(ω,c)-asymptotically periodic solutions to some fractional integro-differential equation

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Abstract

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(ω, c)-asymptotically periodic solutions to some fractional integro-differential equation

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July 30, 2021

Abstract

In this paper, we establish a new composition theorem for (ω, c)-asymptotically periodic functions. Then, we use the Banach contraction principle to investigate the existence and uniqueness of (ω, c)-asymptotically periodic mild solutions to the fractional integro-differential equation $u'(t) = \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-\tau)^{\alpha-2} Au(\tau)d\tau + F(t, u_t)$, $t \geq 0$ and $u_0 = \phi \in \mathcal{B}(\mathbb{X})$, where $\mathcal{B}(\mathbb{X})$ is a linear space of functions defined from $(-\infty, 0] \longrightarrow \mathbb{X}$ and A is a closed but not necessarily bounded linear operator of sectorial type $\varpi < 0$.

Key-words: (ω, c)-asymptotically periodic function, mild solution, Banach space, Banach contraction principle.

Mathematics Subject Classification (2020): 35B10; 46E15; 47D06; 47J35; 93D22.

1 Introduction.

Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|_{\mathbb{X}}$, and let $A : D(A) \subseteq \mathbb{X} \longrightarrow \mathbb{X}$ be a closed (not necessarily bounded) linear operator of sectorial type $\varpi < 0$. We consider the following semilinear fractional integro-differential equation:

$$\begin{cases} u'(t) &= \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-\tau)^{\alpha-2} Au(\tau)d\tau + F(t, u_t), \quad t \geq 0 \\ \\ u_0 &= \phi, \end{cases} \tag{1}$$

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where $1 < \alpha < 2$, the Euler's Gamma function is given by

$$\Gamma(\sigma) = \int_0^{\infty} t^{\sigma-1} e^{-t} dt \quad \text{for} \quad \sigma > 0,$$

ϕ belongs to a space $\mathcal{B}(\mathbb{X})$ to be specified later, $F : \mathbb{R}^+ \times \mathcal{B}(\mathbb{X}) \longrightarrow \mathbb{X}$ is a jointly continuous function. For any $u : \mathbb{R} \longrightarrow \mathbb{X}$, the associated history function $t \longrightarrow u_t$ for $t \geq 0$ is defined as

$$\begin{array}{llll} u_t : & (-\infty, 0] & \longrightarrow & \mathbb{X} \\ & \theta & \longmapsto & u_t(\theta) = u(t+\theta). \end{array}$$

One of the most attractive topics in qualitative theory is the study of the existence of periodic-type solutions to differential equations; this is due to the mathematics interest of qualitative theory and applications in many scientific fields, such as physics, biology and control theory. However, some phenomena in real world are not necessarily periodic, but rather asymptotically periodic.

A periodic function is a function that repeats its values at regular intervals, and is used throughout science to describe oscillations, waves, and other phenomena that exhibit periodicity (see for example [10]). The concept of (ω, c)-asymptotically periodic functions was introduced in 2019 by Edgardo Alvarez et al. [4]. The authors proved that the set of such functions, denoted $AP_{\omega c}(\mathbb{X})$ is a Banach space with the norm $\|h\|_{a\omega c} := \sup\limits_{t \in \mathbb{R}^+} \||c|^\wedge(-t) h(t)\|_{\mathbb{X}}$. A continuous function h is said to be (ω, c)-asymptotically periodic if it can be written as $h = h_1 + h_2$, where h_1 is an (ω, c)-periodic function and h_2 is c-asymptotic. Prior to this new theory, the authors introduced in 2018 the notion of (ω, c)-periodicity [5] and since then, it attracted many researchers.

In 2018, Mengmeng Li et al. [19] proved the existence of (ω, c)-periodic solutions for a nonhomogeneous linear impulsive system by constructing Green functions and adjoint systems, respectively. In the same paper, they studied the existence and uniqueness of (ω, c)-periodic solutions for a semilinear impulsive system via fixed point approach. Recently in 2020, Gisèle Mophou and Gaston M. N'Guérékata [23] studied an existence result of (ω, c)-periodic mild solutions to some fractional differential equation with order $1 < \alpha < 2$. For more results on the (ω, c)-periodicity of solutions we refer to [1, 3, 4, 18, 20, 31] and the references therein.

In 2013, Gisèle Mophou et al. [24] studied the asymptotically anti-periodic mild solutions to the fractional integro-differential equation (1) in Banach spaces. For more results on the asymptotic behavior of solutions of some class of evolutionary systems, see [2, 7, 8, 9, 13, 14, 15, 22, 25, 26, 27, 28, 32, 33]. The main purpose of this paper is to study the existence and uniqueness of (ω, c)-asymptotically periodic mild solution for the system (1) where $c \in \mathbb{C} \backslash \{0\}$ and $\omega > 0$. When $c = 1$, we talk about ω-asymptotically periodicity and several results have been obtained in this case (see [6, 21, 29]) whereas for $c = -1$, we have the asymptotically anti-periodicity, also widely studied (see [11, 17, 24]).

The rest of this paper is organized as follows. Section 2 is devoted to some preliminary results. In particular, we formalize the (ω, c)-asymptotically periodic functions and give some important properties. In Section 3, we establish some important propositions and use the Banach contraction principle to prove that there exists a unique (ω, c)-asymptotically periodic mild solution to the integro-differential equation (1). Finally, Section 4 concludes this work.

2 Some Preliminary Results.

Throughout this paper, $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ will denote a complex Banach space. We denote by

$$\mathcal{C}(\mathbb{R}, \mathbb{X}) := \{h : \mathbb{R} \longrightarrow \mathbb{X} \text{ such that } h \text{ is continuous}\}$$

the space of all $\mathbb{X}$-valued continuous functions on $\mathbb{R}$,

$$\mathcal{BC}(\mathbb{R}, \mathbb{X}) := \{h : \mathbb{R} \longrightarrow \mathbb{X} \text{ such that } h \text{ is bounded and continuous}\}$$

the space of all $\mathbb{X}$-valued bounded and continuous functions on $\mathbb{R}$ and we let

$$C_0(\mathbb{X}) := \{h \in \mathcal{BC}(\mathbb{R}, \mathbb{X}) \text{ such that } \lim_{t \longrightarrow \infty} h(t) = 0\}.$$

In order to describe the phase space, we follow the idea of G. Mophou et al. in [24]. We denote by $(\mathcal{B}(\mathbb{X}), \|\cdot\|_{\mathcal{B}(\mathbb{X})})$ a seminormed linear space of functions defined from $(-\infty, 0] \longrightarrow \mathbb{X}$ satisfying the following fundamental axioms due to Kato and Hale:

(P_1) If $x : (-\infty, T]$ is continuous on $[0, T]$ and $x_0 \in \mathcal{B}(\mathbb{X})$, then for every $t \in [0, T]$ the following conditions hold:

(a) $x_t \in \mathcal{B}(\mathbb{X})$

(b) $\|x(t)\|_{\mathbb{X}} \leq L\|x_t\|_{\mathcal{B}(\mathbb{X})}$

(c) $\|x_t\|_{\mathcal{B}(\mathbb{X})} \leq C_1(t) \sup_{\tau \in [0,t]} \|x(\tau)\|_{\mathbb{X}} + C_2(t)\|x_0\|_{\mathcal{B}(\mathbb{X})}$,

where $L \geq 0$ is a constant, $C_1 : [0, \infty) \longrightarrow [0, \infty)$ is continuous, $C_2 : [0, \infty) \longrightarrow [0, \infty)$ is locally bounded and L, C_1, C_2 are independent of $x(\cdot)$.

