Analytic Solutions of Fractal and Fractional Time Derivative-Burgers-Nagumo Equation

Hamdy Abdel-Gawad[1], Mohammad Tantawy[2], Basma Abdel-Aziz[3], and Ahmet Bekir[4]

[1]Cairo University
[2]Department of Basic Science, Faculty of Engineering at October 6 University, Egypt.
[3]Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
[4]Eskisehir

March 07, 2024

Abstract

The Nagumo equation describes a reaction-diffusion system in biology. Here, it is coupled to Burgers equation, via including convection, which is, namely; Burgers-Nagumo equation BNE. The first objective of this work is to present a theorem to reduce the different versions of the fractional time derivatives FTD to “non autonomous” ordinary ones, that is ordinary derivatives with time dependent coefficients. The second objective is to find the exact solutions of the fractal and fractional time derivative -BNE, that is to solve BNE with time dependent coefficient. On the other hand FTD can be transformed to BNE with constant coefficients via similarity transformations. The unified and extended unified method are used. Self-similar solutions are also obtained. It is found that significant fractal effects hold for smaller order derivatives. While significant fractional effects hold for higher-order derivatives. The solutions obtained show solitary, wrinkle soliton waves, with double kinks, undulated, or with spikes. Further It is shown that wrinkle soliton wave, with double kink configuration holds for smaller fractal order. While in the case of fractional derivative, this holds for higher orders.

Analytic Solutions of Fractal and Fractional Time Derivative- Burgers-Nagumo Equation

H. I. Abdel-Gawad [1], **M. Tantawy** [2*] **B. Abdel-Aziz** [1] **and Ahmet Bekir** [3†]

1-Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
2-Department of Basic Science, Faculty of Engineering at October 6 University, Egypt
3-Neighbourhood of Akcaglan, Imarli Street Number: 28/4, 26030, Eskisehir, Turkey

Abstract

The Nagumo equation describes a reaction-diffusion system in biology. Here, it is coupled to Burgers equation, via including convection, which is, namely; Burgers-Nagumo equation BNE. The first objective of this work is to present a theorem to reduce the different versions of the fractional time derivatives FTD to "non autonomous" ordinary ones, that is ordinary derivatives with time dependent coefficients. The second objective is to find the exact solutions of the fractal and fractional time derivative -BNE, that is to solve BNE with time dependent coefficient. On the other hand FTD can be transformed to BNE with constant coefficients via similarity transformations.The unified and extended unified method are used. Self-similar solutions are also obtained. It is found that significant fractal effects hold for smaller order derivatives. While significant fractional effects hold for higher-order derivatives. The solutions obtained show solitary, wrinkle soliton waves, with double kinks, undulated, or with spikes. Further It is shown that wrinkle soliton wave, with double kink configuration holds for smaller fractal order.While in the case of fractional derivative, this holds for higher orders.

Keywords: Fractal and fractional, Burger-Nagumo equation, reaction diffusion, wrinkle soliton waves.

1 Introduction

Reaction diffusion equations have many applications. In chemistry, they describe general reaction systems and reversible reactions. In biology they describe the population dynamics, predator-prey and competition.The Nagumo equation describes a reaction-diffusion system in biology. Here, it is taken coupled in the Fitzhugh-Nagumo equation that describes the conduction of recovery current to nerve fibers [1-6].

*e–mail: mtantawymath@gmail.com
†e–mail: bekirahmet@gmail.com

The BNE model describes the interaction between reaction mechanisms, convection effects and diffusion transport of nerve pulse propagation in nerve fibers. Further, it deserves to a wall motion in liquid crystals. The Eq. (1) is, in general, not integrable, that is for arbitrary parameters μ, a, σ and b. We mention that it is invariant under translation in x and t. Thus, theoretically traveling wave solutions exist. The traveling waves have contributed to the understanding of physical phenomena complexity, and the characteristics of wave propagation. Here, first the conditions for integrability of Eq. (1) are depicted, second a variety of exact solutions that describe traveling and self-similar waves are obtained. In [7], by reducing the BNE via Cole–Hopf transformation., exact solutions were found. It was shown that they describe kink and periodic waves propagating in the space. In addition, the symmetries of BNE were investigated in [8-10]. However, there are still some unresolved problems as Eq. (1) is not completely integrable. These problems can be solved by studying the behavior of traveling waves via numerical solutions which will provide further investigations. These investigations are embedded in the bifurcation study. We mention that heteroclinic orbits are trajectories that have two distinct equilibrium values, while homoclinic orbits are trajectories that have the same equilibrium state. They correspond to soliton with double kinks and soliton waves configurations respectively[11,12]. The bounded traveling waves mentioned above are completed by the periodic ones. This relationship investigates the bifurcation analysis of a dynamical system, and it is recognizes as an effective method to determine bifurcations of traveling waves. The study of analytical and numerical solutions of the fractional Burgers equation occupies a remarkable area in the literature [13-24].

The BNE reads

$$u_t(x,t) + \mu\, u u_x(x,t) + \sigma\, u_{xx}(x,t) - b\, u(x,t)\,(1-u(x,t))\,(u(x,t)-a) = 0\,, \tag{1}$$

where μ, b are real parameters and $0 < a < 1$. When $\sigma = 1$ and $b = 0$, then Eq. (1) is the Burgers equation. When $b \neq 0$, it is Burgers equation with cubic nonlinear source term. While when $\sigma = -1$, and $\mu \neq 0$ it is the Nagumo equation with convection.

