INVERSE PROBLEMS FOR NONLINEAR NAVIER-STOKES-VOIGT SYSTEM WITH MEMORY

Khonatbek Khompysh[1], A.G. Shakir[1], and A.A. Kabidoldanova[1]

[1]Al-Farabi Kazakh National University

July 24, 2023

Abstract

This paper deals with the unique solvability of some inverse problems for nonlinear Navier-Stokes-Voigt (Kelvin-Voigt) system with memory that governs the flow of incompressible viscoelastic non-Newtonian fluids. The inverse problems that study here, consist of determining a time dependent intensity of the density of external forces, along with a velocity and a pressure of fluids. As an additional information, two types of integral overdetermination conditions over space domain are considered. The system supplemented also with an initial and one of the boundary conditions: stick and slip boundary conditions. For all inverse problems, under suitable assumptions on the data, the global and local in time existence and uniqueness of weak and strong solutions were established.

Posted on 24 Jul 2023 — The copyright holder is the author/funder. All rights reserved. No reuse without permission. — https://doi.org/10.22541/au.169022093.36001393/v1 — This a preprint and has not been peer reviewed. Data may be preliminary.

INVERSE PROBLEMS FOR NONLINEAR NAVIER-STOKES-VOIGT SYSTEM WITH MEMORY

Kh. Khompysh[1,*], A.G. Shakir[1], A.A. Kabidoldanova[1],

[1]*Al-Farabi Kazakh National University, Almaty, Kazakhstan*

Abstract. This paper deals with the unique solvability of some inverse problems for nonlinear Navier-Stokes-Voigt (Kelvin-Voigt) system with memory that governs the flow of incompressible viscoelastic non-Newtonian fluids. The inverse problems that study here, consist of determining a time dependent intensity of the density of external forces, along with a velocity and a pressure of fluids. As an additional information, two types of integral overdetermination conditions over space domain are considered. The system supplemented also with an initial and one of the boundary conditions: stick and slip boundary conditions. For all inverse problems, under suitable assumptions on the data, the global and local in time existence and uniqueness of weak and strong solutions were established.

Keywords: Inverse problem; Navier-Stokes-Voigt system with memory; viscoelastic incompressible fluids; slip and stick boundary conditions; existence and uniqueness.

MSC (2020): 76M21, 35R30; 76D05; 35Q35; 76D03.

1. Introduction

The study of inverse problems of Newtonian hydrodynamics, in particular, for the Navier-Stokes equations, as well as the related with them systems of heat convection and magnetohydrodynamics, have been studied by many authors who have proposed their various approaches and methodologies. One of the main work is the monograph [39] of Prilepko and et al., that included their previous works on the theory of the Navier-Stokes equations since 1989. By this approach, at first the original inverse problems reduce to an equivalent operator equations of second kind, and then use the fixed point principle. As another approach to solving inverse problems of hydrodynamics one can refer to the works of Abylkairov [1]-[2], who used the method of successive approximations. Other approaches which were used the methods such as Carleman estimates and et al for solving inverse problems of hydrodynamics and PDE have been proposed by many authors, for example, we refer readers to [8], [9]- [13], [17], [18], [19], [20], [21], [24], [29] and the references therein. However, all above approaches are applicable only in the case when the corresponding direct problems are unique solvable and their solutions have additional extra smoothness. But, inverse problems

E-mail address: konat_k@mail.ru, ajdossakir@gmail.com, kabasem@mail.ru.
Date: July 22, 2023.
*Corresponding author.

for non-Newtonian hydrodynamics are not sufficiently studied from a mathematical point of view, see for instance [2,5,6,15,25,26], and in although they have important applications in physical point of view. Thus, in this paper, we study the unique solvability of some inverse source problems for a system of integro-differential Navier-Stokes-Voigt system (or so called Kelvin-Voigt system) describing the motion of incompressible non-Newtonian fluids, which taken into account viscoelastic properties. Due to this proposed approach, in this work, we establish the unique solvability of the above inverse problems without using any information about the solvability of the corresponding direct problems.

The statement of problems. Let Ω be a bounded domain in $\mathbb{R}^d,\ d \geq 2$, with a smooth boundary $\partial\Omega$, and $Q_T = \Omega \times (0,T)$, T is a fixed positive constant, and $\Gamma_T = \partial\Omega \times [0,T]$. This paper devoted to recover an intensity of the external forces $f(t)$ addition to a velocity $\mathbf{u}(x,t)$ and a pressure $p(x,t)$, from the system of nonlinear Navier-Stokes-Voigt equations with memory governing the flows of incompressible viscoelastic fluids. More precisely, we study the following inverse problems of determining the functions $(\mathbf{u}(x,t), p(x,t), f(t))$, satisfying the equations

$$\mathbf{u}_t - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \int\limits_0^t K(t-s)\Delta\mathbf{u}(\mathbf{x},s)ds - \nabla p = f(t)\mathbf{g}(\mathbf{x},t) \ \text{ in } \ Q_T, \quad (1.1)$$

$$\mathbf{div}\,\mathbf{u}(\mathbf{x},t) = 0 \ \text{ in } \ Q_T, \quad (1.2)$$

the initial condition

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega, \quad (1.3)$$

and one of the boundary conditions: the stick boundary condition

$$\mathbf{u}(\mathbf{x},t) = 0 \quad \text{on} \quad \Gamma_T \quad (1.4)$$

or the slip boundary condition [28,36,37]:

$$\mathbf{u}_{\mathbf{n}}(\mathbf{x},t) = \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{curl}\,\mathbf{u}\times\mathbf{n} = 0, \ (\mathbf{x},t)\in\Gamma_T \quad (1.5)$$

and the overdetermination condition

$$\int_\Omega \mathbf{u}\omega\mathbf{dx} = e(t), \ t\in[0,T]. \quad (1.6)$$

where $\mathbf{u}_{\mathbf{n}}$ is the normal component of $\mathbf{u}(\mathbf{x},t)$ on $\partial\Omega$, and $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$. Here the bold letters denote vector-valued functions and $\mathbf{u}(\mathbf{x},t) = (u_1, u_2, ..., u_d)$ and $p(\mathbf{x},t)$ is a velocity field and a pressure, respectively, and ν and $\varkappa > 0$ are coefficients of the kinematic viscosity and relaxation of the fluids, respectively. The vector-function $\mathbf{F}(\mathbf{x},t) := f(t)\mathbf{g}(\mathbf{x},t)$ is the density of external force with unknown intensity $f(t)$, and $\mathbf{u}_0(x)$, $\mathbf{g}(\mathbf{x},t)$ and $K(t)$ are given functions.

Moreover, instead of the overdetermination condition (1.6) we will also consider the condition

$$\int_\Omega \mathbf{u}\sigma(\mathbf{x})\mathbf{dx} = \mathbf{e}(\mathbf{t}), \ \text{with} \ \ \sigma(\mathbf{x}) = \omega - \varkappa\boldsymbol{\Delta}\omega, \quad \mathbf{t}\in[\mathbf{0},\mathbf{T}], \quad (1.7)$$

It is obvious that (1.7) can be written as

$$\int_\Omega (\mathbf{u}\omega + \varkappa\nabla\mathbf{u}\nabla\omega)\mathbf{dx} = e(t), \ \ t\in[0,T], \ \text{in the case (1.4)} \quad (1.8)$$

and

$$\int_\Omega (\mathbf{u}\omega + \varkappa \mathbf{curl}\,\mathbf{u}\,\mathbf{curl}\,\omega)\mathbf{dx} = e(t),\ \ t \in [0,T],\ \text{in the case (1.5)}. \tag{1.9}$$

and we will use (1.8) and (1.9) instead of (1.7). Thus, the inverse problems that we investigate in this paper are: (1.1)-(1.4), (1.6) (Inverse problem I); and (1.1)-(1.3), (1.5), (1.6) (Inverse problem II); (1.1)-(1.4), (1.7) (Inverse problem III); (1.1)-(1.3), (1.5), (1.7) (Inverse problem IV).

The system of equations (1.1)-(1.2) is called Navier-Stokes-Voigt system with memory or Kelvin-Voigt system with memory [23], [41], [40], [42], and it describes the flows of an incompressible non-Newtonian fluids with viscoelastic and relaxation properties. For the physical justifications of this models and derivation of the mathematical equation we refer to the works of Oskolkov [34]- [35], Barnes [7], Joseph [22] and Zvyagin, Turbin [42] and the references therein.

The direct problems for the system (1.1)-(1.2) and for their modifications in the various cases of the memory term, have been studied in many papers as [34], [23], [42], [41], and et al., where the existence and uniqueness results for weak and strong solutions were established. The study of direct problems is important and they model the process under known information on physical parameters affecting to the processes. However, in the real, there is might a situation, that one or some of such parameters are unknown or unacceptable for direct measurement during the process, for instance, they located underground or in a high temperature media, and et al [39]. The problems determining such unknown parameters under additional information on solutions are called inverse problems, therefore to investigation of them is also important in both of mathematical and physical point, and they have many applications in the various of branches of sciences and technology. Some inverse problems recovering time depended source for the system (1.1) without the memory term have been studied in [5], [15], [27], [30]. An inverse problem recovering a source $\mathbf{f}(\mathbf{x})$ depending on space variable for the linear system (1.1) (without convective term) recently has been investigated in [26]. An inverse problem for the linear Navier-Stokes-Voigt system (1.1), i.e. neglecting the convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$, recently has been studied by authors in [27], and due to the overdetermination condition (1.6), the uniqueness and existence of solutions were locally established even for linear case. The main goal of this paper is to establish the questions of global and local in time existence and uniqueness of weak and strong solutions to the inverse problems I-IV.

2. Preliminaries

In this section, we introduce the main functional spaces and some useful inequalities related to the boundary conditions (1.4) and (1.5) from [28]. We distinguish vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. The symbol C will denote a generic constant - generally a positive one, whose value will not be specified; it can change from one inequality to another. We denote by $\mathbf{L}^2(\Omega)$ the usual Lebesgue space of square integrable vector-valued functions on Ω, and by $\mathbf{W}^{m,2}(\Omega)$ the Sobolev space of functions in $\mathbf{L}^2(\Omega)$ whose weak derivatives of order not greater than m are in $\mathbf{L}^2(\Omega)$. The norm and inner product in $\mathbf{L}^2(\Omega)$ denoted by $\|\cdot\|_{2,\Omega}$ and $(\cdot,\cdot)_{2,\Omega}$, respectively.