(P_2) For the function $x(\cdot)$ in (P_1), x_t is a $\mathcal{B}(\mathbb{X})$-valued continuous function on $[0, T]$.

(P_3) The space $\mathcal{B}(\mathbb{X})$ is complete.

Remark 2.1 *Note that from (P_1)-(b), we have that*

$$\|x(0)\|_{\mathbb{X}} \leq L\|x_0\|_{\mathcal{B}(\mathbb{X})}. \tag{2}$$

Definition 2.1 *If $x : \mathbb{R} \longrightarrow \mathbb{X}$ is a continuous function on $[\sigma, \infty)$ with $x_\sigma \in \mathcal{B}(\mathbb{X})$ for some $\sigma \in \mathbb{R}$ such that $\|x(t)\|_{\mathbb{X}} \longrightarrow 0$ as $t \longrightarrow \infty$, then $\|x_t\|_{\mathcal{B}(\mathbb{X})} \longrightarrow 0$ as $t \longrightarrow \infty$. In this case, $\mathcal{B}(\mathbb{X})$ is called a fading memory.*

We recall the following definitions and results from M. Pinto et *al.* [4, 5].

Definition 2.2 *[5] A function $h \in \mathcal{C}(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-periodic if there exist $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$ such that*

$$h(t + \omega) \;=\; ch(t) \quad \forall\ t \in \mathbb{R}. \tag{3}$$

Remark 2.2 *When (3) is satisfied, ω is called the c-period of h and we denote by $P_{\omega c}(\mathbb{X})$, the collection of all functions $h \in \mathcal{C}(\mathbb{R}, \mathbb{X})$ which are (ω, c)-periodic. Endowed with the norm*

$$\|h\|_{\omega c} := \sup_{t\in[0,\omega]} \||c|^\wedge(-t)h(t)\|_{\mathbb{X}},$$

$P_{\omega c}(\mathbb{X})$ is a Banach space, where $|c|^\wedge(-t) = |c|^{-t/\omega}$ (see [5]).

We have the following result which give a characterization of an (ω, c)-periodic function.

Proposition 2.1 *[5] Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. A function $h \in \mathcal{C}(\mathbb{R}, \mathbb{X})$ is (ω, c)-periodic if and only if*

$$h(t) = c^\wedge(t)v(t), \tag{4}$$

where $c^\wedge(t) = c^{t/\omega}$ and $v \in P_\omega(\mathbb{X})$ is called the periodic part of h.

Before giving the definition of an (ω, c)-asymptotically periodic function, we need to define the so-called c-asymptotic function.

Definition 2.3 *[4] Let $c \in \mathbb{C}\backslash\{0\}$. A function $h \in \mathcal{C}(\mathbb{R}, \mathbb{X})$ is said to be c-asymptotic if $c^\wedge(-t)h(t) \in C_0(\mathbb{X})$; that is,*

$$\lim_{t\longrightarrow\infty} c^\wedge(-t)h(t) = 0.$$

The collection of those functions will be denoted by $C_{0,c}(\mathbb{X})$.

Definition 2.4 *[4] Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. A function $h \in \mathcal{C}(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-asymptotically periodic if $h = h_1 + h_2$ where $h_1 \in P_{\omega c}(\mathbb{X})$ and $h_2 \in C_{0,c}(\mathbb{X})$. The collection of those functions (with the same period ω for the first component) will be denoted by $AP_{\omega c}(\mathbb{X})$.*

Endowed with the norm

$$\|h\|_{a\omega c} := \sup_{t\in\mathbb{R}^+} \||c|^\wedge(-t)h(t)\|_{\mathbb{X}}, \tag{5}$$

$AP_{\omega c}(\mathbb{X})$ is a Banach space, where $|c|^\wedge(-t) = |c|^{-t/\omega}$ $(See[4])$.
As for the (ω, c)-periodic function, we have the following characterization of an (ω, c)-asymptotically periodic function.

Proposition 2.2 *[4] Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. A function $h \in \mathcal{C}(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-asymptotically periodic if and only if*

$$h(t) = c^\wedge(t)v(t), \tag{6}$$

where $c^\wedge(t) = c^{t/\omega}$ and $v \in AP_{\omega c}(\mathbb{X})$.

We recall the following results on the Nemytskii's superposition operator on (ω, c)- periodic functions obtained in [5].

Theorem 2.1 *Let $F : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$ be a continuous function and $(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C}\backslash\{0\})$ given. For $\varphi \in P_{\omega c}(\mathbb{X})$, if*

$$\mathcal{N}(\varphi)(\cdot) = F(\cdot, \varphi(\cdot))$$

denotes the Nemytskii's superposition operator, then the following are equivalent:

(i) For every $\varphi \in P_{\omega c}(\mathbb{X})$ we have that $\mathcal{N}(\varphi) \in P_{\omega c}(\mathbb{X})$;

(ii) $F(t + \omega, cx) = cF(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{X}$.

Then, we state and prove the following result.

Theorem 2.2 *Let $F : \mathbb{R} \times \mathcal{B}(\mathbb{X}) \longrightarrow \mathbb{X}$ be a continuous function such that*

(H_1) $\exists(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C}\backslash\{0\})$ such that $F(t + \omega, cx) = cF(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathcal{B}(\mathbb{X})$;

(H_2) $\exists K > 0$ such that $\|F(t, x) - F(t, y)\|_{\mathbb{X}} \leq K\|x - y\|_{\mathcal{B}(\mathbb{X})}$ for all $(t, x, y) \in \mathbb{R} \times \mathcal{B}(\mathbb{X}) \times \mathcal{B}(\mathbb{X})$.

Then

$$\mathcal{N}(AP_{\omega c}(\mathcal{B}(\mathbb{X}))) \ \subset \ AP_{\omega c}(\mathbb{X}), \tag{7}$$

where

$$\mathcal{N}(\varphi)(\cdot) := F(\cdot, \varphi(\cdot))$$

denotes the Nemytskii's superposition operator.