2 Fractal derivative

The fractional derivative was introduced in [25].

$$\frac{d}{dt^\alpha} f(t) = Limit_{t_1 \to t} \frac{f(t_1) - f(t)}{t_1^\alpha - t^\alpha},\ t > 0. \tag{2}$$

When $f(t)$ is c continuously differential of first order; $f \epsilon C^1(\mathbb{R}^+)$, (where $\mathbb{R}^+ = [0, \infty)$, the RHS of Eq. (1) reduces to

$$\frac{d}{dt^\alpha} f(t) = \alpha^{-1} t^{1-\alpha} f'(t). \tag{3}$$

From Eq. (2) we find that the fractal derive is nothing else but the conformable fractional derivative up to multiplication by α^{-1}. On the other hand, it is identical to the LHS of Eq. (2) by writing $dt^\alpha = \alpha t^{\alpha-1} dt$. That is dt^α is reducible. This fact suggests defining the fractal derive by

$$\frac{d}{dt^\alpha} f(t) = Limit_{\varepsilon \to 0} \frac{f(t + \varepsilon \alpha^{-1} t^{1-\alpha}) - f(t)}{\varepsilon},\ t > 0. \tag{4}$$

Very recently, some works to study chaotic nonlinear dynamical systems, based on fractal fractional derivatives are carried in [14,15].

3 Fractional derivatives

The Caputo fractional derivative is defined by

$$D_t^{\alpha C} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-t_1)^{-\alpha+(m-1)} f^{(m)}(t_1) dt_1,\ m-1 < \alpha < m\, t > 0\,, \tag{5}$$

provided that f is Holder continuous , $f \epsilon H^{m,\alpha-(m-1)}(\mathbb{R}^+)$ and the integral exists.

The Caputo-Fabrizio fractional derivative CFFD [26,27] is

$$D_t^{\alpha CF} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t e^{\frac{-\alpha}{1-\alpha}(t-t_1)} f'(t_1) dt_1,\ 0 < \alpha < 1\, t > 0\,, \tag{6}$$

provided that $f \epsilon H^{1,\alpha}(\mathbb{R}^+)$. where $M(\alpha)$ in Eq. (6) is a normalization function such that $M(0) = M(1)$ =1.

The Atangana-Baleanu fractional derivative ABFD, in the Caputo sense, is [28-30]

$$D_t^{\alpha AB} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_\alpha(\frac{-\alpha}{1-\alpha}(t-t_1)^\alpha) f'(t_1) dt_1,\ 0 < \alpha < 1\, t > 0\,, \tag{7}$$

where $B(\alpha) > 0$ is the normalization constant that satisfies $B(0) = B(1) = 1$ and $E_\alpha(t)$ is the Mittag–Leffler function and $f \epsilon H^{1,\alpha}(\mathbb{R}^+)$. It is worth noticing that this function is not invariant under the CFD. The function which is invariant is $e_\alpha(t)$ [26] where

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+1)},\ e_\alpha(t) = e_{\alpha,1}(t) = \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n+1)}, \tag{8}$$

and the last function generalizes to

$$e_{\alpha,\beta}(\lambda, t) = \sum_{n=0}^{\infty} \frac{\lambda^n t^{\alpha n}}{\Gamma(\alpha n+\beta)}. \tag{9}$$

We define a new fractional derivative, Gawad's fractional derivative of order β, $0 < \beta < 1$

$$D_t^{\beta G} f(t) = \frac{\beta}{(1-\beta)^{1/\beta}} \int_0^t e^{-\frac{(t-t_1)^\beta}{1-\beta})} f'(t_1) dt_1,\ 0 < \beta < 1\, t > 0 \tag{10}$$

provided that $f \epsilon H^{1,\beta}(\mathbb{R}^+)$. We mention that Eq. (10) reduces to the ordinary derivative when $\beta \to 1^-$, as the Kernel tends to be the Dirak $\delta-$ function.

3.1 Reduction of the fractional derivatives

Here we shall reduce the FD's to ordinary derivatives [31,32].

We consider Eq. (1) when $0 < \alpha < 1$

$$D_{t_1}^{\alpha C} f(t_1) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} (t_1 - t_2)^{-\alpha} f'(t_2) dt_2. \tag{11}$$

Theorem:1 We assume that the integrand in the RHS of Eq. (11) is uni-formally continuous,then the Caputo FD is reduced to

$$D_t^{\alpha C} f(t) = \frac{1}{\Gamma(2-\alpha)} (T-t)^{1-\alpha} f'(t),\ 0 \leq t \leq T_0. \tag{12}$$

Proof: In Eq. (11), by operating by the integral, on t_1 on $[0, t]$, it holds

$$\int_0^t D_{t_1}^{\alpha C} f(t_1)\, dt_1 = \int_0^t (\frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} (t_1 - t_2)^{-\alpha} f'(t_2) dt_2) dt_1\ . \tag{13}$$

By using the assumption, we permute the inner with the outer integral in the RHS of Eq. (13), it becomes

$$\int_0^t (\frac{1}{\Gamma(1-\alpha)} (\int_{t_2}^t (t_1 - t_2)^{-\alpha} dt_1) f'(t_2) dt_2, \tag{14}$$

by evaluating the new inner integral, we get

$$\int_0^t D_{t_1}^{\alpha C} f(t_1) dt_1 = \int_0^t (\frac{1}{\Gamma(1-\alpha)} \frac{(t-t_2)^{1-\alpha}}{1-\alpha}) f'(t_2)\, dt_2. \tag{15}$$

From Eq. (11), it holds that

$${}_0^C D_{t_2}^{\alpha} f(t_2) = \frac{1}{\Gamma(2-\alpha)} (t - t_2)^{1-\alpha} f'(t_2)\ 0 \leq t_2 \leq t. \tag{16}$$