Let us introduce the function spaces regarding to the slip and stick boundary conditions (1.4) and (1.5), respectively (see [32]:

$$\mathbf{H}(\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div}\mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = 0\}; \qquad \mathbf{H}_{\mathbf{n}}(\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div}\mathbf{v} = 0,\ \mathbf{v}_{\mathbf{n}}|_{\partial\Omega} = 0\};$$
$$\mathbf{H}^1(\Omega) \equiv \{\mathbf{v} \in \mathbf{W}^1_2(\Omega) : \operatorname{div}\mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = 0\}; \quad \mathbf{H}^1_{\mathbf{n}}(\Omega) \equiv \{\mathbf{v} \in \mathbf{W}^1_2(\Omega) : \operatorname{div}\mathbf{v} = 0, \mathbf{v}_{\mathbf{n}}|_{\partial\Omega} = 0\};$$
$$\mathbf{H}^2(\Omega) \equiv \{\mathbf{v} \in \mathbf{H}^1(\Omega) \cap \mathbf{W}^{2,2}(\Omega)\}; \quad \mathbf{H}^2_{\mathbf{n}}(\Omega) \equiv \{\mathbf{v} \in \mathbf{H}^1_{\mathbf{n}}(\Omega) \cap \mathbf{W}^{2,2}(\Omega) : (\mathbf{curl}\ \mathbf{v} \times \mathbf{n})|_{\partial\Omega} = 0\}.$$

and for the simplicity, we use the following common notation for both cases

$$\mathbf{V} := \begin{cases} \mathbf{H}(\Omega), \text{ in the case (1.4)}; \\ \mathbf{H}_{\mathbf{n}}(\Omega), \text{ in the case (1.5)}, \end{cases} \mathbf{V}^i := \begin{cases} \mathbf{H}^i(\Omega), \text{ in the case (1.4)}; \\ \mathbf{H}^i_{\mathbf{n}}(\Omega), \text{ in the case (1.5)},\ i = 1, 2. \end{cases} \tag{2.1}$$

The inner product and the norm in $\mathbf{H}^1_{\mathbf{n}}(\Omega)$ is $(\operatorname{rot}\mathbf{v}, \operatorname{rot}\mathbf{u})_{2,\Omega}$ and $\|\mathbf{v}\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} := \|\mathbf{curl}\ \mathbf{v}\|_{2,\Omega}$, respectively. According to [28, 32] and the references cited in them (see for example [16]), for any function $\mathbf{u} \in \mathbf{H}^1_{\mathbf{n}}(\Omega)$ (for $\mathbf{H}(\Omega)$ is well known from Navier-Stokes theory), the usual Ladyzhenskaya and Poincare inequalities and following inequalities are hold:
Poincare inequality

$$\|\mathbf{u}\|_{2,\Omega} \le C_1(\Omega)\, \|\nabla\mathbf{u}\|_{2,\Omega}\,,\ \mathbf{u} \in \mathbf{H}^1_{\mathbf{n}}(\Omega); \tag{2.2}$$

$$N_1(\Omega)\, \|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)} \le \|\mathbf{curl}\,\mathbf{u}\|_{2,\Omega} \le N_2(\Omega)\, \|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)}\,, \quad \forall\mathbf{u} \in \mathbf{H}_{\mathbf{n}}(\Omega); \tag{2.3}$$

$$N_3(\Omega)\, \|\mathbf{u}\|_{\mathbf{W}^{2,2}(\Omega)} \le \|\Delta\mathbf{u}\|_{2,\Omega} = \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}\|_{2,\Omega} \le N_4(\Omega)\, \|\mathbf{u}\|_{\mathbf{W}^{2,2}(\Omega)}\,, \quad \forall\mathbf{u} \in \mathbf{H}^2_{\mathbf{n}}(\Omega); \tag{2.4}$$

Ladyzhenskaya's inequalities [31, 32]

$$\|\mathbf{u}\|^4_{4,\Omega} \le 2\, \|\mathbf{u}\|^2_{2,\Omega}\, \|\nabla\mathbf{u}\|^2_{2,\Omega}\,; \tag{2.5}$$

in case $d = 2$, and

$$\|\mathbf{u}\|^4_{4,\Omega} \le (4/3)^{\frac{3}{2}}\, \|\mathbf{u}\|_{2,\Omega}\, \|\nabla\mathbf{u}\|^3_{2,\Omega}\,; \tag{2.6}$$

in case $d = 3$, and

$$\|\mathbf{u}\|_{6,\Omega} \le (48)^{\frac{1}{6}}\, \|\nabla\mathbf{u}\|_{2,\Omega}\,,\ d = 3. \tag{2.7}$$

Let us introduce now the bilinear and continuous form $\mathbf{a}$ on $\mathbf{H}^1$, associated with the operator $-\Delta$:

$$\mathbf{a}(\mathbf{v}, \mathbf{u}) = \begin{cases} (\nabla\mathbf{v}, \nabla\mathbf{u})_{2,\Omega}\,, & \forall\mathbf{v}, \mathbf{u} \in \mathbf{H}^1(\Omega), \quad \text{in the case (1.4)} \\ (\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{u})_{2,\Omega}\,, & \forall\mathbf{v}, \mathbf{u} \in \mathbf{H}^1_{\mathbf{n}}(\Omega), \quad \text{in the case (1.5)} \end{cases} \tag{2.8}$$

It is clear that $\mathbf{a}(\mathbf{u}, \mathbf{u}) = \|\nabla\mathbf{u}\|_{2,\Omega} = \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}$ is a norm on $\mathbf{H}^1(\Omega)$, which is equivalent to $\mathbf{W}^{1,2}(\Omega)$-norm. In particular, due to Fridrichs inequality and (2.3), in $\mathbf{H}^1_{\mathbf{n}}$ the norm $\|\mathbf{u}\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} = \|\mathbf{curl}\,\mathbf{u}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)}$, and therefore equivalent to the norm $\|\nabla\mathbf{u}\|_{2,\Omega}$.

Thus, $\mathbf{a}$ defines an isomorphism A from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{-1}(\Omega)$,

$$\langle A\mathbf{v}, \mathbf{u}\rangle \equiv \mathbf{a}(\mathbf{v}, \mathbf{u}), \quad \forall\mathbf{v}, \mathbf{u} \in \mathbf{H}^1(\Omega), \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathbf{H}^1$ and $\mathbf{H}^{-1}$. There hold the following continuous inclusions

$$\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega), \tag{2.10}$$

where each of the first two spaces is dense in the next one.

It follows from (2.4) also that in $\mathbf{H}^2_{\mathbf{n}}$ the norm $\|\Delta\mathbf{u}\|_{2,\Omega} = \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{u}\|_{\mathbf{W}^{2,2}(\Omega)}$.

Regarding to the sliding condition (1.5), we have the following Green formulas (see [28,32]):

$$(-\Delta \mathbf{v}, \mathbf{u})_{2,\Omega} = -(\nabla \mathbf{div}\, \mathbf{v}, \mathbf{u})_{2,\Omega} + \left(\mathbf{curl}^2\, \mathbf{v}, \mathbf{u}\right)_{2,\Omega} = -\int_{\partial\Omega} \mathbf{div}\, \mathbf{v} \cdot \mathbf{u}_n \, dS +$$
$$(\mathbf{div}\, \mathbf{v}, \mathbf{div}\, \mathbf{u})_{2,\Omega} + \int_{\partial\Omega} \mathbf{u} \cdot (\mathbf{curl}\, \mathbf{v} \times \mathbf{n}) \, dS + (\mathbf{curl}\, \mathbf{v}, \mathbf{curl}\, \mathbf{u})_{2,\Omega} = (\mathbf{curl}\, \mathbf{v}, \mathbf{curl}\, \mathbf{u})_{2,\Omega} \tag{2.11}$$

in the case $d = 3$, and

$$(-\Delta \mathbf{v}, \mathbf{u})_{2,\Omega} = (\mathbf{div}\, \mathbf{v}, \mathbf{div}\, \mathbf{u})_{2,\Omega} + \left(\overline{\mathbf{curl}}\, (\mathbf{curl}\, \mathbf{v}), \mathbf{u}\right)_{2,\Omega} =$$
$$\int_{\partial\Omega} (\mathbf{curl}\, \mathbf{v} \times \mathbf{n})\, \mathbf{u} \, dS + (\mathbf{curl}\, \mathbf{v}, \mathbf{curl}\, \mathbf{u})_{2,\Omega} = (\mathbf{curl}\, \mathbf{v}, \mathbf{curl}\, \mathbf{u})_{2,\Omega}\,, \tag{2.12}$$

in the case $d = 2$, where $\overline{\mathbf{curl}}\, \varphi$ is the vector $(\varphi_{x_2}, -\varphi_{x_1})_{2,\Omega}$ for the scalar function φ.

The following nonlinear Gronwall's inequality will be used to establish the first and second estimates below.

Lemma 1. *If $y : \mathbb{R}^+ \longrightarrow [0, \infty)$ is a continuous function such that*

$$y(t) \leq C_1 \int_0^t y^\mu(s) ds + C_2, \quad t \in \mathbb{R}^+, \quad \mu > 1$$

for some positive constants C_1 and C_2, then

$$y(t) \leq C_2 \left(1 - (\mu - 1) C_1 C_2^{\mu-1} t\right)^{-\frac{1}{\mu-1}} \quad \text{for} \ \ 0 \leq t < t_{\max} := \frac{1}{(\mu - 1) C_1 C_2^{\mu-1}}.$$

Proof. See *e.g.* [4]. □

3. Inverse problems I-II

In this section we work with the inverse problems I and II: define a weak and a strong solutions; reduce them to the corresponding an equivalent nonlocal problems which we will study in the next sections.