Proof. Let $\varphi \in AP_{\omega c}(\mathcal{B}(\mathbb{X}))$. Then there exist $\varphi_1 \in P_{\omega c}(\mathcal{B}(\mathbb{X}))$ and $\varphi_2 \in C_{0,c}(\mathcal{B}(\mathbb{X}))$ such that $\varphi = \varphi_1 + \varphi_2$. We note that

$$\mathcal{N}(\varphi)(\cdot) = \mathcal{N}(\varphi_1)(\cdot) + \mathcal{N}(\varphi)(\cdot) - \mathcal{N}(\varphi_1)(\cdot). \tag{8}$$

According to the hypothesis (H_2) we have:

$$\begin{aligned}\|c^\wedge(-t)\left(\mathcal{N}(\varphi)(t)-\mathcal{N}(\varphi_1)(t)\right)\|_{\mathbb{X}} &= |c^\wedge(-t)|\,\|(F(t,\varphi(t))-F(t,\varphi_1(t)))\|_{\mathbb{X}}\\ &\leq K\,|c^\wedge(-t)|\,\|\varphi(t)-\varphi_1(t)\|_{\mathcal{B}(\mathbb{X})}\\ &= K\,\|c^\wedge(-t)\varphi_2(t)\|_{\mathcal{B}(\mathbb{X})}\ \longrightarrow 0 \text{ as } t\longrightarrow\infty\end{aligned}$$

because $\varphi_2 \in C_{0,c}(\mathcal{B}(\mathbb{X}))$. So, $\lim\limits_{t\longrightarrow\infty} c^\wedge(-t)\left(\mathcal{N}(\varphi)(t)-\mathcal{N}(\varphi_1)(t)\right) = 0$ and we deduce that

$$\mathcal{N}(\varphi)(\cdot)-\mathcal{N}(\varphi_1)(\cdot)\in C_{0,c}(\mathbb{X}). \tag{9}$$

In addition, according to Theorem 2.1 we have

$$\mathcal{N}(\varphi_1)(\cdot)\in P_{\omega c}(\mathbb{X}). \tag{10}$$

Hence, from (8), (9) and (10) we deduce that $\mathcal{N}(\varphi)(\cdot)\in AP_{\omega c}(\mathbb{X})$ and this concludes the proof of our theorem. ■

Now, let's recall some definitions of sectorial type operator and its generated solution operator.

Definition 2.5 *[12, 24] A closed linear operator A with domain $D(A)$ dense in a Banach space $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ is said to be sectorial of type ϖ and angle θ if there exist constants ϖ, $M>0$ and an angle $\theta\in\left(0,\dfrac{\pi}{2}\right)$ such that its resolvent exists outside the sector*

$$\varpi+S_\theta := \left\{\lambda+\varpi \text{ such that } \lambda\in\mathbb{C} \text{ and } |arg(-\lambda)|<\theta\right\}, \tag{11}$$

$$\|(\lambda-A)^{-1}\|_{\mathbb{X}} \leq \frac{M}{|\lambda-\varpi|},\quad \lambda\notin\varpi+S_\theta. \tag{12}$$

Definition 2.6 *[12, 24] Let $\alpha>0$ and A be a closed linear operator densely defined in $\mathbb{X}$. Let $\rho(A)$ be the resolvent set of A. Then A is called the generator of a solution operator if there exists $\varpi\in\mathbb{R}$ and a strongly continuous function $E_\alpha:\mathbb{R}^+\longrightarrow\mathcal{B}(\mathbb{X})$ such that $\{\lambda^\alpha : Re\lambda>\varpi\}\ \subset\ \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}x=\int_0^\infty e^{-\lambda t}E_\alpha(t)x\,dt,\quad Re\lambda>\varpi,\ x\in\mathbb{X}. \tag{13}$$

In this case, E_α is called the solution operator generated by A.
If we assume that A is sectorial with $0\leq\theta\leq\pi\left(1-\dfrac{\alpha}{2}\right)$, then A is the generator of the following solution operator:

$$E_\alpha(t)=\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}d\lambda,\quad t\geq 0, \tag{14}$$

where γ is a suitable path outside the sector $\varpi+S_\theta$.

Proposition 2.3 *[12, 24] Let $1 < \alpha < 2$ and let $A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ be a sectorial operator in a complex Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, satisfying (11) and (12) for some $M > 0$, $\varpi < 0$ and $0 \leq \theta < \pi\left(1 - \dfrac{\alpha}{2}\right)$. Then there exists $\Lambda(\theta, \alpha) > 0$ depending only on θ and α such that*

$$\|E_\alpha(t)\|_{\mathcal{L}(\mathbb{X})} \leq \frac{\Lambda(\theta,\alpha)M}{1 + |\varpi| t^\alpha}, \quad t \geq 0. \tag{15}$$

To prove the existence and uniqueness of (ω, c)-asymptotically periodic solution to (1), we use the following fixed-point theorem.

Theorem 2.3 *[16, 30] (Banach contraction principle) Assume (U, d) to be a non-empty complete metric space, let $0 \leq K < 1$ and let the mapping $F : U \longrightarrow U$ satisfy the inequality*

$$d(Fu, Fv) \leq K d(u, v) \quad \textit{for every} \quad u, v \in U.$$

Then, F has a unique fixed point; that is, there exists a unique $u^* \in U$ such that $Fu^* = u^*$. Furthermore, for any $u_0 \in U$, the sequence $\left(F^j u_0\right)_{j=1}^{\infty}$ converges to the fixed point u^*.

3 Existence and Uniqueness Result.

In this section, we give the definition of a mild solution to (1) and prove under suitable assumptions via the Banach contraction principle that (1) has a unique (ω, c)-asymptotically periodic solution.

Definition 3.1 *[24] A function $u \in \mathcal{BC}(\mathbb{R}, \mathbb{X})$ is said to be a mild solution to problem (1) if it satisfies the following:*

$$u(t) = \begin{cases} E_\alpha(t)\phi(0) + \displaystyle\int_0^t E_\alpha(t-\tau)F(\tau, u_\tau)d\tau, & t \in \mathbb{R}^+, \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases} \tag{16}$$

Proposition 3.1 *Let $u \in P_{\omega c}(\mathbb{X})$. Then the function*

$$v(t) := \int_{-\infty}^t E_\alpha(t-\tau)u(\tau)d\tau$$

belongs to $P_{\omega c}(\mathbb{X})$.