By letting $t \to T$and $t_2 \to t$, Eq. (16) leads to Eq. (12)..□

By the same way, that is by permuting the inner with the outer integral and evaluating the new inner integral, the other different versions of the fractional derivatives are transformed to

$$\begin{array}{c} D_t^{\alpha CF} f(t) = \frac{2}{(2-\alpha)} (1 - e^{-\frac{\alpha}{1-\alpha}(T_0 - t)}) f'(t) \\ 0 \leq t \leq T_0, \end{array} \quad , \tag{17}$$

$$D_t^{\alpha AB} f(t) = \frac{B(\alpha)}{(1-\alpha)} (T-t)^{\alpha} e_{\alpha,2}(-\frac{\alpha}{1-\alpha}, (T-t)) f_0'(t), \tag{18}$$

where $e_{\alpha,\beta}(\sigma, x)$is the generalized Mittag-leffler function.(see [33])

$$e_{\alpha,\beta}(\sigma,x)=\sum_{n=0}^{\infty}\frac{\sigma^n x^{\alpha n}}{\Gamma(\alpha n+\beta)}. \tag{19}$$

To discuss the convergence of the series in Eq. (19), we use the identity

$$\Gamma(\alpha(n+1)+\beta)=\frac{\Gamma(\alpha n+\beta)\mathit{\Gamma}(\alpha)}{B(\alpha n+\beta,\alpha)}, \tag{20}$$

where $B(a,b)$ is the Beta function. By inserting Eq. (20) into Eq. (19),it can be easily shown that the series converges for $x>0$.

Also, the Gawad's FD reduces to

$$\begin{gathered} D_t^{\beta G} f(t)=\gamma(\tfrac{1}{\beta},\tfrac{1}{1-\beta}(T-t)^{\beta})f'(t)\ , \\ 0\leq t\leq T,\ 0<\beta<1, \end{gathered} \tag{21}$$

where $\gamma(m,t)$ is the incomplete lower Gamma function. We mention that

$$\gamma(a,x)+\mathit{\Gamma}(a,x)=\Gamma(a),\ a>0,\ x>0,\ \mathit{\Gamma}(a,x)=\int_x^{\infty}e^{-y}y^{a-1}dy. \tag{22}$$

By using Eq. (11), we can prove the following theorem.

Theorem 2: The GFD satisfies the following:

(i) $D_t^{\beta G}(f(t){+}g(t))=D_t^{\beta G}f(t)+D_t^{\beta G}g(t)$.

(ii) $D_t^{\beta G}(f(t)g(t))=f(t)D_t^{\beta G}g(t)+g(t)D_t^{\beta G}f(t)$.

(iii) $D_t^{\beta G}(\frac{f(t)}{g(t)}))=\frac{g(t)D_t^{\beta G}f(t)-f(t)D_t^{\beta G}g(t)}{g(t)^2}$.

By using (20) the function $f(t)$ which is invariant under the FD $D_t^{\beta G}$, that is $D_t^{\beta G}e_{\beta,G}(t)=e_{\beta,G}(t)$, is found directly:

$$e_{\beta,G}(t)=e^{\int_0^t \frac{1}{\gamma(\frac{1}{\beta},\frac{\lambda}{1-\beta}T_0-t_1)^{\beta})}dt_1}. \tag{23}$$

4 Solutions of the fractal and fractional time-derivative BNE

By using the results of section 3, the fractal and fractional time-derivative BNE Eq. (1)) is transformed to

$$p(t)u_t(x,t)+\mu u(x,t)u_x(x,t))+\sigma u_{xx}(x,t)-bu(x,t)(1-u(x,t))(u(x,t)-a)=0\ ., \tag{24}$$

where p(t) takes one of the following forms:

(a) $CFD:\quad p(t)=\frac{1}{\Gamma(2-\alpha)}(T_0-t)^{1-\alpha}$.

(b) CFFD: $\quad p(t)=\frac{2}{(2-\alpha)}(1-e^{-\frac{\alpha}{1-\alpha}(T_0-t)})$.

(c) GFD: $p(t) = \frac{2}{\lambda+2}\gamma(\frac{1}{\beta}, \lambda(T_0 - t)^{\beta})$.

(d) ABFD: $\frac{B(\alpha)}{(1-\alpha)}(T-t)^{\alpha} e_{\alpha,2}(-\frac{\alpha}{1-\alpha}, (T-t))$.

It is wort noticing that Eq. (24) is non autonomous PDE. It suggests to introduce the similarity transformations $u(x,t) = \tilde{u}(x,\tau)$ and $\tau = \int_0^t \frac{1}{p(s)} ds$, thus (24) reduces to

$$\tilde{u}_{\tau}(x,\tau) + \mu\tilde{u}(x,\tau)\tilde{u}_x(x,\tau)) + \sigma\tilde{u}_{xx}(x,\tau) - b\tilde{u}(x,\tau)(1-\tilde{u}(x,\tau))(\tilde{u}(x,\tau)-a) = 0\,. \quad (25)$$

To find the exact solutions of Eq.(25), we can use Eq. (1) and simply let $t \to \tau$. Thus our attention is focusing on finding the solutions of Eq. (1).

The unified, extended and generalized unified methods have been proposed in [34-36]. The unified method asserts that, the solutions of a nonlinear evolution equation can be written in the forms of polynomial or rational functions in auxiliary functions, that satisfy appropriate auxiliary equations.

Here, we are concerned with finding rational solutions of Eq. (1). First we find the traveling waves solutions. To this end we use the transformations $u(x,t) = U(z)$ and $z = x - ct$. Single, multi and coupled waves solutions of Eq. (1) are obtained as follows:

(i) Single traveling wave solutions:

Here, we write the solution in the form

$$U(z) = \frac{a_1 g(z) + a_0}{b_1 g(z) + b_0},\; g'(z) = c_1 g(z) + c_0, \quad (26)$$

where the second equation in Eq.(26) is the auxiliary equation.