Definition 1. *The pair of functions $(\boldsymbol{u}(x,t), f(t))$ is a weak solution to the inverse problem (1.1)-(1.4), (1.6) and (1.1)-(1.3), (1.5), (1.6), if:*

(1) $\mathbf{u} \in \mathbf{L}^\infty(0, T; \mathbf{V}^1(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{V}^1(\Omega))$, $\mathbf{u}_t \in \mathbf{L}^2(0, T; \mathbf{V}^1(\Omega))$, $f(t) \in L^2[0, T]$;
(2) $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *a.e. in* Ω;
(3) For every $\boldsymbol{\varphi} \in \boldsymbol{V}^1(\Omega)$ and for a.a. $t \in (0, T)$ holds

$$\frac{d}{dt}\Big((\mathbf{u}(t), \varphi)_{2,\Omega} + \varkappa \mathbf{a}\,(\mathbf{u}(t), \varphi)\Big) + \nu \mathbf{a}\,(\mathbf{u}(t), \varphi) + ((\boldsymbol{u}(t) \cdot \nabla) \boldsymbol{u}(t), \varphi)_{\mathbf{2},\mathbf{\Omega}} =$$
$$f(t)\,(\mathbf{g}(t), \varphi)_{2,\Omega} - \int_0^t K(t-s) \mathbf{a}\,(\mathbf{u}(s), \varphi)\, ds. \tag{3.1}$$

Definition 2. *The pair of functions $(\boldsymbol{u}(x,t), f(t))$ is called a strong solution to the inverse problems (1.1)-(1.4), (1.6) and (1.1)-(1.3), (1.5), (1.6), if:*

(1) $\mathbf{u} \in \mathbf{L}^\infty(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))\cap\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))$, $\mathbf{u}_t \in \mathbf{L}^2(0,T;\mathbf{V}^2(\Omega))$, $f(t) \in L^2[0,T]$;

(2) Each equation holds in the distribution sense in the their corresponding domain.

Remark 1. *The pressure p, as usual, was not included in the definition of a weak solution. It can be uniquely recovered from equation (1.2) by using de Rhaam's lemma, after existence of $\mathbf{u}$, f as in [3].*

3.1. **Reformulation of problem: an equivalent nonlocal problems.** Assume that data of the problem satisfy the following conditions

$$\mathbf{u}_0(\mathbf{x}) \in \mathbf{V}^1(\Omega); \tag{3.2}$$

$$\exists k_0 \in \mathbb{R} : 0 < k_0 < \infty, \text{ such that } |g_0(t)| = \left|(\mathbf{g}(t),\omega)_{2,\Omega}\right| \geq k_0 > 0, \ \forall t \geq 0; \tag{3.3}$$

$$\mathbf{g}(\mathbf{x},t) \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)); \tag{3.4}$$

$$\omega(\mathbf{x}) \in \mathbf{V}^1(\Omega), \ e(t) \in W_2^1([0,T]); \tag{3.5}$$

$$(\mathbf{u}_0,\omega)_{2,\Omega} = e(0); \tag{3.6}$$

$$K(t) \in L^2([0,T]) : \quad \|K(t)\|_{L^2([0,T])} \equiv K_0 < \infty. \tag{3.7}$$

Let us now multiply (1.1) by $\omega(\mathbf{x})$ and integrate over Ω. Integrating by parts and using (1.6) and the assumption (3.3), we define $f(t)$

$$\begin{aligned} f(t) = & \frac{1}{g_0(t)}\Big(e'(t) + \varkappa \mathbf{a}(\mathbf{u}_t,\omega) + \\ & \nu \mathbf{a}(\mathbf{u},\omega) - ((\mathbf{u}\cdot\nabla)\omega,\mathbf{u})_{2,\Omega} + \int_0^t K(t-s)\mathbf{a}(\mathbf{u},\omega)\,ds\Big) := F(\mathbf{u},t). \end{aligned} \tag{3.8}$$

Hence, substituting (3.8) into (1.1), we reformulate the problem: Find $\mathbf{u}$ and p from the nonlocal direct problem for the system

$$\begin{aligned} & \mathbf{u}_t - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \int_0^t K(t-s)\Delta\mathbf{u}(s)ds - \nabla p = F(\mathbf{u},t)\mathbf{g}(\mathbf{x},t), \\ & \mathbf{div}\mathbf{u}(\mathbf{x},t) = 0, \ (\mathbf{x},t) \in Q_T, \end{aligned} \tag{3.9}$$

supplemented with the initial and boundary conditions (1.3) and (1.4) or (1.5), where $F(\mathbf{u},t) = f(t)$ given by (3.8). The following lemma is valid.

Lemma 2. *Assume that the conditions* (3.3)-(3.6) *are fulfilled. Then the solvability of the inverse problem (1.1)-(1.4), (1.6) ((1.1)-(1.3), (1.5), (1.6)) is equivalent to the nonlocal direct problem (3.9), (1.3)-*(1.4) *((3.9), (1.3), (1.5)), i.e. if $(\boldsymbol{u},p,f)$ is a solution of the inverse problem (1.1)-(1.4), (1.6) ((1.1)-(1.3), 1.5), (1.6)), then the pair $(\boldsymbol{u},p)$ is the solution of the nonlocal direct problem (3.9), (1.3)-*(1.4) *((3.9), (1.3), (1.5)), and vise versa, if the pair $(\boldsymbol{u},p)$ is a solution of the nonlocal direct problem (3.9), (1.3)-*(1.4) *((3.9), (1.3), (1.5)), then this pair $(\boldsymbol{u},p)$ together with the function $f(t)$ defined by* (3.8) *gives a solution of the inverse problem (1.1)-(1.4), (1.6) ((1.1)-(1.3), (1.5), (1.6)) .*

Remark 2. *A weak and strong solution to the problem (1.1)-(1.4), (3.8) ((1.1)-(1.3), (1.5), (3.8)) can be defined analogically as in Definition 1 and 2.*

Proof. In fact, the first part of the proof has been done when we derived (3.9) from (1.1)-(1.2) and (1.6). Let us prove the second part, i.e. let $(\mathbf{u}, p)$ be a solution of the nonlocal direct problem (3.9), (1.3)-(1.4) ((3.9), (1.3), (1.5)) with $F(\mathbf{u}, t) = f(t)$ defined by (3.8). It means that $(\mathbf{v}, f)$ satisfies the equations (1.1)-(1.4) ((1.1)-(1.3), (1.5)). In order to complete the proof, it is enough to prove that $\mathbf{u}$ satisfies the overdetermination condition (1.6).

Let us assume that for contradiction, i.e. the overdetermination condition (1.6) doesn't hold. Suppose that

$$(\mathbf{u}, \boldsymbol{\omega})_{2,\Omega} = e_1(t),\ \ t \geq 0. \tag{3.10}$$

where $e_1(t) \neq e(t)$ for all $t \geq 0$. Thus, by the conditions (3.5), (3.6) and the definition of solution, we have $e_1(t) \in W_2^1([0, T])$ and

$$e_1(0) = (\mathbf{u}_0, \boldsymbol{\omega})_{2,\Omega} = e(0). \tag{3.11}$$

Multiply (3.9) by $\boldsymbol{\omega}$ and integrating by parts and using (3.10) and (3.8), we get

$$e_1'(t) + \varkappa \mathbf{a}\left(\mathbf{u}_t, \omega\right) + \nu \mathbf{a}\left(\mathbf{u}, \omega\right) - ((\mathbf{u} \cdot \nabla)\omega, \mathbf{u})_{2,\Omega} + \int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}, \omega\right) ds = F(t, \mathbf{u}) g_0(t), \tag{3.12}$$

where $F(t, \mathbf{v})$ is defined in (3.8). Plugging (3.8) into (3.12) we obtain

$$e_1'(t) = e'(t). \tag{3.13}$$

It follows from the Cauchy problem (3.13) and (3.11) that $e_1(t) \equiv e(t)$ for all $t > 0$, which contradicts to above assumption. $\square$

4. Existence of weak solutions of the nonlocal problems (3.9), (1.3)-(1.4) and (3.9), (1.3), (1.5)

By Lemma 2, we will study the nonlocal problems (3.9), (1.3)-(1.4) and (3.9), (1.3), (1.5) instead of original inverse problems I: (1.1)-(1.4), (1.6) and II: (1.1)-(1.3), (1.5), (1.6), respectively.

Theorem 1. *Assume that the conditions (3.2)-(3.7) are fulfilled and there exists a positive constant m such that*

$$\frac{\varkappa}{k_0^2} \sup_{t\in[0,T]} \|\mathbf{g}(t)\|_{2,\Omega}^2 \|\omega\|_{\mathbf{V}^1(\Omega)}^2 \leq m < 2. \tag{4.1}$$

Then there exist $T_1 \in (0, T]$ and at least one weak solution to the nonlocal direct problems (3.9), (1.3)-(1.4) and (3.9), (1.3), (1.5) in Q_{T_1}, where T_1 is defined at (4.19) below. Moreover, weak solutions satisfy the estimate

$$\|\mathbf{u}\|_{\mathbf{L}^\infty(0,T_1;\mathbf{L}^2(\Omega)\cap\mathbf{V}^1(\Omega))}^2 + \|\mathbf{u}\|_{\mathbf{L}^2(0,T_1;\mathbf{V}^1(\Omega))}^2 + \|\mathbf{u}_t\|_{\mathbf{L}^2(0,T_1;\mathbf{L}^2(\Omega)\cap\mathbf{V}^1(\Omega))}^2 \leq C, \tag{4.2}$$

where C is a constant depending on data of the problem.

Remark 3. *The assumption* (4.1) *has been appeared due to the obtaining a priory estimates regarding to the overdetermination condition (1.6) with a general form. It can be removed if consider the overdetermination condition* (1.7) *with a special testing function $\sigma(\mathbf{x}) = \omega - \varkappa \boldsymbol{\Delta}\omega$ instead of (1.6) or with the special RHS $\mathbf{g}(\mathbf{x}, t) = \sigma(\mathbf{x})$, see Theorem 4 and 8, below.*

Proof. We use Galerkin's method, i.e. first construct approximate solutions, establish some a priori estimates, and passage to the limit.