Proof. Let $\omega > 0$ be given. We have

$$v(t+\omega) = \int_{-\infty}^{t+\omega} E_\alpha(t+\omega-\tau)u(\tau)d\tau.$$

By the change of variable $z = \tau - \omega$ we have

$$v(t+\omega) = \int_{-\infty}^{t} E_\alpha(t-z)u(z+\omega)dz. \tag{17}$$

Since u is an (ω, c)-periodic function, then there exists $c \in \mathbb{C}\backslash\{0\}$ such that $u(z+\omega) = cu(z)$. So, (17) becomes

$$\begin{aligned} v(t+\omega) &= c\int_{-\infty}^{t} E_\alpha(t-z)u(z)dz \\ &= cv(t). \end{aligned}$$

Therefore $v \in P_{\omega c}(\mathbb{X})$. ■

Proposition 3.2 *Let $u \in AP_{\omega c}(\mathbb{X})$. Then the function*

$$v(t) := \int_0^t E_\alpha(t-\tau)u(\tau)d\tau$$

belongs to $AP_{\omega c}(\mathbb{X})$ if $|c| \geq 1$.

Proof. By definition, $u \in AP_{\omega c}(\mathbb{X})$ means that there exists $u_1 \in P_{\omega c}(\mathbb{X})$ and $u_2 \in C_{0,c}(\mathbb{X})$ such that $u = u_1 + u_2$. Then,

$$\begin{aligned} v(t) &= \int_0^t E_\alpha(t-\tau)(u_1(\tau)+u_2(\tau))d\tau \\ &= \int_0^t E_\alpha(t-\tau)u_1(\tau)d\tau + \int_0^t E_\alpha(t-\tau)u_2(\tau)d\tau \\ &= \int_{-\infty}^t E_\alpha(t-\tau)u_1(\tau)d\tau - \int_{-\infty}^0 E_\alpha(t-\tau)u_1(\tau)d\tau + \int_0^t E_\alpha(t-\tau)u_2(\tau)d\tau \\ &= v_1(t) + v_2(t), \end{aligned}$$

where

$$\begin{aligned} v_1(t) &= \int_{-\infty}^t E_\alpha(t-\tau)u_1(\tau)d\tau, \\ v_2(t) &= \int_0^t E_\alpha(t-\tau)u_2(\tau)d\tau - \int_{-\infty}^0 E_\alpha(t-\tau)u_1(\tau)d\tau. \end{aligned}$$

By Proposition 3.1 we have clearly $v_1 \in P_{\omega c}(\mathbb{X})$. Now, we need to prove that $v_2 \in C_{0,c}(\mathbb{X})$.
Let $t > 0$ and $\varepsilon > 0$ be given. Since $u_2 \in C_{0,c}(\mathbb{X})$,

$$\exists T > 0 : \forall \tau > T,\ \|c^\wedge(-\tau)u_2(\tau)\|_{\mathbb{X}} < \varepsilon. \tag{18}$$

Actually, to prove that $v_2 \in C_{0,c}(\mathbb{X})$, we need to consider two cases: $t > T$ and $t < -T$. We will give the proof for $t > T$ since the second case ($t < -T$) can be obtained by using similar arguments.
So, for $t > T$ we have

$$v_2(t) = \int_0^T E_\alpha(t-\tau)u_2(\tau)d\tau + \int_T^t E_\alpha(t-\tau)u_2(\tau)d\tau - \int_{-\infty}^0 E_\alpha(t-\tau)u_1(\tau)d\tau \tag{19}$$

and

$$\|c^\wedge(-t)v_2(t)\|_{\mathbb{X}} \leq \sum_{i=1}^{3} I_i(t), \tag{20}$$

where

$$\begin{aligned}
I_1(t) &= \left\| c^\wedge(-t)\int_0^T E_\alpha(t-\tau)u_2(\tau)d\tau \right\|_{\mathbb{X}};\\
I_2(t) &= \left\| c^\wedge(-t)\int_T^t E_\alpha(t-\tau)u_2(\tau)d\tau \right\|_{\mathbb{X}};\\
I_3(t) &= \left\| c^\wedge(-t)\int_{-\infty}^0 E_\alpha(t-\tau)u_1(\tau)d\tau \right\|_{\mathbb{X}}.
\end{aligned}$$

Now, we have:

$$\begin{aligned}
I_1(t) &= \left\| \int_0^T c^{-(t-\tau)/\omega} E_\alpha(t-\tau) c^{-\tau/\omega} u_2(\tau)d\tau \right\|_{\mathbb{X}}\\
&\leq \int_0^T \left| c^{-(t-\tau)/\omega} \right| \|E_\alpha(t-\tau)\|_{\mathcal{L}(\mathbb{X})} \left\| c^{-\tau/\omega}u_2(\tau) \right\|_{\mathbb{X}} d\tau
\end{aligned}$$

We note that $t > T$, if $|c| \geq 1$ and $0 \leq \tau \leq T$ implies that $\left|c^{-(t-\tau)/\omega}\right| \leq 1$. So, taking into account (15) we obtain

$$\begin{aligned}
I_1(t) &\leq \sup_{0\leq\tau\leq T}\left\{ \left\| c^{-\tau/\omega}u_2(\tau) \right\|_{\mathbb{X}} \right\} \int_0^T \frac{\Lambda(\theta,\alpha)M}{1+|\varpi|(t-\tau)^\alpha} d\tau\\
&\leq \sup_{0\leq\tau\leq T}\left\{ \left\| c^{-\tau/\omega}u_2(\tau) \right\|_{\mathbb{X}} \right\} \int_0^T \frac{\Lambda(\theta,\alpha)M}{|\varpi|(t-\tau)^\alpha} d\tau\\
&= \frac{\Lambda(\theta,\alpha)M}{|\varpi|} \sup_{0\leq\tau\leq T}\left\{ \left\| c^{-\tau/\omega}u_2(\tau) \right\|_{\mathbb{X}} \right\} \int_0^T (t-\tau)^{-\alpha} d\tau\\
&= \frac{\Lambda(\theta,\alpha)M}{|\varpi|(\alpha-1)} \left[\frac{1}{(t-T)^{\alpha-1}} - \frac{1}{t^{\alpha-1}} \right] \sup_{0\leq\tau\leq T}\left\{ \left\| c^{-\tau/\omega}u_2(\tau) \right\|_{\mathbb{X}} \right\}.
\end{aligned}$$

Thus,

$$I_1(t) \leq \frac{\Lambda(\theta,\alpha)M}{|\varpi|(\alpha-1)} \left[\frac{1}{(t-T)^{\alpha-1}} - \frac{1}{t^{\alpha-1}} \right] \sup_{0\leq\tau\leq T}\left\{ \left\| c^{-\tau/\omega}u_2(\tau) \right\|_{\mathbb{X}} \right\}.$$

Observing on the one hand that,

$$\sup_{0\leq\tau\leq T} \left\| c^{-\tau/\omega}u_2(\tau) \right\|_{\mathbb{X}} < \infty$$