By inserting Eq. (26) into Eq. (1), and by setting the coefficients of $g(z)^j, j = 0,1,2,..$equal to zero, we get

$$\begin{gathered} a_1 = a b_1,\; a_0 = \tfrac{1}{c_1}\left(b_0 c_1 + (-1+a) b_1 c_0\right), \\ c_1 = \tfrac{c\,(-1+a)}{((1+a)\alpha^2\sigma)}, b = \tfrac{c\,(-2c+(1+a)\alpha\,\mu)}{(1+a)^2\alpha^2\sigma} \end{gathered} \quad (27)$$

and

$$\begin{gathered} U(z) = \tfrac{P_1}{Q_1}, \\ P_1 = (-1+a)b_0 c + (-1+a)a A_0 b_1 c\, e^{\frac{(-1+a)cz}{(1+a)\alpha^2\sigma}} - (1+a)b_1 c_0\alpha^2\sigma\,, \\ Q_1 = (-1+a)b_0 c + (-1+a)A_0 b_1 c\, e^{\frac{(-1+a)cz}{(1+a)\alpha^2\sigma}} - (1+a)b_1 c_0\alpha^2\sigma, \end{gathered} \quad (28)$$

This solution is solitary wave solutions.

(ii) Multi waves solutions:

These solutions are found by using multi auxiliary functions and equations.

For two waves interactions the solution of Eq. (1) is

$$U(z) = \frac{a_1 g_1(z)+a_2 g_2(z)+a_3 g_1(z) g_2(z)+a_0}{b_1 g_1(z)+b_2 g_2(z)+b_3 g_1(z0 g_2(z)+b_0}\,,$$
$$g_1'(z) = c_1 g_1(z) + c_0,\; g_2'(z) = d_1 g_2(z) + d_0. \tag{29}$$

By inserting Eq. (29) into Eq. (1), and by the same way, we have

$$c = \frac{1}{4(1+a)\alpha\,\mu},\; d_1 = \frac{(1+a)\mu}{4\alpha\sigma},\; d_0 = \frac{(1+a)(-a_0+ab_0)\mu}{4(-1+a)b_2\alpha\sigma}\,,$$
$$b = \frac{\mu^2}{8\sigma},\; c_1 = \frac{a\mu}{2\alpha\sigma},\; c_0 = \frac{(a_0-b_0)c_1}{(-1+a)b1})\,,$$

and

$$U(z) = \frac{P_1}{Q_1}\,,$$
$$P_1 = a((1-a)A_1 b_1 e^{\frac{a\mu\mu z}{2\alpha\sigma}} + b_0(-1+(-1+a)A_2 b_2 e^{\frac{(1+a)z\mu}{4\alpha\sigma}}$$
$$Q_1 = A_1 b_1 (1-a) e^{\frac{a\mu\mu z}{2\alpha\sigma}} + (a^2-a)A_2 b_2 b_0 e^{\frac{(1+a)z\mu}{4\alpha\sigma}}) - ab_0. \tag{30}$$

For three waves interactions the solution is

$$U(z) = \frac{a_0+a_1 g_1(z)+a_2 g_2(z)+a_3 g_3(z)}{b_0+b_1 g_1(z)+b_2 g_2(z)+b_3 g_3(z)},\; g_1'(z) = c_1 g_1(z),$$
$$g_2'(z) = d_1 g_2(z),\; g_3'(z) = r_1 g_3(z). \tag{31}$$

By substituting from Eqs. (30) into Eq. (1), we have the following

$$c = \alpha(a\mu - (d_1+2r_1)\alpha\sigma),\; b = \frac{-r_1(c+\alpha(-a\,\mu+r_1\alpha\,\sigma))}{(-1+a)a},$$
$$a_3 = ab_3,\; c_1 = d_1 + r_1,\; a_0 = ab_0,\; a_2 = ab_2. \tag{32}$$

and the solution is

$$U(z) = \frac{P}{Q}\,,$$
$$P = A_1 e^{(d_1+r_1)z}(a_1 + A_3 a_3 e^{r_1 z} + a(b_0 + A_2 b_2 e^{d_1 z}) + A_3 b_3 e^{r_1 z}))\,,$$
$$Q = (b_0 + A_2 b_2 e^{d_1 z}) + A_3 b_3 e^{r_1 z} + A_1 e^{(d_1+r_1)z})(b_1 + A_3 b_3 e^{r_1 z})). \tag{33}$$

(iii) Exact solutions: Here we consider the solution

$$U(z) = \frac{a_1 g_1(z)+a_2 g_2(z)+a_0}{b_1 g_1(z)+b_2 g_2(z)+b_0}\,, \tag{34}$$
$$g_1'(z) = \alpha_1 g_1(z) + \alpha_2 g_2(z),\; g_2'(z) = \beta_1 g_1(z) + \beta_2 g_2(z).$$

By inserting Eqs. (34) into Eq. (1), we get

$$
\begin{gathered}
a_0 = a\,b_0,\ c = \tfrac{1}{b_2}\alpha(a_2\mu + \alpha(-2b_1\alpha_2 + b_2(\alpha_1 - \beta_2))\sigma)\,, \\
b = -\tfrac{1}{a^2 a_1^2}\alpha(-b_2\beta_1 + a_1\beta_2)(a\,a_1\mu + 2\alpha(-b_2\beta_1 + a_1\beta_2)\sigma)\,, \\
a_2 = b_2,\ \alpha_1 = \tfrac{ab_1b_2\beta_1 + a_1^2\beta_2 + a_1((-2+a)b_2\beta_1 - ab_1\beta_2)}{(a_1(a_1 - ab_1))}\,, \\
\alpha_2 = -\tfrac{b_2\beta_2}{a_1},\ b_1 = \tfrac{(a\alpha_2 - (-1+a)b_2^2\beta_1)}{aa_1\alpha_2}.
\end{gathered} \tag{35}
$$