4.1. **Galerkin's approximations.** Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be an orthonormal family in $\mathbf{L}^2(\Omega)$ formed by functions of $\mathbf{V}$ whose linear combinations are dense in $\mathbf{V}^1(\Omega)$. Given $n \in \mathbb{N}$, let us consider the n-dimensional space $\mathbf{X}^n$ spanned by φ_k, $k = 1, \dots, n$, respectively. For each $n \in \mathbb{N}$, we search for approximate solutions to the problem (1.1)-(1.4), (3.8) in the form

$$\mathbf{u}^n(x,t) = \sum_{j=1}^{n} c_j^n(t)\varphi_j(x), \quad \varphi_j \in \mathbf{X}^n, \tag{4.3}$$

where unknown coefficient $c_j^n(t)$, $j = 1, ..., n$ are defined as solutions of the following system of ordinary differential equations (ODE) derived from

$$\begin{gathered}\frac{d}{dt}\Big((\mathbf{u}^n,\varphi_k)_{2,\Omega} + \varkappa \mathbf{a}\left(\mathbf{u}^n,\varphi_k\right)\Big) + \nu \mathbf{a}\left(\mathbf{u}^n,\varphi_k\right) - ((\mathbf{u}^n\cdot\nabla)\varphi_k,\mathbf{u}^n)_{2,\Omega} = \\ -\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n,\varphi_k\right)ds + F(\mathbf{u}^n,t)\left(\mathbf{g}(\mathbf{x},t),\varphi_k\right)_{2,\Omega}, \quad k = 1,\ 2,\ \dots,\ n,\end{gathered} \tag{4.4}$$

where

$$F_1(\mathbf{u}^n,t) = \frac{1}{g_0}\left(e' + \varkappa \mathbf{a}\left(\mathbf{u}^n_t,\omega\right) + \nu \mathbf{a}\left(\mathbf{u}^n,\omega\right) - ((\mathbf{u}^n\cdot\nabla)\omega,\mathbf{u}^n)_{2,\Omega} + \int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n,\omega\right)ds\right). \tag{4.5}$$

The system (4.4) is supplemented with the following Cauchy data

$$\mathbf{u}^n(0) = \mathbf{u}_0^n \quad \text{in } \Omega, \tag{4.6}$$

where

$$\mathbf{u}_0^n = \sum_{j=1}^{n}\left(\mathbf{u}_0,\varphi_j\right)_{2,\Omega}\varphi_j$$

is sequence in $\mathbf{L}^2(\Omega) \cap \mathbf{V}^1(\Omega)$ respectively such that

$$\mathbf{u}_0^n \to \mathbf{u}_0(x) \text{ strong as } n \to \infty \text{ in } \mathbf{L}^2(\Omega) \cap \mathbf{V}^1(\Omega). \tag{4.7}$$

According to a general theory of ordinary differential equations, the system (4.4)-(4.6) has a solution $c_j^n(t)$ in $[0, t_0]$. By a priori estimates which we shall establish below, the solution can be extended to $[0, T_0] \subset [0, T]$, where $[0, T_0]$ is a maximal time interval, such that a priori estimates are hold.

4.2. **A priori estimates.** Multiply the $k-$th equation of (4.4) by $c_k^n(t)$ and $\frac{dc_k^n(t)}{dt}$ and sum up by k from 1 to n. The we have

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\right) + \nu\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} = \\ -\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n(t)\right)ds + F_1(\mathbf{u}^n,t)\left(\mathbf{g},\mathbf{u}^n\right)_{2,\Omega}\end{gathered} \tag{4.8}$$

$$
\begin{gathered}
\frac{\nu}{2}\frac{d}{dt}\left\|\mathbf{u}^n\right\|^2_{\mathbf{V}^1(\Omega)}+\left\|\mathbf{u}^n_t(t)\right\|^2_{2,\Omega}+\varkappa\left\|\mathbf{u}^n_t(t)\right\|^2_{\mathbf{V}^1(\Omega)}= \\
-\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n_t(t)\right)ds+F_1(\mathbf{u}^n,t)\left(\mathbf{g},\mathbf{u}^n_t\right)_{2,\Omega}+\left(\left(\mathbf{u}^n\cdot\nabla\right)\mathbf{u}^n_t,\mathbf{u}^n\right)_{2,\Omega},
\end{gathered}
\tag{4.9}
$$

respectively. Adding (4.8) and (4.9), we obtain

$$
\begin{gathered}
\frac{1}{2}\frac{d}{dt}\left(\left\|\mathbf{u}^n\right\|^2_{2,\Omega}+(\nu+\varkappa)\left\|\mathbf{u}^n\right\|^2_{\mathbf{V}^1(\Omega)}\right)+\nu\left\|\mathbf{u}^n\right\|^2_{\mathbf{V}^1(\Omega)}+\left\|\mathbf{u}^n_t(t)\right\|^2_{2,\Omega}+\varkappa\left\|\mathbf{u}^n_t(t)\right\|^2_{\mathbf{V}^1(\Omega)}= \\
-\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n(t)\right)ds-\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n_t(t)\right)ds+ \\
F_1(\mathbf{u}^n,t)\left(\mathbf{g},\mathbf{u}^n_t\right)_{2,\Omega}+F_1(\mathbf{u}^n,t)\left(\mathbf{g},\mathbf{u}^n\right)_{2,\Omega}+\left(\left(\mathbf{u}^n\cdot\nabla\right)\mathbf{u}^n_t,\mathbf{u}^n\right)_{2,\Omega}:=\sum_{i=1}^5 I_i.
\end{gathered}
\tag{4.10}
$$

Next, using Hölder and Young inequalities with suitable powers, we estimate terms on the right hade side of (4.10).

$$
|I_1|\leq\frac{\nu}{2}\left\|\mathbf{u}^n\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{K_0^2}{2\nu}\int\limits_0^t\left\|\mathbf{u}^n(s)\right\|^2_{\mathbf{V}^1(\Omega)}ds;
\tag{4.11}
$$

$$
|I_2|\leq\frac{\varepsilon_1}{4}\left\|\mathbf{u}^n_t\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{K_0^2}{\varepsilon_1}\int\limits_0^t\left\|\mathbf{u}^n(s)\right\|^2_{\mathbf{V}^1(\Omega)}ds;
\tag{4.12}
$$

$$
\begin{gathered}
|I_3|\leq|F_1|\left\|\mathbf{g}\right\|_{2,\Omega}\left\|\mathbf{u}^n\right\|_{2,\Omega}\leq\frac{\left\|\mathbf{g}\right\|_{2,\Omega}\left\|\mathbf{u}^n\right\|_{2,\Omega}}{k_0}\Big[|e'(t)|+\varkappa\left\|\mathbf{u}^n_t\right\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)}+ \\
\nu\left\|\mathbf{u}^n\right\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)}+\left\|\mathbf{u}^n\right\|^2_{4,\Omega}\|\omega\|_{\mathbf{V}^1(\Omega)}+\|\omega\|_{\mathbf{V}^1(\Omega)}K_0\left(\int\limits_0^t\left\|\mathbf{u}^n(s)\right\|^2_{\mathbf{V}^1(\Omega)}ds\right)^{\frac{1}{2}}\Big]\leq \\
\left(\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2}\right)\left\|\mathbf{u}^n\right\|^2_{2,\Omega}+\frac{\varkappa^2}{2\varepsilon_2k_0^2}\left\|\mathbf{g}\right\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}^n_t\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{1}{2\varepsilon_3k_0^2}\left\|\mathbf{g}\right\|^2_{2,\Omega}\times \\
\left[|e'(t)|^2+\|\omega\|^2_{\mathbf{V}^1(\Omega)}\left(\nu^2\left\|\mathbf{u}^n\right\|^2_{\mathbf{V}^1(\Omega)}+C^2\left\|\mathbf{u}^n\right\|^4_{\mathbf{V}^1(\Omega)}+K_0^2\int\limits_0^t\left\|\mathbf{u}^n(s)\right\|^2_{\mathbf{V}^1(\Omega)}ds\right)\right],
\end{gathered}
\tag{4.13}
$$

$$
\begin{gathered}
|I_4|\leq\left(\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2}\right)\left\|\mathbf{u}^n_t\right\|^2_{2,\Omega}+\frac{\varkappa^2}{2\varepsilon_2k_0^2}\left\|\mathbf{g}\right\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}^n_t\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{\left\|\mathbf{g}\right\|^2_{2,\Omega}}{2\varepsilon_3k_0^2}\times \\
\left[|e'(t)|^2+\|\omega\|^2_{\mathbf{V}^1(\Omega)}\left(C^2\left\|\mathbf{u}^n\right\|^4_{\mathbf{V}^1(\Omega)}+\nu^2\left\|\mathbf{u}^n\right\|^2_{\mathbf{V}^1(\Omega)}+K_0^2\int\limits_0^t\left\|\mathbf{u}^n(\tau)\right\|^2_{\mathbf{V}^1(\Omega)}d\tau\right)\right],
\end{gathered}
\tag{4.14}
$$

$$
|I_5|\leq\left\|\mathbf{u}^n_t\right\|_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}^n\right\|^2_{4,\Omega}\leq\frac{\varepsilon_1}{4}\left\|\mathbf{u}^n_t\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{C^2(\Omega)}{\varepsilon_1}\left\|\mathbf{u}^n\right\|^4_{\mathbf{V}^1(\Omega)}.
\tag{4.15}
$$

Plugging (4.11)-(4.15) into (4.10), we get

$$\frac{d}{dt}\left(1+\|\mathbf{u}^n\|^2_{2,\Omega}+(\nu+\varkappa)\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\right)+\nu\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}+\alpha\|\mathbf{u}^n_t(t)\|^2_{2,\Omega}+\beta\|\mathbf{u}^n_t(t)\|^2_{\mathbf{V}^1(\Omega)}\leq$$
$$C_1\int_0^t\left(1+\|\mathbf{u}^n\|^2_{2,\Omega}+(\nu+\varkappa)\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\right)ds+C_2\left(1+\|\mathbf{u}^n\|^2_{2,\Omega}+(\nu+\varkappa)\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\right)^2+$$
$$(C_3+C_4(t))\left(1+\|\mathbf{u}^n\|^2_{2,\Omega}+(\nu+\varkappa)\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\right)+C_5(t), \tag{4.16}$$

where $\alpha:=2\left(1-\frac{\varepsilon_2+\varepsilon_3}{2}\right)$; $\quad\beta:=2\left(\varkappa-\frac{\varepsilon_1}{2}-\frac{\varkappa^3}{\varepsilon_2k_0^2}\sup_{t\in[0,T]}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\right)$;

$$C_1:=\frac{1}{\nu+\varkappa}\left(\frac{K_0^2}{\nu}+\frac{2K_0^2}{\varepsilon_1}+\frac{2K_0^2}{\varepsilon_3k_0^2}\sup_{t\in[0,T]}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\right);$$
$$C_2:=\frac{2C^2}{(\nu+\varkappa)^2}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_3k_0^2}\sup_{t\in[0,T]}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\right);$$
$$C_3:=\frac{1}{\nu+\varkappa}\left(\varepsilon_3+\varepsilon_2+\frac{2\nu^2}{\varepsilon_3k_0^2}\sup_{t\in[0,T]}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\right);$$
$$C_4(t):=\frac{2}{\varepsilon_3k_0^2}\sup_{t\in[0,T]}\|\mathbf{g}\|^2_{2,\Omega}\left|e'(t)\right|^2.$$