(because $u_2 \in C_{0,c}(\mathbb{X})$), and on the other hand that,

$$\lim_{t\to\infty} \left[\frac{1}{(t-T)^{\alpha-1}} - \frac{1}{t^{\alpha-1}} \right] = 0,$$

we deduce that

$$\lim_{t\to\infty} I_1(t) = 0$$

Using (15), we have also

$$\begin{aligned}
I_2(t) &= \left\| \int_T^t c^{-(t-\tau)/\omega} E_\alpha(t-\tau) c^{-\tau/\omega} u_2(\tau) d\tau \right\|_{\mathbb{X}} \\
&\leq \int_T^t \left| c^{-(t-\tau)/\omega} \right| \| E_\alpha(t-\tau) \|_{\mathcal{L}(\mathbb{X})} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} d\tau \\
&\leq \int_T^t \left| c^{-(t-\tau)/\omega} \right| \frac{\Lambda(\theta,\alpha)M}{1+|\varpi|(t-\tau)^\alpha} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} d\tau
\end{aligned}$$

which in view of (18) and the fact that $1+|\varpi|(t-\tau)^\alpha \geq 1$, gives

$$\begin{aligned}
I_2(t) &\leq \varepsilon \int_T^t \left| c^{-(t-\tau)/\omega} \right| \frac{\Lambda(\theta,\alpha)M}{1+|\varpi|(t-\tau)^\alpha} d\tau \\
&\leq \varepsilon \Lambda(\theta,\alpha) M \int_T^t |c|^{-(t-\tau)/\omega} d\tau.
\end{aligned}$$

Observing that on the one hand that $|c|^{-(t-\tau)/\omega} = e^{\frac{\tau-t}{\omega}\ln(|c|)}$ and on the other hand that,

$$\begin{aligned}
\int_T^t e^{\frac{\tau-t}{\omega}\ln(|c|)} d\tau &= \frac{\omega}{\ln(|c|)} \left(e^{\frac{t-t}{\omega}\ln(|c|)} - e^{\frac{T-t}{\omega}\ln(|c|)} \right) \\
&= \frac{\omega}{\ln(|c|)} \left(1 - e^{\frac{T-t}{\omega}\ln(|c|)} \right) \\
&\leq \frac{\omega}{\ln(|c|)},
\end{aligned}$$

we deduce that for any $\varepsilon > 0$,

$$\begin{aligned}
I_2(t) &\leq \varepsilon \Lambda(\theta,\alpha) M \int_T^t |c|^{-(t-\tau)/\omega} d\tau \\
&\leq \frac{\omega}{\ln(|c|)} \varepsilon \Lambda(\theta,\alpha) M.
\end{aligned}$$

Consequently,

$$\lim_{t\to\infty} I_2(t) = 0$$

if $|c| \geq 1$.
Finally, making the change of variable $s = t - \tau$ and using (15), we have:

$$\begin{aligned}
I_3(t) &= \left\| \int_{-\infty}^{0} c^{-t/\omega} E_\alpha(t-\tau) u_1(\tau) d\tau \right\|_{\mathbb{X}} \\
&= \left\| \int_{t}^{\infty} c^{-t/\omega} E_\alpha(s) u_1(t-s) ds \right\|_{\mathbb{X}} \\
&\leq \int_{t}^{\infty} |c|^{-s/\omega} \|E_\alpha(s)\|_{\mathcal{L}(\mathbb{X})} \left\| c^{-(t-s)/\omega} u_1(t-s) \right\|_{\mathbb{X}} ds \\
&\leq \frac{\Lambda(\theta,\alpha)M}{|\varpi|} \left(\sup_{s\in[t,\infty]} \left\| c^{-(t-s)/\omega} u_1(t-s) \right\|_{\mathbb{X}} \right) \int_{t}^{\infty} |c|^{-s/\omega} s^{-\alpha} ds.
\end{aligned}$$

Observing on the one hand that, for $|c| \geq 1$, we have $|c|^{-s/\omega} \leq 1$, and on the other hand that,

$$\left(\sup_{s\in[t,\infty]} \left\| c^{-(t-s)/\omega} u_1(t-s) \right\|_{\mathbb{X}} \right) < \infty$$

because $u_1 \in P_{\omega c}(\mathbb{X})$; we deduce that

$$I_3(t) \leq \frac{\Lambda(\theta,\alpha)M}{|\varpi|(\alpha-1)} \left(\sup_{s\in[t,\infty]} \left\| c^{-(t-s)/\omega} u_1(t-s) \right\|_{\mathbb{X}} \right) \frac{1}{t^{\alpha-1}}.$$

Hence,

$$\lim_{t\to\infty} I_3(t) = 0.$$

We have just proved that for $t > T$, $v_2 \in C_{0,c}(\mathbb{X})$. As mentioned above, similar arguments can be made if $t < -T$. Hence, if $|c| \geq 1$, we have $v_2 \in C_{0,c}(\mathbb{X})$ and therefore, $v \in AP_{\omega c}(\mathbb{X})$. ∎

Proposition 3.3 *Let $u \in AP_{\omega c}(\mathbb{X})$. Then the function*

$$v(t) := \int_{-\infty}^{t} E_\alpha(t-\tau) u(\tau) d\tau$$

belongs to $AP_{\omega c}(\mathbb{X})$ if $|c| \geq 1$.

Proof. By definition, $u \in AP_{\omega c}(\mathbb{X})$ means that there exists $u_1 \in P_{\omega c}(\mathbb{X})$ and $u_2 \in C_{0,c}(\mathbb{X})$ such that $u = u_1 + u_2$. Then,

$$\begin{aligned}
v(t) &= \int_{-\infty}^{t} E_\alpha(t-\tau)(u_1(\tau) + u_2(\tau)) d\tau \\
&= \int_{-\infty}^{t} E_\alpha(t-\tau) u_1(\tau) d\tau + \int_{-\infty}^{t} E_\alpha(t-\tau) u_2(\tau) d\tau \\
&= v_1(t) + v_2(t),
\end{aligned}$$

where

$$\begin{aligned} v_1(t) &= \int_{-\infty}^{t} E_\alpha(t-\tau)u_1(\tau)d\tau, \\ v_2(t) &= \int_{-\infty}^{t} E_\alpha(t-\tau)u_2(\tau)d\tau. \end{aligned}$$

By Proposition 3.1, we have clearly $v_1 \in P_{\omega c}(\mathbb{X})$. Now, we need to prove that $v_2 \in C_{0,c}(\mathbb{X})$.
Using (15) and the fact that $1 + |\varpi|(t-\tau)^\alpha \geq 1$, we have:

$$\begin{aligned} \|c^\wedge(-t)v_2(t)\|_{\mathbb{X}} &= \left\| \int_{-\infty}^{t} c^{-(t-\tau)/\omega} E_\alpha(t-\tau) c^{-\tau/\omega} u_2(\tau) d\tau \right\|_{\mathbb{X}} \\ &\leq \int_{-\infty}^{t} \left| c^{-(t-\tau)/\omega} \right| \|E_\alpha(t-\tau)\|_{\mathcal{L}(\mathbb{X})} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} d\tau \\ &\leq \left(\sup_{\tau \in (-\infty,t]} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} \right) \Lambda(\theta,\alpha) M \int_{-\infty}^{t} |c|^{-(t-\tau)/\omega} d\tau \\ &= \left(\sup_{\tau \in (-\infty,t]} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} \right) \Lambda(\theta,\alpha) M \int_{-\infty}^{t} e^{\frac{\tau-t}{\omega}\ln(|c|)} d\tau. \end{aligned}$$

