$

which is (B42). Due to (B47), we can represent

$$\tilde{f}_* = \check{\Delta}(b\mathcal{F}_{\Psi_*}) + \tilde{f}_{1*} = \nabla\cdot\mathring{E}\nabla(b\mathcal{F}_{\Psi_*}) - \gamma^*\Psi_{**} \tag{B52}$$

where $\tilde{f}_{1*} \in H^{-1}_{\partial\Omega}$ defined in (2) and hence, due to e.g.[15] Theorem 2.10 can be in turn represented as $\tilde{f}_{1*} = -\gamma^*\Psi_{**}$, with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$. Then (B47) is satisfied and hence due to Lemma 3, Eq. (B51) reduces to

$$\Psi_{**} = -\frac{b}{a}T_a^+\mathcal{F}_{\Psi_*} - (\gamma^+\mathcal{F}_{\Psi_*})\partial_n b = T_b^+\mathcal{F}_{\Psi_*} - (\gamma^+\mathcal{F}_{\Psi_*})\partial_n b \tag{B53}$$

and (B52) to (B41). Now Eq. (B45) can be written in the form

$$V_\Delta\Psi_* = \mathcal{H}_\Delta \quad \text{in } \Omega, \tag{B54}$$

where

$$\mathcal{H}_\Delta := -b\mathcal{F}_{\Psi_*} + \mathcal{P}_\Delta\tilde{f}_* = -b\mathcal{F}_{\Psi_*} + \mathcal{P}_\Delta\big[\check{\Delta}(b\mathcal{F}_{\Psi_*}) + \gamma^*\big(T_b^+\mathcal{F}_{\Psi_*} - (\gamma^+\mathcal{F}_{\Psi_*})\partial_n b\big)\big] \tag{B55}$$

is harmonic function in Ω due to (B46). The trace of equation (B54) gives

$$\mathcal{V}_\Delta\Psi_* = \gamma^+\mathcal{H}_\Delta \quad \text{on } \partial\Omega. \tag{B56}$$

Since $\mathcal{V}_\Delta$ is an isomorphism in $H^{-\frac{1}{2}}(\partial\Omega)$,

$$\begin{aligned} \Psi_* &= \mathcal{V}_\Delta^{-1}\gamma^+\mathcal{H}_\Delta \\ &= \mathcal{V}_\Delta^{-1}\gamma^+\big\{-b\mathcal{F}_{\Psi_*} + \mathcal{P}_\Delta\big\{\check{\Delta}(b\mathcal{F}_{\Psi_*}) + \gamma^*\big(T_b^+\mathcal{F}_{\Psi_*} - (\gamma^+\mathcal{F}_{\Psi_*})\partial_n b\big)\big\} \qquad \text{on } \partial\Omega, \end{aligned}$$

which is (B44). Now we have to prove that Ψ_* given by (B44) with $\mathcal{H}_\Delta$ in (B55) and $\tilde{f}_*$ by (B43) do satisfy (B41). The potential $V_\Delta\Psi_*$ with Ψ_* given by (B44) is a harmonic function, and one can check that $\mathcal{H}_\Delta$ given by (B55) is also harmonic. Since (B56) implies that they coincide on the boundary, the two harmonic functions should coincide also in the domain, i.e. (B54) holds true, which implies (B41). □

Considering a couple $(F, \Phi)^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ and employing Lemma 8 for $\mathcal{F}_{\Psi_*} = F + W_b\Phi \in H^1(\Omega)$, we arrive at the following statement.

Corollary 4. For any couple

$$(F, \Phi)^\top = \mathcal{F}_* \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

there exists a triple

$$(\tilde{f}_*, \Psi_*, \Phi_*)^\top = \widetilde{\mathcal{C}}_{\Phi *}\mathcal{F}_* \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\mathcal{P}_b\tilde{f}_* + V_b\Psi_* - W_b\Phi_* = F \ \text{ in } \Omega^+, \quad \Phi_* = \Phi \quad \text{on } \partial\Omega.$$

Moreover, the operator $\widetilde{\mathcal{C}}_{\Phi *} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is linear and continuous.