Now we choose $\varepsilon_i,\ i=1,2,3$ such that $\alpha,\beta>0$, and C_i to be finite. It is possible because (4.1):

$$\frac{\varkappa}{k_0^2}\sup_{t\in[0,T]}\|\mathbf{g}(t)\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\leq m<2,$$

but $\varepsilon_1,\varepsilon_2$ cannot be chosen such that $m>2$, because $\varepsilon_2<2$ due to $\alpha>0$. Thus, choosing $\varepsilon_i,\ i=1,2,3$ with suitable values, and integrating (4.16) by s from 0 to t, and using (4.7), we obtain the following integral inequality

$$z(t)+\nu\|\mathbf{u}^n\|^2_{L^2(0,T;\mathbf{V}^1(\Omega))}+\alpha\|\mathbf{u}^n_t(t)\|^2_{L^2(Q_T)}+\beta\|\mathbf{u}^n_t(t)\|^2_{L^2(0,T;\mathbf{V}^1(\Omega))}\leq$$
$$C_5\int_0^t z^2(s)ds+C_6, \tag{4.17}$$

where $z(t):=1+\|\mathbf{u}^n\|^2_{2,\Omega}+(\nu+\varkappa)\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}$ and

$$C_5=\max\{C_1T+C_3;C_2\};\quad C_6=\|\mathbf{u}_0\|^2_{2,\Omega}+(\varkappa+\nu)\|\mathbf{u}_0\|^2_{\mathbf{V}^1(\Omega)}+\int_0^T|C_4(t)|dt.$$

Then, applying the Lemma 1 for the function $z(t)$, we obtain from (4.17)

$$z(t)\leq\frac{C_6}{1-C_5C_6t}\equiv K_1<\infty \tag{4.18}$$

for

$$0\leq t\leq T_1<T_\star:=\frac{1}{C_5C_6}. \tag{4.19}$$

Thus, for all $t \leq T_1 < T_\star$, (4.18) yields

$$\|\mathbf{u}^n\|_{2,\Omega}^2 + (\nu + \varkappa)\,\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 \leq K_1. \tag{4.20}$$

Applying the estimate (4.20) to the right hand side of (4.17) and taking the supremum by $t \in [0, T_1]$, we obtain the following estimate

$$\begin{gathered}\sup_{t\in[0,T_1]}\left(\|\mathbf{u}^n\|_{2,\Omega}^2 + \|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2\right) + \|\mathbf{u}^n\|_{L^2(0,T_1;\mathbf{V}^1(\Omega))}^2 + \|\mathbf{u}_t^n(t)\|_{L^2(Q_{T_1})}^2 + \\ \|\mathbf{u}_t^n(t)\|_{L^2(0,T_1;\mathbf{V}^1(\Omega))}^2 \leq C := C(\nu, \varkappa, \alpha, \beta, T_1, K_1, C_5, C_6).\end{gathered} \tag{4.21}$$

4.3. **Passage to the limit.** By means of reflexivity and up to some subsequences, the estimate (4.2) implies that

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0, T_1; \mathbf{V}^1(\Omega)), \quad \text{as } n \to \infty, \tag{4.22}$$

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \quad \text{weak-}* \text{ in } L^\infty(0, T_1; \mathbf{V}^1(\Omega)), \quad \text{as } n \to \infty, \tag{4.23}$$

$$\mathbf{u}_t^n \rightharpoonup \mathbf{u}_t \quad \text{weakly in } L^2(0, T_1; \mathbf{V}^1(\Omega)), \quad \text{as } n \to \infty. \tag{4.24}$$

On the other hand, from the estimate (4.2), we have

$$\mathbf{u}^n \ \text{ is uniformly bounded in } \ L^2(0, T_1; \mathbf{W}_0^{1,2}(\Omega)), \tag{4.25}$$

$$\mathbf{u}_t^n \ \text{ is uniformly bounded in } \ \mathbf{L}^2(Q_{T_1}). \tag{4.26}$$

Then, due to the compact embedding $\mathbf{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$, we can use the Aubin-Lions compactness lemma so that

$$\mathbf{u}^n \longrightarrow \mathbf{u} \quad \text{strongly in } L^2(0, T_1; \mathbf{L}^2(\Omega)), \quad \text{as } n \to \infty. \tag{4.27}$$

Let be $\zeta(t) \in C_0^\infty([0, T_1])$. Multiplying the equation of (4.4) by $\zeta(t)$, integrating the resulting equations between 0 and T_1, we obtain

$$\begin{gathered}\int_{Q_{T_1}} \mathbf{u}_t^n \cdot \varphi_k \zeta \, d\mathbf{x}dt + \varkappa \int_0^{T_1} \mathbf{a}\left(\mathbf{u}_t^n, \varphi_k \zeta\right) dt + \int_{Q_{T_1}} (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n \cdot \varphi_k \zeta \, d\mathbf{x}dt + \\ \nu \int_0^{T_1} \mathbf{a}\left(\mathbf{u}^n, \varphi_k \zeta\right) dt = \int_0^{T_1}\!\!\int_0^{\tau} K(\tau - s)\mathbf{a}\left(\mathbf{u}^n, \varphi_k \zeta\right) ds d\tau + \int_0^{T_1} F(\mathbf{u}^n, t) \int_\Omega \mathbf{g}\varphi_k \zeta \, d\mathbf{x}\, dt\end{gathered} \tag{4.28}$$

for $k \in \{1, \ldots, n\}$. Then, fixing k, we can pass in equation (4.28) to the limit $n \to \infty$, by using the convergence results (4.22)-(4.27). Then, we obtain

$$\begin{gathered}\int_{Q_{T_1}} \mathbf{u}_t \cdot \varphi_k \zeta \, d\mathbf{x}dt + \varkappa \int_0^{T_1} \mathbf{a}\left(\mathbf{u}_t, \varphi_k \zeta\right) dt + \int_{Q_{T_1}} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \varphi_k \zeta \, d\mathbf{x}dt + \\ \nu \int_0^{T_1} \mathbf{a}\left(\mathbf{u}, \varphi_k \zeta\right) dt = \int_0^{T_1}\!\!\int_0^{\tau} K(\tau - s)\mathbf{a}\left(\mathbf{u}, \varphi_k \zeta\right) ds d\tau + \int_0^{T_1} F(\mathbf{u}, t) \int_\Omega \mathbf{g}\varphi_k \zeta \, d\mathbf{x}\, dt\end{gathered} \tag{4.29}$$

for $k \in \{1, \ldots, n\}$.

Here, for the convective term we have

$$(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n \rightharpoonup (\mathbf{u} \cdot \nabla)\mathbf{u} \quad \text{weakly in } \ \mathbf{L}^2(Q_{T_1}), \quad \text{as } n \to \infty. \tag{4.30}$$

In fact, writing the corresponding integrals in (4.30) as

$$\int_{Q_{T_1}} [(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n-(\mathbf{u}\cdot\nabla)\mathbf{u}]\,d\mathbf{x}dt=\int_{Q_{T_1}}[(\mathbf{u}^n-\mathbf{u})\cdot\nabla]\,\mathbf{u}^n\,d\mathbf{x}dt-\int_{Q_{T_1}}(\mathbf{u}\cdot\nabla)(\mathbf{u}^n-\mathbf{u})\,d\mathbf{x}dt,$$

we see that the first right-hand side integral converges to zero by application of Hölder's inequality together with (4.2) and (4.27):

$$\int_{Q_{T_1}}[(\mathbf{u}^n-\mathbf{u})\cdot\nabla]\,\mathbf{u}^n\,d\mathbf{x}dt\leq\|\mathbf{u}^n-\mathbf{u}\|_{\mathbf{L}^2(Q_{T_1})}\|\mathbf{u}^n\|_{\mathbf{L}^2(0,T_1;\mathbf{V}^1(\Omega))}\leq$$
$$\sqrt{C}\|\mathbf{u}^n-\mathbf{u}\|_{\mathbf{L}^2(Q_{T_1})}\longrightarrow 0,\quad \text{as } n\to\infty,$$

The second integral converges to zero, due to (4.22) and because $\mathbf{u}\in\mathbf{L}^2(Q_{T_1})$. Likewise, the nonlocal term

$$\int_0^{T_1}F(\mathbf{u},t)\int_\Omega \mathbf{g}\varphi_k\zeta\,d\mathbf{x}=\int_0^{T_1}\frac{1}{g_0(t)}\Big[e'(t)+\varkappa\mathbf{a}(\mathbf{u}_t^n,\omega)+\nu\mathbf{a}(\mathbf{u}^n,\omega)-((\mathbf{u}^n\cdot\nabla)\omega,\mathbf{u}^n)_{2,\Omega}+$$
$$\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\omega\right)\int_\Omega \mathbf{g}\varphi_k\zeta\,d\mathbf{x}ds\Big]\rightharpoonup\int_0^{T_1}F(\mathbf{u},t)\int_\Omega\mathbf{g}\varphi_k\zeta\,d\mathbf{x}dt \text{ in } L^2([0,T_1]) \text{ as } n\to\infty.$$

It is obvious that the second, third and fifth terms due to the (4.24), and (4.22), and (4.23), respectively. The fourth term converges due to (4.30), and the first term is trivial. By linearity, the equation (4.29) holds for any finite linear combination of $\varphi_1,\dots,\varphi_n$ and by a continuity argument, they are still true for any $\varphi\zeta\in\mathbf{L}^2(0,T_1;\mathbf{V}^1(\Omega))$ with $\zeta\in C_0^\infty([0,T_1])$. Moreover, all terms in the equation (4.29) is absolutely continuous as functions of t defined by integrals over $[0,T_1]$. So we obtain the following equalities which hold for a.e. $t\in[0,T_1]$ and for any $\varphi\in\mathcal{V}$, respectively

$$\begin{aligned}&\int_\Omega\Big[\mathbf{u}_t(t)+\Big(\mathbf{u}(t)\cdot\nabla\Big)\mathbf{u}(t)\Big]\cdot\varphi\,d\mathbf{x}+\nu\mathbf{a}\left(\mathbf{u}(t),\varphi\right)+\varkappa\mathbf{a}\left(\mathbf{u}_t(t),\varphi\right)=\\&\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}(s),\varphi\right)ds+F\left(\mathbf{u}(t),t\right)\int_\Omega\mathbf{g}(t)\varphi\,d\mathbf{x}.\end{aligned}\tag{4.31}$$

Therefore, the proof of Theorem 1 is completed. □

5. Existence of strong solutions of inverse problems I and II

Theorem 2. *Let all conditions of Theorem 1 be fulfilled. In addition, assume that*

$$\mathbf{u}_0(\mathbf{x})\in\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega).\tag{5.1}$$

Then the nonlocal direct problems (3.9), (1.3)-(1.4) and (3.9), (1.3), (1.5) have at least one strong solution $(\mathbf{u}(\mathbf{x},t),p(\mathbf{x},t))$ in Q_{T_1}. Therefore, corresponding inverse problems (1.1)-(1.4), (1.6) and (1.1)-(1.3), (1.5), (1.6) have at least one strong solution and the estimate is hold

$$\|\mathbf{u}\|^2_{\mathbf{L}^\infty(0,T_1;\mathbf{V}^1\cap\mathbf{V}^2(\Omega))}+\|\mathbf{u}_t\|^2_{\mathbf{L}^2(0,T_1;\mathbf{V}^1\cap\mathbf{V}^2(\Omega))}\leq C<\infty.\tag{5.2}$$

where T_1 defined by (4.19) and C is positive constant depending on data of the problem.