Since

$$\begin{aligned} \int_{T}^{t} e^{\frac{\tau-t}{\omega}\ln(|c|)} d\tau &= \frac{\omega}{\ln(|c|)} \left(e^{\frac{t-t}{\omega}\ln(|c|)} - e^{\frac{T-t}{\omega}\ln(|c|)} \right) \\ &= \frac{\omega}{\ln(|c|)} \left(1 - e^{\frac{T-t}{\omega}\ln(|c|)} \right) \\ &\leq \frac{\omega}{\ln(|c|)}, \end{aligned}$$

we deduce that

$$\begin{aligned} \lim_{T\to-\infty} \int_{T}^{t} e^{\frac{\tau-t}{\omega}\ln(|c|)} d\tau &= \int_{-\infty}^{t} e^{\frac{\tau-t}{\omega}\ln(|c|)} d\tau \\ &\leq \frac{\omega}{\ln(|c|)}. \end{aligned}$$

It then follows that

$$\begin{aligned} \|c^\wedge(-t)v_2(t)\|_{\mathbb{X}} &\leq \left(\sup_{\tau \in (-\infty,t]} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} \right) \Lambda(\theta,\alpha) M \int_{-\infty}^{t} e^{\frac{\tau-t}{\omega}\ln(|c|)} d\tau \\ &\leq \left(\sup_{\tau \in (-\infty,t]} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} \right) \Lambda(\theta,\alpha) M \frac{\omega}{\ln(|c|)}. \end{aligned}$$

Consequently,
$$\lim_{t\to\infty} \|c^\wedge(-t)v_2(t)\|_{\mathbb{X}} = 0,$$
because u_2 being in $C_{0,c}(\mathbb{X})$ means that
$$\lim_{t\to\infty} \sup_{\tau\in(-\infty,t]} \left\| c^{-\tau/\omega} u_2(\tau) \right\|_{\mathbb{X}} = 0.$$
Therefore, $v \in AP_{\omega c}(\mathbb{X})$. $\blacksquare$

Lemma 3.1 *Let $\mathcal{B}(\mathbb{X})$ be a fading memory and $u \in AP_{\omega c}(\mathbb{X})$ such that $u_0 \in \mathcal{B}(\mathbb{X})$. Let also $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. Then the function $\mathbb{R}^+ \longrightarrow \mathcal{B}(\mathbb{X})$, $t \longmapsto u_t$ is also in $AP_{\omega c}(\mathbb{X})$.*

Proof. Let $u \in AP_{\omega c}(\mathbb{X})$. Then there exists $x \in P_{\omega c}(\mathbb{X})$ and $y \in C_{0,c}(\mathbb{X})$ such that $u = x + y$. fix $t \geq 0$. Then for any $\theta \leq 0$,
$$u_t(\theta) = x_t(\theta) + y_t(\theta) = x(\theta + t) + y(\theta + t).$$
We then have
$$x_t(\omega + \theta) = x(t + \omega + \theta) = cx(t + \theta) = cx_t$$
This means that $x_t \in P_{\omega c}(\mathbb{X})$.
Now, if we set $\tilde{y}_t(\theta) = c^\wedge(-\theta)y_t(\theta)$, then
$$\begin{aligned}
\tilde{y}_t(\theta) &= c^\wedge(-\theta)y(\theta + t) \\
&= c^{-\theta/\omega}y(\theta + t) \\
&= c^{-t/\omega}c^{-(\theta+t)/\omega}y(\theta + t).
\end{aligned}$$
But $y \in C_{0,c}(\mathbb{X})$ means that
$$\lim_{\theta\to-\infty} c^{-(\theta+t)/\omega}y(\theta + t) = 0.$$
Consequently,
$$\lim_{\theta\to-\infty} \tilde{y}_t(\theta) = \lim_{\theta\to-\infty} c^\wedge(-\theta)y_t(\theta) = 0$$
This implies that $y_t \in C_{0,c}(\mathbb{X})$ and since $x_t \in P_{\omega c}(\mathbb{X})$, we conclude that $u_t \in AP_{\omega c}(\mathbb{X})$. $\blacksquare$
The following result gives us sufficient conditions to obtain a unique (ω, c)-asymptotically periodic mild solution to the fractional integro-differential equation (1).

Theorem 3.1 *Under the assumptions (H_1) and (H_2) of Theorem 2.2, suppose also that $\tilde{C}_1 := \sup_{0\leq t<\infty} C_1(t) < \infty$ and $|c| \geq 1$. If*
$$K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta, \alpha)M\tilde{C}_1 \frac{\frac{\pi}{\alpha}}{\sin(\frac{\pi}{\alpha})} < 1, \tag{21}$$
then Problem (1) has a unique (ω, c)-asymptotically periodic mild solution.

Proof. We apply the Banach contraction principle.
Let $u \in AP_{\omega c}(\mathbb{X})$. According to Lemma 3.1 and Theorem 2.2, it is clear that for any $\tau \geq 0$, $F(\tau, u_\tau) \in AP_{\omega c}(\mathbb{X})$. In addition, by Proposition 3.2 we deduce that $\int_0^t E_\alpha(t-\tau)F(\tau,u_\tau)d\tau \in AP_{\omega c}(\mathbb{X})$. Moreover, we note that

$$\|c\hat{}(-t)E_\alpha(t)\phi(0)\|_{\mathbb{X}} \quad \leq \quad \|\phi(0)\|_{\mathbb{X}} \frac{\Lambda(\theta,\alpha)M}{\left|c^{t/\omega}\right|(1+|\varpi|t^\alpha)} \longrightarrow 0 \;\text{ as }\; t \longrightarrow \infty.$$

This means that $E_\alpha(t)\phi(0) \in C_{0,c}(\mathbb{X}) \subseteq AP_{\omega c}(\mathbb{X})$ and hence $E_\alpha(t)\phi(0) \in AP_{\omega c}(\mathbb{X})$. Therefore, we have proved that for all $t \geq 0$ and $u \in AP_{\omega c}(\mathbb{X})$, $E_\alpha(t)\phi(0) + \int_0^t E_\alpha(t-\tau)F(\tau,u_\tau)d\tau \in AP_{\omega c}(\mathbb{X})$. To prove the uniqueness, it suffices to consider the part of the solution on $t \geq 0$. To achieve this, let us define the operator

$$\begin{array}{rccl} G: & AP_{\omega c}(\mathbb{X}) & \longrightarrow & AP_{\omega c}(\mathbb{X}) \\ & u & \longmapsto & (Gu)(t) = E_\alpha(t)\phi(0) + \int_0^t E_\alpha(t-\tau)F(\tau,u_\tau)d\tau, \quad t \geq 0. \end{array} \tag{22}$$