Thus the solution which is given into the first equation in (34) is

$$
\begin{gathered}
U(z) = \tfrac{P}{Q},\ P = a\beta_2(ab_0 + a_1g_1(z) + b_2g_2(z))\,, \\
Q = ((-1+a)b_2\beta_1 + a_1\beta_2)g_1(z) + a\beta_2(b_0 + b_2g_2(z)).
\end{gathered} \tag{36}
$$

By solving the auxiliary equations in Eq. (34) and by using Eq. (35), the exact solution is given by

$$
\begin{gathered}
U(z) = \tfrac{P}{Q},\ P = (a\beta_2(a_1^2\beta_2A_1 - b_2\beta_1(ab_0 + b_2A_2) + a_1(-A_1b_2\beta_1 + \\
ab_0\beta_2 + b_2\beta_2A_2)), \\
Q = (-A_1b_2e^{-\frac{b_2z\beta1}{a_1}})\beta1((-1+a)b_2e^{z\beta_2}\beta_1 + a_1(ae^{\frac{b_2z\beta1}{a_1}} + e^{z\beta_2} - ae^{z\beta_2}\beta_2) \\
+\beta_2(a(-b_2b_0\beta_1 - A_2b_2^2e^{-\frac{b_2z\beta1}{a_1}} + \beta_2))\beta_1 + a_1b_0\beta_2 + a_1A_2b_2e^{-\frac{b_2z\beta1}{a_1}} + \beta_2))\beta_2 \\
+a_1b_2\beta_1A_1) + (b_2\beta_1 - a_1\beta_2)(A_2b_2e^{-\frac{b_2z\beta1}{a_1}} + \beta_2)) - a_1A_1 - b_2A_2)).
\end{gathered} \tag{37}
$$

5 Self similar solutions

To get these solutions we use the similarity transformations $z := x\,\omega(t)$, $t := t$ and $u(x,t) = U(z,t)$. By introducing these later transformations into Eq.(1), it reduces to

$$
U_t + \mu\,\omega(t)UU_z + \sigma\,\omega(t)^2U_{zz} - b\,U(1-U)(U-a) = 0\,. \tag{38}
$$

We write the solution of Eq. (38), with time-dependent coefficients, and appropriate auxiliary equations as follows

$$
\begin{gathered}
U(z) = \tfrac{a_1(t)\,g(z,t) + a_0(t)}{b_1(t)\,g(z,t) + b_0(t)},\quad g_z(z,t) = \gamma(c_2g(z,t)^2 + c_1g(z,t) + c_0)\,, \\
g_t(z,t) = h(t)(c_2g(z,t)^2 + c_1g(z,t) + c_0)\,.
\end{gathered} \tag{39}
$$

In Eq. (37), we mention that the compatibility equation $g_{zt}(z,t) = g_{tz}(z,t)$ holds. By inserting Eq. (37) into Eq. (36), we obtain a set of equations in $a_1'(t)$, $b_1'(t)$, $a_0'(t)$ and $a_1(t)$, are given by

$$a_1'(t) = \frac{1}{b_1(t)^2}/(-ba_1(t)^3 + a_1(t)^2((1+a)bb_1(t) - c_2\gamma\mu b_0(t)\omega(t))$$

$$+c_2a_0(t)b_1(t)(2c_2\gamma^2\sigma b_0(t)\omega(t)^2 + b_1(t)(h(t) - c_1\gamma^2\sigma\omega(t)^2))$$

$$+a_1(t)(-abb_1(t)^2 - 2c_2^2\gamma^2\sigma b_0(t)^2\omega(t)^2 + b_1(t)(-c2b_0(t)h(t)$$

$$+c_2\gamma\,\mu\,a_0(t)\,\omega(t) + c_1c_2\gamma^2\sigma b_0(t)\,\omega(t)^2 + b_1'(t)))\,,$$

$$b_1'(t) = \frac{c_2}{6\mu}(b_0(t)(6\mu h(t) + 4\gamma\,\omega(t)(1+a)\mu^2) + b_1(t)(c_1b_0(t) \\ (-6\mu h(t) + \gamma\omega(t)(-2\mu^2(1+a) + 3\mu b_0'(t))) \tag{40}$$

$$a_0'(t) = \frac{a_0(t)}{9\sigma b_0(t)^2}(-\mu^2a_0(t)^2 + (1+a)\mu^2a_0(t)b_0(t) \\ +b_0(t)(-a\mu^2b_0(t) + 9\sigma b_0'(t))),$$

$$a_1(t) = \frac{1}{2\mu^2b_0(t)}(2\mu^2a_0(t)b_1(t) + 3\gamma(-2\mu)\sigma b_0(t) \\ (-c_1b_1(t) + c_2b_0(t))\omega(t))$$

It is worth noticing that the computations, in this case, are not straight forward as (i) the equations obtained are nonlinear and (ii) compatibility equations have to be satisfied. For example, we have equations in $a_1'(t)$ and $a_1(t)$. Thus the compatible equation reads $a_1'(t) - (a_1(t))' = 0$. The result for this later equation is too lengthy to be produced here. To solve this equation, we find that the condition for integrability is $b = \frac{\mu^2}{9\sigma}$, and

$$c_0 = 0,\ \gamma = \frac{1}{3\mu},\ h(t) = A_0\omega(t),\ b_0'(t) = \frac{1}{6}(1+6A_0)c_1b_0(t)\omega(t),$$

$$a_0(t) = \frac{b_0(t)}{6\gamma\mu^2}(\mu((1+a)\gamma\mu(1+3\gamma\mu) + A_0(-3+9\gamma\mu) - 9c_1\gamma^2\sigma\omega(t)) \tag{41}$$

$$\omega'(t) = \frac{1}{108\sigma}\omega(t)(4(1-a+a^2)\mu^2 - 27c_1^2\gamma^2\sigma^2\omega^2(t)).$$