Proof. We prove the existence of a strong solutions to these problems by using the special basis, associated to the eigenfunctions of the spectral problem for Stokes operator

$$\mathbb{A}\varphi_k := -\mathbb{P}\Delta\varphi_k = \lambda_k\ \varphi_k,\quad \varphi_k(x) \in \mathbf{H}^1(\Omega) \cap \mathbf{H}^2(\Omega), \tag{5.3}$$

in the case (1.4), i.e. for problem (3.9), (1.3)-(1.4), where $\mathbb{P} : \mathbf{L}^2(\Omega) \to \mathbf{H}(\Omega)$ is the Leray projector, and

$$\mathbb{A}\varphi_k := -\Delta\varphi_k = -\mathbf{curl}\ \mathbf{curl}\varphi_k = \lambda_k\ \varphi_k,\quad \varphi_k \in \mathbf{H}^1_{\mathbf{n}}(\Omega) \cap \mathbf{H}^2_{\mathbf{n}}(\Omega) \tag{5.4}$$

in the case (1.5), for problem (3.9), (1.3), (1.5). The latter is due to the fact (see [32])

$$(\Delta\varphi, \nabla p)_{2,\Omega} = 0 \text{ for any } \varphi \in \mathbf{H}^1_{\mathbf{n}} \cap \mathbf{H}^2_{\mathbf{n}}(\Omega),\ p \in W^{1,2}(\Omega),\ \text{and } \mathbf{L}^2(\Omega) = \mathbf{H}_{\mathbf{n}}(\Omega) \oplus \mathbf{G}(\Omega).$$

It is known from [31] and [32], that the system $\{\varphi_k\}_{k\in\infty}$ of eigenfunctions are orthogonal in $\mathbf{V}$ and an orthonormal basis in $\mathbf{V}^1(\Omega) \cap \mathbf{V}^2(\Omega)$.

Let us first consider the (1.1)-(1.4), (1.6) (Inverse problem I), the problem (1.1)-(1.3), (1.5), (1.6) (Inverse problem II) is similar. In this case, all first and second estimates are true for strong solution. Thus, in order to complete the proof this theorem, it is sufficient to get more strong estimates, i.e. estimate $\Delta\mathbf{u}^n$ and $\Delta\mathbf{u}^n_t$.

Thus, multiplying both sides of (4.4) by $\lambda_k \frac{d c^n_k(t)}{dt}$ and sum up by k from 1 to n, we obtain

$$\begin{gathered}\frac{\nu}{2}\frac{d}{dt}\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} + \varkappa\|\mathbb{A}\mathbf{u}^n_t\|^2_{2,\Omega} + \|\mathbf{u}^n_t\|^2_{\mathbf{V}^1(\Omega)} = -\left((\mathbf{u}^n\cdot\nabla)\,\mathbf{u}^n, \mathbb{A}\mathbf{u}^n_t\right) - \\ -\int_0^t K(t-s)\,(\mathbb{A}\mathbf{u}^n(s), \mathbb{A}\mathbf{u}^n_t(t))_{2,\Omega}\,ds + F_1(\mathbf{u}^n,t)\,(\mathbf{g}, -\mathbb{A}\mathbf{u}^n_t)_{2,\Omega} \equiv I_4,\end{gathered} \tag{5.5}$$

where $F(\mathbf{u}^n,t)$ given by (4.5) and it can be estimated as follow

$$\begin{gathered}|F_1|^2 \le \frac{5}{k_0^2}\Big[|e'(t)|^2 + \\ \|\omega\|^2_{\mathbf{V}^1(\Omega)}\left(\varkappa^2\|\mathbf{u}^n_t\|^2_{\mathbf{V}^1(\Omega)} + \nu^2\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} + C^4(\Omega)\|\mathbf{u}^n\|^4_{\mathbf{V}^1(\Omega)} + K_0^2\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)}\,ds\right)\Big].\end{gathered} \tag{5.6}$$

Using Hölder and Young inequalities and (5.6), estimate I_4

$$|I_4| \le \frac{\varkappa}{2}\|\mathbb{A}\mathbf{u}^n_t\|^2_{2,\Omega} + \frac{3}{2\varkappa}\left[C\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} + K_0^2\int_0^t\|\mathbb{A}\mathbf{u}^n(s)\|^2_{2,\Omega}\,ds + |F_1|^2\|\mathbf{g}\|^2_{2,\Omega}\right]. \tag{5.7}$$

Plugging (5.7) into (5.5), we get

$$\nu\frac{d}{dt}\left\|\mathbb{A}\mathbf{u}^n\right\|_{2,\Omega}^2+\varkappa\left\|\mathbb{A}\mathbf{u}_t^n\right\|_{2,\Omega}^2+\left\|\mathbf{u}_t^n\right\|_{\mathbf{V}^1(\Omega)}^2\leq \frac{3}{\varkappa}\left[C\left\|\mathbf{u}^n\right\|_{\mathbf{V}^1(\Omega)}^2\left\|\mathbb{A}\mathbf{u}^n\right\|_{2,\Omega}^2+K_0^2\int\limits_0^t\left\|\mathbb{A}\mathbf{u}^n(s)\right\|_{2,\Omega}^2ds+\left|F_1\right|^2\left\|\mathbf{g}\right\|_{2,\Omega}^2\right]. \tag{5.8}$$

Integrating (5.8) by s from 0 to t, and applying already obtained estimates, we get

$$\nu\left\|\mathbb{A}\mathbf{u}^n\right\|_{2,\Omega}^2+\varkappa\int\limits_0^t\left\|\mathbb{A}\mathbf{u}_t^n(s)\right\|_{2,\Omega}^2ds\leq C_8+C_9\int\limits_0^t\nu\left\|\mathbb{A}\mathbf{u}^n(s)\right\|_{2,\Omega}^2ds, \tag{5.9}$$

where

$$C_8:=\nu\left\|\mathbb{A}\mathbf{u}_0\right\|_{2,\Omega}^2+\frac{3}{\varkappa}\left\|\mathbf{g}\right\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}^2\int\limits_0^{T_1}\left|F_1(s)\right|^2ds<\infty,$$

$$C_9:=\frac{3}{\nu\varkappa}\left(C(\Omega)\sup_{t\in[0,T_1]}\left\|\mathbf{u}^n\right\|_{\mathbf{V}^1(\Omega)}^2+K_0^2T\right)<\infty.$$

Thus, apply Grönwall's lemma to (5.9) to obtain

$$\left\|\mathbb{A}\mathbf{u}^n(t)\right\|_{2,\Omega}^2\leq\frac{1}{\nu}C_8e^{C_9T_1},\quad\forall t\in(0,T_1). \tag{5.10}$$

Taking the suptemum both sides of (5.9) by $t\in[0,T_1]$ and using (5.10), we get

$$\sup_{t\in[0,T_1]}\left\|\mathbb{A}\mathbf{u}^n\right\|_{2,\Omega}^2+\left\|\mathbb{A}\mathbf{u}_t^n\right\|_{\mathbf{L}^2(Q_{T_1})}^2\leq C:=C(\nu,\varkappa,C_8,C_9,T_1)<\infty. \tag{5.11}$$

□

6. Uniqueness of weak and strong solutions

Theorem 3. *Let all conditions of Theorem 1 be fulfilled. Let* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *be two weak and strong solutions of (1.1)-(1.4), (3.8) corresponding to same given data. Then* $\mathbf{u}_1\equiv\mathbf{u}_2$ *for all* $(x,t)\in Q_{T^*}$*, i.e. weak and strong solutions is unique, where* T^* *is a maximal time such that weak and strong solutions exist in* $(0,T^*)$.