From the above calculations, it is clear that the operator G is well defined.
Now, let $u, v \in AP_{\omega c}(\mathbb{X})$ be solutions of system (1). We have $u_0 = v_0 = \phi$ and by the hypothesis (H_2) and the condition (P_1)-(c) one has:

$$\begin{aligned}
& \||c|\hat{}(-t)\left((Gu)(t)-(Gv)(t)\right)\|_{\mathbb{X}} \\
= \; & \left\| \int_0^t |c|^{-(t-\tau)/\omega}|c|^{-\tau/\omega} E_\alpha(t-\tau)\left(F(\tau,u_\tau)-F(\tau,v_\tau)\right)d\tau \right\|_{\mathbb{X}} \\
\leq \; & K\int_0^t |c|^{-(t-\tau)/\omega}\|E_\alpha(t-\tau)\|_{\mathcal{L}(\mathbb{X})}|c|^{-\tau/\omega}\|u_\tau - v_\tau\|_{\mathcal{B}(\mathbb{X})}d\tau \\
\leq \; & K\int_0^t e^{(-(t-\tau)/\omega)\ln(|c|)}\|E_\alpha(t-\tau)\|_{\mathcal{L}(\mathbb{X})}|c|^{-\tau/\omega}C_1(\tau)\sup_{0\leq\eta\leq\tau}\|u(\eta)-v(\eta)\|_{\mathbb{X}}d\tau \\
\leq \; & K\int_0^t \|E_\alpha(t-\tau)\|_{\mathcal{L}(\mathbb{X})}C_1(\tau)\sup_{0\leq\eta\leq\tau}\||c|^{-\eta/\omega}(u(\eta)-v(\eta))\|_{\mathbb{X}}d\tau
\end{aligned}$$

because $|c| \geq 1$ implies that $e^{(-(t-\tau)/\omega)\ln(|c|)} \leq 1$ and $\eta \leq \tau$ implies that $|c|^{-\tau/\omega} \leq |c|^{-\eta/\omega}$. So, taking into account (15) we obtain

$$
\begin{aligned}
& \| |c|^\wedge(-t)\left((Gu)(t)-(Gv)(t)\right)\|_{\mathbb{X}} \\
\leq \quad & K\Lambda(\theta,\alpha)M\tilde{C}_1 \sup_{\eta\in\mathbb{R}^+} \| |c|^\wedge(-\eta)(u(\eta)-v(\eta))\|_{\mathbb{X}} \int_0^t \frac{1}{1+|\varpi|(t-\tau)^\alpha}d\tau \\
= \quad & K\Lambda(\theta,\alpha)M\tilde{C}_1 \|u-v\|_{a\omega c} \int_0^t \frac{1}{1+|\varpi|(t-\tau)^\alpha}d\tau.
\end{aligned}
$$

By the change of variable $z=|\varpi|(t-\tau)^\alpha$, the above inequality gives:

$$
\begin{aligned}
& \| |c|^\wedge(-t)\left((Gu)(t)-(Gv)(t)\right)\|_{\mathbb{X}} \\
\leq \quad & \frac{1}{\alpha}K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta,\alpha)M\tilde{C}_1\|u-v\|_{a\omega c}\int_0^{|\varpi|t^\alpha}\frac{z^{\frac{1}{\alpha}-1}}{1+z}dz \\
\leq \quad & \frac{1}{\alpha}K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta,\alpha)M\tilde{C}_1\|u-v\|_{a\omega c}\int_0^{\infty}\frac{z^{\frac{1}{\alpha}-1}}{(1+z)^{\frac{1}{\alpha}+1-\frac{1}{\alpha}}}dz \\
= \quad & \frac{1}{\alpha}K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta,\alpha)M\tilde{C}_1\|u-v\|_{a\omega c}B\left(\frac{1}{\alpha},1-\frac{1}{\alpha}\right) \\
= \quad & \frac{1}{\alpha}K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta,\alpha)M\tilde{C}_1\|u-v\|_{a\omega c}\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) \\
= \quad & K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta,\alpha)M\tilde{C}_1\frac{\frac{\pi}{\alpha}}{\sin(\frac{\pi}{\alpha})}\|u-v\|_{a\omega c}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|Gu-Gv\|_{a\omega c} \quad &= \quad \sup_{t\in\mathbb{R}^+} \| |c|^\wedge(-t)\left((Gu)(t)-(Gv)(t)\right)\|_{\mathbb{X}} \\
&\leq \quad K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta,\alpha)M\tilde{C}_1\frac{\frac{\pi}{\alpha}}{\sin(\frac{\pi}{\alpha})}\|u-v\|_{a\omega c}.
\end{aligned}
$$

Hence when $K|\varpi|^{-\frac{1}{\alpha}}\Lambda(\theta,\alpha)M\tilde{C}_1\frac{\frac{\pi}{\alpha}}{\sin(\frac{\pi}{\alpha})}<1$, we deduce by the Banach contraction principle that G has a unique mild solution $u\in AP_{\omega c}(\mathbb{X})$. In other words, Problem (1) has a unique (ω,c)-asymptotically periodic mild solution. ■

4 Conclusion.

This paper presents a fractional integro-differential equation with parameter $1<\alpha<2$ and a closed (but not necessarily bounded) operator of sectorial type

$\varpi < 0$. We have applied the concept of (ω, c)-asymptotically periodic functions and have established some important propositions and lemmas. In our study, we used the latter results and the Banach contraction principle to prove, under some (sufficient) conditions which we have clearly specified, that there exists a unique (ω, c)-asymptotically periodic mild solution to the system (1) provided that $|c| \geq 1$. For the fractional integro-differential equation (1), more existence results can be further developed using various fixed point theorems.