Now the compatibility equation $a_1'(t) - (a_1(t))' = 0$, we get

$$a = \frac{1}{2},\ b = \frac{\mu^2}{9\sigma},\ \gamma = \frac{1}{3\mu},\ c_0 = 0,\ b_0(t) = B_0e^{c_1/6(1+6A_0)\int_0^{t_1}\omega(t_2)dt_2},$$

$$a_1(t) = \frac{8bc_2a_0(t)b_1(t) + \gamma(-\mu+\sqrt{\mu^2-8b\sigma})((c_1^2-k_0^2)b_1^2(t) - 4c_1c_2b_1(t)b_0(t) + 4c_2^2b_0^2(t))\omega(t)}{8bc_2b_0(t)}\ . \tag{42}$$

$$b_1(t) = B_1 + \frac{(1+6A_0)c_2}{6}\int_0^t\omega(t_1)e_1^{\frac{c_1(1+6A_0)}{6}\int_0^{t_1}\omega(t_2)dt_2}dt_1,$$

and

$$\omega'(t) = -\frac{(\omega(t)(\mu^4 + 3c_1\mu\sigma\omega(t) + 2c_1^2\sigma\omega(t)^2)}{18\mu^2\sigma}, c = \alpha(a\mu - (d_1+2r_1)\alpha\sigma) \tag{43}$$

$$g(z,t) = -\frac{c_1e^{c_1(z\,\gamma + A_0\int_0^t\omega(t_1)dt_1)}}{-1 + c_2e^{c_1(z\,\gamma + A_0\int_0^t\omega(t_1)dt_1)}},\ z = \alpha x - c\tau. \tag{44}$$

By substituting from Eqs. (40)-(43) into the first equation in Eq. (37), we get the required solution. It is too lengthy to be produced here.

6 Solutions of the fractal and fractional time derivative BNE

To derive the solutions of the time- fractal and fractional BNE, we use the similarity transformations $u(x,t) = \tilde{u}(x,\tau)$, where τ takes one of the following forms:

(a) Fractal D $\quad \tau = t^{\beta}$.

(b) CFD: $\quad \tau = \frac{\Gamma(2-\alpha)(T^{\alpha}-(T-t)^{\alpha})}{\alpha} \quad 0 \le t \le T$.

(c) CFFD: $\quad \tau = \frac{(2-\alpha)(1-\alpha)}{2\alpha} Log(\frac{e^{\frac{\alpha}{1-\alpha}(T-t)}-1}{e^{\frac{\alpha}{1-\alpha}T}-1}) \quad 0 \le t \le T$.

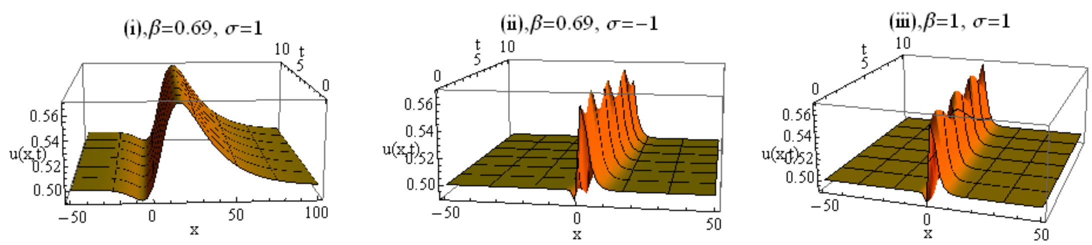
(d) GFD: $\quad \tau = \frac{(\lambda+2)}{2} \int_0^t \frac{1}{\gamma(\frac{1}{\beta},\lambda(T-t_1)^{\beta})} dt_1,\ 0 \le t \le T, 0 < \beta < 1$.

The solutions of Eq. (1) derived in sections (4) and (5) hold but t is replaced by τ.

6.1 Case of fractal BNE

(i) **Case of three waves interactions.**

In this case we have $z = x - c\tau, \tau = \tau = t^{\beta},\ c = \alpha(a\mu - (d_1 + 2r_1)\alpha\sigma)$. Numerical results of the solutions of Eq. (33) are shown in figures 1 (i), (ii), and (iii). They are displayed against x and$t$ for different values of the fractal order β and by varying the parameter σ.



Figures 1 (i), (ii),and (iii). The solutions , given by (33), are displayed against x and t when $b_0 = 3, b = 5, b = 7, b_3 = 2, \mu = 0.7, A_1 = 0.3, A_2 = 0.7,\ \alpha = 5, , \tau = t^{\beta}, a = 0.5, A_3 = 0.8, d_1 = -0.4, r_1 = 0.3, a_3 = 2.5; a_1 = 1.5$.

These figures show that for smaller fractal order derivative, soliton wave with double kinks hold when $\sigma = 1$ in (i). While (iii) shows wrinkle soliton in time and soliton with double kinks in space. This holds for higher fractal order.

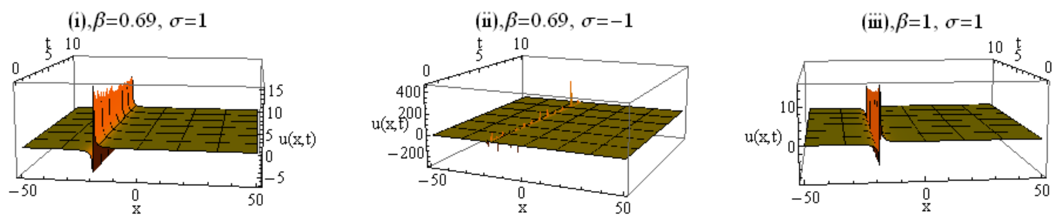
Numerical results of the solutions in Eq. (33) are shown in figures 2 (i), (ii), and (iii). They are displayed against x and t for different values of the fractional order α and by varying the parameter σ.

(ii) **Case of coupled two wave**s.