Proof. Subtracting the equation (3.9) for $\mathbf{u}_2$ to the equation for $\mathbf{u}_1$, and taking inner product at (3.9) with $\mathbf{u}:=\mathbf{u}_1-\mathbf{u}_2$ and $\mathbf{u}_t$ in $\mathbf{L}_2(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\mathbf{u}\right\|_{2,\Omega}^2+\varkappa\left\|\mathbf{u}\right\|_{\mathbf{V}^1(\Omega)}^2\right)+\nu\left\|\mathbf{u}\right\|_{\mathbf{V}^1(\Omega)}^2=-((\mathbf{u}\cdot\nabla)\mathbf{u}_1,\mathbf{u})_{2,\Omega}-\int\limits_0^tK(t-\tau)\mathbf{a}\left(\mathbf{u}(t),\mathbf{u}(\tau)\right)d\tau+$$
$$\frac{1}{g_0(t)}\left[\varkappa\mathbf{a}\left(\mathbf{u}_t,\omega\right)_{2,\Omega}+\nu\mathbf{a}\left(\mathbf{u},\omega\right)+\left(\left(\mathbf{u}\cdot\nabla\right)\omega,\mathbf{u}_1\right)_{2,\Omega}+\left(\left(\mathbf{u}_2\cdot\nabla\right)\omega,\mathbf{u}\right)+\int\limits_0^tK(t-\tau)\mathbf{a}\left(\mathbf{u}(\tau),\omega\right)d\tau\right]\left(\mathbf{g},\mathbf{u}\right)_{2,\Omega}, \tag{6.1}$$

$$
\begin{gathered}
\frac{\nu}{2}\frac{d}{dt}\left\|\mathbf{u}\right\|^2_{\mathbf{V}^1(\Omega)}+\left\|\mathbf{u}_t(t)\right\|^2_{2,\Omega}+\varkappa\left\|\mathbf{u}_t(t)\right\|^2_{\mathbf{V}^1(\Omega)}=-\left(\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}_1,\mathbf{u}_t\right)_{2,\Omega}- \\
\left(\left(\mathbf{u}_2\cdot\nabla\right)\mathbf{u},\mathbf{u}_t\right)_{2,\Omega}-\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n_t(t)\right)ds+\frac{1}{g_0(t)}\left[\varkappa\mathbf{a}\left(\mathbf{u}_t,\omega\right)+\right. \\
\left.\nu\mathbf{a}\left(\mathbf{u},\omega\right)+\left(\left(\mathbf{u}\cdot\nabla\right)\omega,\mathbf{u}_1\right)+\left(\left(\mathbf{u}_2\cdot\nabla\right)\omega,\mathbf{u}\right)+\int\limits_0^t K(t-\tau)\mathbf{a}\left(\mathbf{u}(\tau),\omega\right)d\tau\right]\left(\mathbf{g},\mathbf{u}_t\right)_{2,\Omega}
\end{gathered}
\tag{6.2}
$$

Adding results, we get

$$
\begin{gathered}
\frac{1}{2}\frac{d}{dt}\left(\left\|\mathbf{u}\right\|^2_{2,\Omega}+\left(\nu+\varkappa\right)\left\|\mathbf{u}\right\|^2_{\mathbf{V}^1(\Omega)}\right)+\nu\left\|\mathbf{u}\right\|^2_{\mathbf{V}^1(\Omega)}+\left\|\mathbf{u}_t(t)\right\|^2_{2,\Omega}+\varkappa\left\|\mathbf{u}_t(t)\right\|^2_{\mathbf{V}^1(\Omega)}= \\
-\left(\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}_1,\mathbf{u}\right)_{2,\Omega}+\left(\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}_1,\mathbf{u}_t\right)_{2,\Omega}+\left(\left(\mathbf{u}_2\cdot\nabla\right)\mathbf{u},\mathbf{u}_t\right)_{2,\Omega}- \\
\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}(t),\mathbf{u}(s)\right)ds+\frac{1}{g_0(t)}\left[\varkappa\mathbf{a}\left(\mathbf{u}_t,\omega\right)_{2,\Omega}+\nu\mathbf{a}\left(\mathbf{u},\omega\right)+\right. \\
\left.\left(\left(\mathbf{u}\cdot\nabla\right)\omega,\mathbf{u}_1\right)_{2,\Omega}+\left(\left(\mathbf{u}_2\cdot\nabla\right)\omega,\mathbf{u}\right)+\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}(s),\omega\right)ds\right]\left(\mathbf{g},\mathbf{u}\right)_{2,\Omega}+ \\
\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n_t(t)\right)ds-\frac{1}{g_0(t)}\left[\varkappa\mathbf{a}\left(\mathbf{u}_t,\omega\right)+\nu\mathbf{a}\left(\mathbf{u},\omega\right)+\right. \\
\left.\left(\left(\mathbf{u}\cdot\nabla\right)\omega,\mathbf{u}_1\right)+\left(\left(\mathbf{u}_2\cdot\nabla\right)\omega,\mathbf{u}\right)+\int\limits_0^t K(t-s)\mathbf{a}\left(\mathbf{u}(s),\omega\right)ds\right]\left(\mathbf{g},\mathbf{u}_t\right)_{2,\Omega}=\sum_{i=1}^{7}R_i.
\end{gathered}
\tag{6.3}
$$

Using the Hölder and Young inequalities estimate the terms on the right hand side of (6.1), we obtain

$$
\left|R_1\right|=\left|-\left(\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}_1,\ \mathbf{u}\right)_{2,\Omega}\right|\leq\left\|\mathbf{u}_1\right\|_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}\right\|^2_{4,\Omega}\leq C(\Omega)\left\|\mathbf{u}_1\right\|_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}\right\|^2_{\mathbf{V}^1(\Omega)},
\tag{6.4}
$$

$$
\left|R_2\right|\leq\left\|\mathbf{u}\right\|_{4,\Omega}\left\|\mathbf{u}_1\right\|_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}_t\right\|_{4,\Omega}\leq\frac{\varepsilon_0}{8}\left\|\mathbf{u}_t\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{2C^2}{\varepsilon_0}\left\|\mathbf{u}_1\right\|^2_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}\right\|^2_{\mathbf{V}^1(\Omega)},
\tag{6.5}
$$

$$
\left|R_3\right|\leq\left\|\mathbf{u}_2\right\|_{4,\Omega}\left\|\mathbf{u}\right\|_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}_t\right\|_{4,\Omega}\leq\frac{\varepsilon_0}{8}\left\|\mathbf{u}_t\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{2C^2}{\varepsilon_0}\left\|\mathbf{u}_2\right\|^2_{\mathbf{V}^1(\Omega)}\left\|\mathbf{u}\right\|^2_{\mathbf{V}^1(\Omega)},
\tag{6.6}
$$

$$
\left|R_4\right|\leq\left\|\mathbf{u}\right\|_{\mathbf{V}^1(\Omega)}K_0\left(\int\limits_0^t\left\|\mathbf{u}(s)\right\|^2_{\mathbf{V}^1(\Omega)}ds\right)^{\frac{1}{2}}\leq\frac{\nu}{2}\left\|\mathbf{u}(t)\right\|^2_{\mathbf{V}^1(\Omega)}+\frac{K_0^2}{2\nu}\int\limits_0^t\left\|\mathbf{u}(s)\right\|^2_{\mathbf{V}^1(\Omega)}ds,
\tag{6.7}
$$

$$
\begin{aligned}
&|R_5| \le \frac{\|\mathbf{g}\|_{2,\Omega}\|\mathbf{u}\|_{2,\Omega}}{k_0}\Big[\varkappa\|\mathbf{u}_t\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)} + \nu\|\mathbf{u}\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)} + \\
&\|\mathbf{u}\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)}\|\mathbf{u}_1\|_{\mathbf{V}^1(\Omega)} + \|\mathbf{u}_2\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)}\|\mathbf{u}\|_{\mathbf{V}^1(\Omega)} + \\
&\|\omega\|_{\mathbf{V}^1(\Omega)} K_0\left(\int_0^t \|\mathbf{u}^n(\tau)\|^2_{\mathbf{V}^1(\Omega)}\,d\tau\right)^{\frac12}\Bigg] \le \frac{\|\mathbf{g}\|^2_{2,\Omega}}{2k_0^2}\|\mathbf{u}\|^2_{2,\Omega} + \frac{\|\omega\|^2_{\mathbf{V}^1(\Omega)}}{2}\Big[\nu\|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)} + \\
&\|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)}\|\mathbf{u}_1\|^2_{\mathbf{V}^1(\Omega)} + \|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)}\|\mathbf{u}_2\|^2_{\mathbf{V}^1(\Omega)} + K_0^2\int_0^t \|\mathbf{u}(\tau)\|^2_{\mathbf{V}^1(\Omega)}\,d\tau\Bigg] + \\
&\frac{\varepsilon_0}{8}\|\mathbf{u}_t\|^2_{\mathbf{V}^1(\Omega)} + \frac{2\varkappa^2}{\varepsilon_0 k_0^2}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\|\mathbf{u}\|^2_{2,\Omega}
\end{aligned}
\tag{6.8}
$$

$$
|R_6| \le \|\mathbf{u}_t\|_{\mathbf{V}^1(\Omega)} K_0\left(\int_0^t \|\mathbf{u}(s)\|^2_{\mathbf{V}^1(\Omega)}\,ds\right)^{\frac12} \le \frac{\varepsilon_0}{8}\|\mathbf{u}_t(t)\|^2_{\mathbf{V}^1(\Omega)} + \frac{2K_0^2}{\varepsilon_0}\int_0^t \|\mathbf{u}(s)\|^2_{\mathbf{V}^1(\Omega)}\,ds,
\tag{6.9}
$$

$$
\begin{aligned}
&|R_7| \le \frac{\|\mathbf{g}\|_{2,\Omega}\|\mathbf{u}_t\|_{2,\Omega}}{k_0}\Big[\varkappa\|\mathbf{u}_t\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)} + \nu\|\mathbf{u}\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)} + \\
&\|\mathbf{u}\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)}\|\mathbf{u}_1\|_{\mathbf{V}^1(\Omega)} + \|\mathbf{u}_2\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)}\|\mathbf{u}\|_{\mathbf{V}^1(\Omega)} + \\
&\|\omega\|_{\mathbf{V}^1(\Omega)} K_0\left(\int_0^t \|\mathbf{u}^n(\tau)\|^2_{\mathbf{V}^1(\Omega)}\,d\tau\right)^{\frac12}\Bigg] \le \frac{\varepsilon_1}{4}\|\mathbf{u}_t\|^2_{2,\Omega} + \frac{\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}}{\varepsilon_1 k_0^2}\Big[\nu\|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)} + \\
&\|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)}\|\mathbf{u}_1\|^2_{\mathbf{V}^1(\Omega)} + \|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)}\|\mathbf{u}_2\|^2_{\mathbf{V}^1(\Omega)} + K_0^2\int_0^t \|\mathbf{u}(\tau)\|^2_{\mathbf{V}^1(\Omega)}\,d\tau\Bigg] + \\
&\frac{\varepsilon_1}{4}\|\mathbf{u}_t\|^2_{2,\Omega} + \frac{\varkappa^2}{\varepsilon_1 k_0^2}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\|\mathbf{u}_t\|^2_{\mathbf{V}^1(\Omega)}
\end{aligned}
\tag{6.10}
$$