References

[1] L. Abadias, E. Alvarez and R. Grau, (ω, c)-*Periodic Mild Solutions to Non-Autonomous Abstract Differential Equations*, Mathematics, Multidisciplinary Digital Publishing Institute, vol. 9 (5), pp. 474-489 (2021)

[2] R. P. Agarwal, C. Cuevas, H. Soto and M. El-Gebeily *Asymptotic periodicity for some evolution equations in Banach spaces*, Nonlinear Analysis: Theory, Methods & Applications, Elsevier, vol. 74 (5), pp. 1769-1798 (2011)

[3] E. Alvarez, S. Castillo and M. Pinto, (ω, c)-*Pseudo periodic functions, first order Cauchy problem and Lasota-Wazewska model with ergodic and unbounded oscillating production of red cells*, Boundary Value Problems, Springer, vol. 2019 (106), (2019)

[4] E. Alvarez, S. Castillo and M. Pinto, (ω, c)-*asymptotically periodic functions, first-order Cauchy problem, and Lasota-Wazewska model with unbounded oscillating production of red cells*, Mathematical Methods in the Applied Sciences, Wiley Online Library, vol. 43 (1), pp. 305-319 (2020)

[5] E. Alvarez, A. Gómez and M. Pinto Jiménez, (ω, c)-*periodic functions and mild solutions to abstract fractional integro-differential equations*, Electronic Journal of Qualitative Theory of Differential Equations, niv Szeged, vol. 16, pp. 1-8 (2018)

[6] P. Amster and L. Idels, *Periodic solutions in general scalar non-autonomous models with delays*, Nonlinear Differential Equations and Applications NoDEA, Springer, vol. 20 (5), pp. 1577-1596 (2013)

[7] J. Blot, P. Cieutat and G. M. N'Guérékata, *S-asymptotically ω-periodic functions and applications to evolution equations*, African Diaspora Journal of Mathematics. New Series, Mathematical Research Publishers, vol. 12 (2), pp. 113-121 (2011)

[8] A. Caicedo, C. Cuevas and H. R. Henríquez, *Asymptotic periodicity for a class of partial integro-differential equations*, International Scholarly Research Notices, Hindawi, vol. 2011 (2011)

[9] A.Caicedo, C. Cuevas, G. Mophou and G. M. N'Guérékata, *Asymptotic behavior of solutions of some semilinear functional differential and integro-differential equations with infinite delay in Banach spaces*, Journal of the Franklin Institute, Elsevier, vol. 349 (1), pp. 1-24 (2012)

[10] T. Cazenave and A. Haraux, *Oscillatory phenomena associated to semilinear wave equations in one spatial dimension*, Transactions of the American Mathematical Society, vol. 300 (1), pp. 207-233 (1987)

[11] Y. Chen, *Anti-periodic solutions for semilinear evolution equations*, Journal of mathematical analysis and applications, Elsevier, vol. 315 (1), pp. 337-348 (2006)

[12] E. Cuesta, *Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations*, Conference Publications, American Institute of Mathematical Sciences, vol. 2007 (Special), pp. 277-285 (2007)

[13] J. M. Cushing, *Forced asymptotically periodic solutions of predator-prey systems with or without hereditary effects*, SIAM Journal on Applied Mathematics, SIAM, vol. 30 (4), pp. 665-674 (1976)

[14] W. Dimbour, G. Mophou and G. M. N'Guérékata, *S-asymptotically periodic solutions for partial differential equations with finite delay*, Electronic Journal of Differential Equations, vol. 2011 (117), pp. 1-12 (2011)

[15] W. Dimbour and G. M. N'Guérékata, *S-asymptotically ω-periodic solutions to some classes of partial evolution equations*, Applied Mathematics and Computation, Elsevier, vol. 218 (14), pp. 7622-7628 (2012)

[16] A. Granas and J. Dugundji, *Fixed point theory*, Science & Business Media, Springer (2013)

[17] A. Haraux, *Anti-periodic solutions of some nonlinear evolution equations*, Manuscripta Mathematica, Springer, vol. 63 (4), pp. 479-505 (1989)

[18] M. Kéré, G. M. N'Guérékata and E. R. Oueama-Guengai *An Existence Result of (ω, c)-almost Periodic Mild Solutions to Some Fractional Differential Equations*, PanAmerican Mathematical Journal, vol. 31 (3), pp. 11-20 (2021)

[19] M. Li, J. R. Wang and M. Feckan, *(ω, c)-Periodic Solutions for Impulsive Differential Systems*, Communications in Mathematical Analysis, Mathematical Research Publishers, vol. 21 (2), pp. 35-45 (2018)

[20] K. Liu, J. R. Wang, D. O'Regan and M. Fečkan, *A New Class of (ω, c)-Periodic Non-instantaneous Impulsive Differential Equations*, Mediterranean Journal of Mathematics, Springer, vol. 17 (5), pp. 1-22 (2020)

[21] J. Mawhin, *Periodic solutions of nonlinear functional differential equations*, Journal of Differential Equations, Elsevier, vol. 10 (2), pp. 240-261 (1971)

[22] A. Meril, G. Mophou and G. M. N'Guérékata, *Asymptotic behavior of bounded mild solutions of some functional differential and fractional differential equations*, Differential and Integral Equations, Khayyam Publishing, vol. 24 (5/6), pp. 401-416 (2011)

[23] G. Mophou and G. M. N'Guérékata, *An existence result of (ω, c)-periodic mild solutions to some fractional differential equation*, Nonlinear Studies-The International Journal, Elsevier, vol. 27 (1), pp. 167-175 (2020)

[24] G. Mophou, G. M. N'Guérékata and V. Valmorin, *Asymptotic behavior of mild solutions of some fractional functional integro-differential equations*, African Diaspora Journal of Mathematics, Mathematical Research Publishers, vol. 16 (1), pp. 70-81 (2013)

[25] G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, Springer, New York, Second edition, (2021)

[26] G. M. N'Guérékata, *An asymptotic theorem for abstract differential equations*, Bulletin of the Australian Mathematical Society, MCambridge University Press, vol. 33 (1), pp. 139-144 (1986)

[27] G. M. N'Guérékata, *Existence and uniqueness of almost automorphic mild solutions to some semilinear abstract differential equations*, Semigroup Forum, Springer, vol. 69 (1), pp. 80-86 (2004)

[28] G. M. N'Guérékata, *Quelques remarques sur les fonctions asymptotiquement presque-automorphes*, Ann. sc. math. Québec, vol. 7 (2) (1983)

[29] E. R. Oueama-Guengai and G. M. N'Guérékata, *On S-asymptotically ω-periodic and Bloch periodic mild solutions to some fractional differential equations in abstract spaces*, Mathematical Methods in the Applied Sciences, Wiley Online Library, vol. 41 (18), pp. 9116-9122 (2018)

[30] D. R. Smart, *Fixed point theorems*, Press., Cambridge (1980)

[31] J. R. Wang, L. Ren and Y. Zhou, *(ω, c)-Periodic solutions for time varying impulsive differential equations*, Advances in Difference Equations, Springer, vol. 2019 (1), pp. 1-9 (2019)

[32] F. Wei and K. Wang, *Asymptotically periodic solution of N-species cooperation system with time delay*, Nonlinear Analysis: Real World Applications, Elsevier, vol. 7 (4), pp. 591-596 (2006)

[33] L. Zhongchao, *Asymptotically periodic solutions of a class of second order nonlinear differential equations*, Proceedings of the American Mathematical Society, JSTOR, pp. 693-699 (1987)