Here, we have

$$c = \frac{1}{b_2}\alpha(a_2\mu + \alpha(-2b_1\alpha_2 + b_2(\alpha_1 - \beta_2))\sigma),\ z = x - c\tau,\ \tau = t^{\beta}.$$

In figures 3 (i), (ii) and (iii) the solutions (Eq. (37)) are displayed against x and t for different values of the fractal order β and by varying the parameter σ.



Figures 2 (i), (ii) and (iii). The solutions , given by (37), are displayed against x and t when $\alpha_1 = 0.2, \beta_1 = 0.5, \beta_2 = -0.4, b_0 = 5, b_1 = 3, b = 0.2,\ \mu = 0.7, A_1 = 0.3, A_2 = 0.7, \alpha = 0.5, a_1 = 1.3, a = 0.5$.

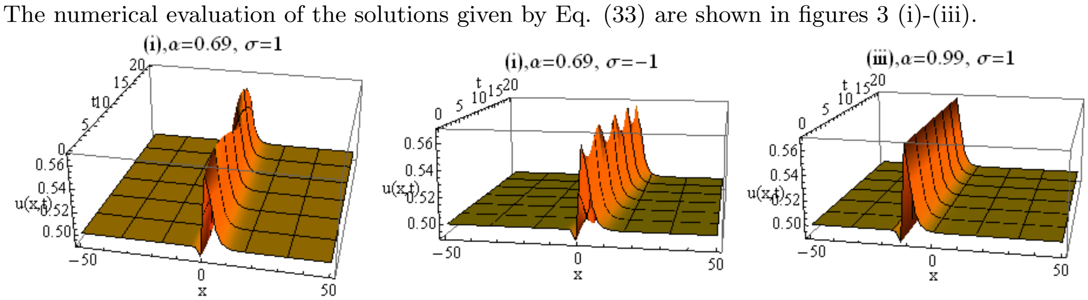
These figures in (i), (ii) and (iii) show soliton wave with double kinks and spikes. In (ii) the solution takes high values when $\sigma = -1$.

6.2 The case of CFD.

Here, we have $z = x - c\tau,\ \tau = (\alpha(T^{\alpha} - (T-t)^{\alpha}))/\Gamma(2-\alpha$.

(i) **Case of three waves interactions.**

The numerical evaluation of the solutions given by Eq. (33) are shown in figures 3 (i)-(iii).

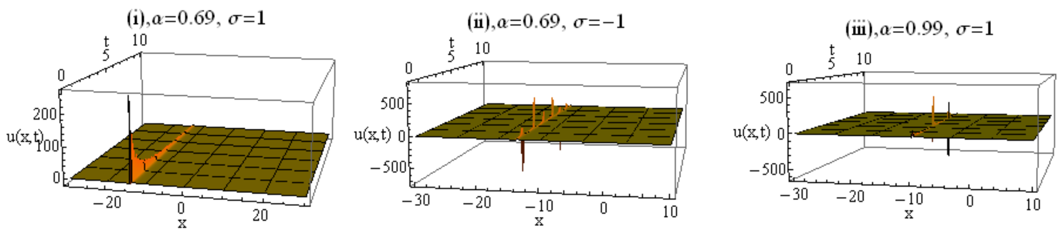


Figures 3 (i), (ii),and (iii). The solutions , given by (33), are displayed against x and t when $b_0 = 3, b = 5, b = 7, b_3 = 2, \mu = 0.7, A_1 = 0.3, A_2 = 0.7,\ d_1 = -0.4, r_1 = 0.3, a_3 = 2.5; a_1 = 1.5,$.

These figures show that for smaller fractional order derivative, undulated soliton wave hold when $\sigma = 1$in (i). While (iii) shows soliton wave with spikes, wrinkle soliton and double kinks when $\sigma = -1$. In (iii) soliton wave with double kinks for higher fractional order.

(i) **Case of coupled two waves.**

The numerical results of the solutions in Eq. (37) are carried. They are shown in the following in Fig. 4.



Figures 4 (i), (ii),and (iii). The solutions , given by (36), are displayed against x and t when $\alpha_1 = 0.2, \beta_1 = 0.5, \beta_2 = -0.4, b_0 = 5, b_1 = 3, b = 0.2$, $\mu = 0.7, A_1 = 0.3, A_2 = 0.7, \alpha = 0.5, a_1 = 1.3, a := 0.5, T = 20$.

Fig. (i) shows that a spike occurs for a small time value. While Figs (ii) and (iii) show multi spikes.

7 Conclusions

The fractal and fractional time-derivative BNEs are studied, where exact solutions are obtained. It is found that the fractal and fractional time-derivative can be reduced to non autonomous ordinary ones. Thus, the fractal and fractional BNE are transformed to (i) a PDE with time dependent coefficient or (ii) a PDE with constant coefficients via similarity transformations. the exact solutions are obtained by using the unified and extended unified methods.

They are obtained are classified to describe traveling and self-similar waves. Numerical computation of the traveling waves are carried. They reveal various geometric configurations:

(a) Soliton wave with double kinks hold when $\sigma = 1$ for smaller fractal order derivative, and wrinkle soliton wave with double kinks for higher fractional order. This holds in time.

(b) Soliton with spikes are propagating in time.

(c) Soliton wave with double kinks and spikes. in the case of the C FD and the solution is high when $\sigma = -1$.

The authors declare that there is no conflict of interest.

References

[1] McKean H. P. Nagumo's equation. Adv Math 4(3) (1970) 209–223.

[2] Nagumo J, Arimoto S, Yoshizawa S. An active pulse transmission line simulating nerve axon. Proc IRE 50(10) (1962) 2061–2070.

[3] Tonnelier A . The McKean's caricature of the FitzHugh-Nagumo model I. The space-clamped system. SIAM J Appl Math 63(2) (2003) 459–484.

[4] Hodgkin, A.L. and A.F. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve. J Physiol, . 117(4) (1952) p. 500–44.

[5] FitzHugh R, Mathematical models of threshold phenomena in the nerve membrane. Bull Math Biophys. 17(4) (1955) 257–278.

[6] FitzHugh R, Impulses and physiological states in theoretical models of nerve membrane. Biophys. J 1 (1961) 445–466.

[7] Yefimova O. Y. , Kudryashov N. A. , Exact solutions of the Burgers–Huxley equation. J. Appl. Math. Mech. 68 (2004) 413–420.

[8] Estévez P. G., Non-classical symmetries and the singular manifold method: the Burgers and the Burgers–Huxley equations. J. Phys. A 27 (1994) 2113–2127.

[9] Estévez P. G., Gordoa P. R., Nonclassical symmetries and the singular manifold method: theory and six examples. Stud. Appl. Math. 95 (1995) 73–113.

[10] Nourazar, S.S., Soori, M. and Nazari-Golshan, A. , On the Exact Solution of Burgers-Huxley Equation Using the Homotopy Perturbation Method. J. Appl. Math. Phys. 3 (2015) 285–294.

[11] Katzengruber B. , Krupa M., Szmolyan P., Bifurcation of traveling waves in extrinsic semiconductors. Physica D 144 (2000) 1–19.

[12] Li J. B., Dai H.. H., On the Study of Singular Nonlinear Traveling Wave Equation: Dynamical System Approach. Science Press, Beijing, (2007).

[13] Chen W., Time-space fabric underlying anomalous diffusion, Chaos, Solitons and Fractals 28 (2006) 923–929.

[14] W. Chen, L.-J. Ye, H.-G. Sun, Fractional diffusion equations by the Kansa method. Comput. Math. Appl. 59 (2010) 1614–1620.

[15] E. Sousa, Finite difference approximations for a fractional advection diffusion problem. J. Comput. Phys. 228 (2009) 4038–4054.

[16] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. , Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006).

[17] Atangana, A. Convergence and stability analysis of a novel iteration method for fractional biological population equation. Neural Comput. Appl. 25(5) (2014) 1021–1030 .

[18] Alibaud, N., Imbert, C., Karch, G., Asymptotic properties of entropy solutions to fractal Burgers equation. SIAM J. Math. Anal. 42(1) (2010) 354–376 .

[19] Karch, G., Miao, C., Xu, X., On convergence of solutions of fractal Burgers equation toward rarefaction waves. SIAM J. Math. Anal. 39(5) (2008) 1536–1549 .

[20] Chan, C.H., Czubak, M., Silvestre, L., Eventual regularization of the slightly supercritical fractional Burgers equation. Discrete Contin. Dyn. Syst. 27(2) (2010) 847–861

[21] Syam M. I., Abu Obayda D., Alshamsi W., Al-Wahashi N., Alshehhi M., Generalized solutions of the fractional Burger's equation. Res. Phys. 15 (2019,)102525.

[22] Xu, Y., Agrawal, O.P., Numerical solutions and analysis of diffusion for new generalized fractional Burgers equation. Fract. Calc. Appl. Anal. 16(3) (2013) 709–736.

[23] Wu, G.C., Baleanu, D,. Variational iteration method for the Burgers' flow with fractional derivatives–new Lagrange multipliers. Appl. Math. Mod. 37(9) (2013) 6183– 6190.

[24] Chen, Y., An, H.L., Numerical solutions of coupled Burgers equations with time-and space-fractional derivatives. Appl. Math. Comput. 200(1) (2008) 87–95.

[25] Chena W., Suna H.,Zhanga X., Koroak D., Anomalous diffusion modeling by fractal and fractional derivatives. Comput. and Math. with Appli. 59 (2010) 1754–1758.

[26] Caputo M., Fabrizio M., A new definition of fractional derivative without singular Kernel. Progr. Fract. Differ. Appl. 1 (2015) 73–85.

[27] Losada J., Nieto J. J., Properties of the new fractional derivative without singular Kernel. Progr. Fract. Differ. Appl. 1 (2015) 87–92.

[28] Yusuf A., Qureshi S, S. F. Shah S. F., Mathematical analysis for an autonomous financial dynamical system via classical and modern fractional operators., Chaos, Solitons and Fractals 132 (2020) 109552.

[29] Atangana A., D. Baleanu D., New fractional derivatives with non-local and non-singular kernel: theory and applications to heat transfer model Therm Sci, 20 (2016) 763-769.

[30] J. Singh, D.Kumar , Z. Hammouch , A. Atangana., A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. Appl. Math. and Comput. 316 (2018) 504–515.

[31] Abdel-Gawad H. I. Tantawy M., Traveling Wave Solutions of DNA-Torsional Model of Fractional Order, Appl. Math. Inf. Sci. Lett. 6 (2) (2018) 85–89.

[32] Abdel-Gawad H. I. Tantawy M. Baleanu. D., Fractional KdV and Boussenisq-Burger's equations, reduction to PDE and stability approaches, Math. Meth. Appl. Sci. 43 (7) (2020) 4125–4135.

[33] Abdel-Gawad H. I., Approximate solutions of nonlinear fractional equations, Appl. Math. and Comp. 215 (2010) 4094–4100.

[34] Abdel-Gawad H.I., Towards a Unified Method for Exact Solutions of Evolution Equations. An Application to Reaction Diffusion Equations with Finite Memory Transport, J. Stat. Phys. 147(2012) 506–521.

[35] Abdel-Gawad H. I. Tantawy M. and Abo-Elkhair R. E., On the extension of solutions of the real to complex KdV equation and A mechanism for the constructions of rogue waves. Wave Random Complex. 26(3) (2016) 397–406.

[36] Abdel-Gawad, H. I., El-Azab, N. and Osman M., Exact solution of the space -dependent KdV equation, JPSP , 82 (2013) 044004.