Plugging (6.4)-(6.10) into (6.3), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\mathbf{u}\|^2_{2,\Omega} + (\varkappa+\nu)\|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)}\right) + \nu\|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)} + \alpha\|\mathbf{u}_t\|^2_{\mathbf{V}^1(\Omega)} + \beta\|\mathbf{u}_t\|^2_{2,\Omega} \le \\
&a_1\|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)} + a_2\int_0^t \|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)}\,ds + a_3\|\mathbf{u}\|^2_{2,\Omega},
\end{aligned}
\tag{6.11}
$$

where $\alpha := 2\left(\varkappa - \frac{\varepsilon_0}{2} - \frac{\varkappa^2}{\varepsilon_1 k_0^2}\sup\limits_{t\in[0,T]}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}\right),\ \beta := 2\left(1 - \frac{\varepsilon_1}{2}\right);$

$$
\begin{aligned}
a_1 := 2C\sup_{t\in[0,T^*]}\|\mathbf{u}_1\|_{2,\Omega} + \frac{4C^2}{\varepsilon_0}\sup_{t\in[0,T^*]}\|\mathbf{u}_1\|^2_{2,\Omega} + \frac{4C^2}{\varepsilon_0}\sup_{t\in[0,T^*]}\left\|\mathbf{u}_2^2\right\|_{2,\Omega} + \\
+\left(\frac{2\sup\limits_{t\in[0,T]}\|\mathbf{g}\|^2_{2,\Omega}\|\omega\|^2_{\mathbf{V}^1(\Omega)}}{\varepsilon_1 k_0^2} + \|\omega\|^2_{\mathbf{V}^1(\Omega)}\right)\left(\nu^2 + \sup_{t\in[0,T^*]}\|\mathbf{u}_1\|^2_{2,\Omega} + \sup_{t\in[0,T^*]}\|\mathbf{u}_2\|^2_{2,\Omega}\right);
\end{aligned}
$$

$$a_2 := \frac{K_0^2}{\nu} + \|\omega\|^2_{\mathbf{V}^1(\Omega)} K_0^2 + \frac{2 \sup\limits_{t\in[0,T]} \|\mathbf{g}\|^2_{2,\Omega} \|\omega\|^2_{\mathbf{V}^1(\Omega)}}{\varepsilon_1 k_0^2} K_0^2 + \frac{4K_0^2}{\varepsilon_0};$$
$$a_3 := \frac{1}{k_0^2} \sup_{t\in[0,T]} \|\mathbf{g}\|^2_{2,\Omega} \left(1 + \frac{4\varkappa^2}{\varepsilon_0} \|\omega\|^2_{\mathbf{V}^1(\Omega)}\right).$$

Using estimates for $\mathbf{u}_i$, $i = 1, 2$, and choosing $\varepsilon_i, i = 0, 1$ with suitable values as we did as obtaining a priory estimates above, we can make $\alpha, \beta, a_1, a_2, a_3$ to be positive and finite constants, and it is possible due to the assumption $\frac{\varkappa}{k_0^2} \sup\limits_{t\in[0,T]} \|\mathbf{g}(t)\|^2_{2,\Omega} \|\omega\|^2_{\mathbf{V}^1(\Omega)} \leq m < 2$. Thus, integrating (6.11) by τ from 0 to $t \in [0, T^*]$ and simplifying, we derive

$$y(t) \leq a \int_0^t y(\tau) d\tau, \tag{6.12}$$

where $y(t) := \|\mathbf{u}\|^2_{2,\Omega} + (\varkappa+\nu) \|\mathbf{u}\|^2_{\mathbf{V}^1(\Omega)}$, $a := \max\left\{\frac{1}{\varkappa+\nu}(a_1 + Ta_2), a_3\right\}$. Due to the conditions to the Theorem 3 and then by Grönwall's lemma, it follows from (6.12) that $y(t) \equiv 0$ for all $t \in [0, T^*]$, i.e. $\mathbf{u}_1 \equiv \mathbf{u}_2$. □

7. Inverse problems III and IV

In this section, we consider the inverse problems (1.1)-(1.4), (1.8) (Inverse problem III); and (1.1)-(1.3), (1.5), (1.9) (Inverse problem IV), regarding to the overdetermination condition (1.9). Due to notation (2.8), the conditions (1.8) and (1.9) can be written as following common form for both problem IP3 and IP4.

$$(\mathbf{u}(t), \omega)_{2,\Omega} + \mathbf{a}(\mathbf{u}(t), \omega) = e(t), \quad t \in [0, T]. \tag{7.1}$$

For these inverse problems, the corresponding equivalent nonlocal direct problems have the following form

$$\begin{aligned} &\mathbf{div}\, \mathbf{u}(\mathbf{x}, t) = 0, \\ &\mathbf{u}_t - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} - \int_0^t K(t-s)\Delta\mathbf{u}(\mathbf{x}, s) ds + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = F_2(\mathbf{u}, t)\mathbf{g}(\mathbf{x}, t), \ Q_T. \end{aligned} \tag{7.2}$$

with the initial condition (1.3), and boundary conditions (1.4) or (1.5), where

$$F_2(\mathbf{u}, t) \equiv \frac{1}{g_0(t)} \left(e'(t) + \nu\mathbf{a}(\mathbf{u}, \omega) - ((\mathbf{u}\cdot\nabla)\omega, \mathbf{u})_{2,\Omega} + \int_0^t K(t-\tau)\mathbf{a}(\mathbf{u}, \omega) d\tau \right) = f(t). \tag{7.3}$$

Analogical as Lemma 2, one can prove the following the lemma.

Lemma 3. *Let the assumptions* (3.3)-(3.5) *be fulfilled and*

$$(\mathbf{u}_0, \omega)_{2,\Omega} + \mathbf{a}(\mathbf{u}_0, \omega) = e(0). \tag{7.4}$$

Then the solvability of the inverse problems (1.1)-(1.4), (1.8) and (1.1)-(1.3), (1.5), (1.9) is equivalent to the solvability of the corresponding nonlocal direct problems (7.2), (1.3)-(1.4) and (7.2), (1.3), (1.5), respectively.

7.1. **Existence and uniqueness of weak solutions of the inverse problems (1.1)-(1.4), (1.8) and (1.1)-(1.3), (1.5), (1.9).** By Lemma 3, below we study the nonlocal direct problems (7.2), (1.3)-(1.4) and (7.2), (1.3), (1.5) instead of the corresponding inverse problems (7.2), (1.3)-(1.4) and (7.2), (1.3), (1.5).

Theorem 4. *Let the conditions (3.2)-(3.5), (3.7), (7.4) be fulfilled. Then there exists $T_2 \in (0,T]$, such that the nonlocal direct problems (7.2), (1.3)-(1.4) and (7.2), (1.3), (1.5) have unique weak solution in the Q_{T_2}, where T_2 is defined at (7.14) below. Moreover, the weak solutions satisfy the estimate*

$$\|\mathbf{u}\|^2_{\mathbf{L}^\infty(0,T_2;\mathbf{L}^2(\Omega)\cap\mathbf{V}^1(\Omega))} + \|\mathbf{u}\|^2_{\mathbf{L}^2(0,T_2;\mathbf{V}^1(\Omega))} + \|\mathbf{u}_t\|^2_{\mathbf{L}^2(0,T_2;\mathbf{L}^2(\Omega)\cap\mathbf{V}^1(\Omega))} \le C, \tag{7.5}$$

where C is a constant depending on data of the problem.

Proof. As we have note in Remark 3, in this case, above a priory estimates can be established without the assumption (4.1). It is obvious that, the Galerkin's approximations (4.3) with unknown coefficients $c_j^n(t)$, $j = 1, ..., n$ will be defined from the system of equations

$$\begin{gathered}(\mathbf{u}^n_t, \varphi_k)_{2,\Omega} + \varkappa \mathbf{a}\left(\mathbf{u}^n_t, \varphi_k\right) + \nu \mathbf{a}\left(\mathbf{u}^n, \varphi_k\right) - ((\mathbf{u}^n\cdot\nabla)\varphi_k, \mathbf{u}^n)_{2,\Omega} = \\ F_2(\mathbf{u}^n, t)\left(\mathbf{g}(\mathbf{x},t), \varphi_k\right)_{2,\Omega} - \int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n, \varphi_k\right) ds, \quad k = 1,\ 2,\ \dots,\ n,\end{gathered} \tag{7.6}$$

with the Cauchy data (4.6), where $F_2(\mathbf{u}^n, t)$ given by (7.3). □

Lemma 1. *Assume that the conditions (3.2)-(3.5), (3.7), (4.7) and (7.4) are fulfilled. Then there exits a finite time $T_2 \in [0,T]$ such that the following a priori estimate is valid for all $t \in (0, T_2)$*

$$\|\mathbf{u}^n\|^2_{\mathbf{L}^\infty(0,T_2;\mathbf{V}^1(\Omega))} + \|\mathbf{u}^n\|^2_{\mathbf{L}^2(0,T_2;\mathbf{V}^1(\Omega))} \le M_0 < \infty, \tag{7.7}$$

where M_0 is a positive constant depending on data of the problem.

Proof. Multiply (7.6) by $c_k^n(t)$ and summing with respect to k, from 1 to n, and integrating over Ω. We have

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\right) + \nu\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} = J_1 + J_2 \tag{7.8}$$

where

$$J_1 := -\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s), \mathbf{u}^n(t)\right) ds \quad \text{and} \quad J_2 := F_2(\mathbf{u}^n, t)\left(\mathbf{g}(\mathbf{x},t), \mathbf{u}^n\right)_{2,\Omega}.$$

Using the Hölder and Young inequalities we estimate each term on the right hand side of (7.8)

$$\begin{aligned}|F_2(\mathbf{u}^n, t)| \le &\frac{1}{k_0}\Big[|e'(t)| + \nu\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)} + \\ &C^2(\Omega)\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}\|\omega\|_{\mathbf{V}^1(\Omega)} + \|\omega\|_{\mathbf{V}^1(\Omega)} K_0\left(\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)} ds\right)^{\frac{1}{2}}\Big].\end{aligned} \tag{7.9}$$

$$|J_1| \leq \|\mathbf{u}^n(t)\|_{\mathbf{V}^1(\Omega)} K_0 \left(\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)} \, ds \right)^{\frac{1}{2}} \leq \frac{\nu}{2} \|\mathbf{u}^n(t)\|^2_{\mathbf{V}^1(\Omega)} + \frac{K_0^2}{2\nu} \int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)} \, ds, \tag{7.10}$$

$$\begin{aligned} |J_2| \leq \|\mathbf{g}\|_{2,\Omega} \|\mathbf{u}^n\|_{2,\Omega} |F_2(\mathbf{u}^n, t)| \leq \frac{C^3(\Omega)}{k_0} \|\mathbf{g}\|_{2,\Omega} \|\omega\|_{\mathbf{V}^1(\Omega)} \|\mathbf{u}^n\|^3_{\mathbf{V}^1(\Omega)} + \frac{3}{2} \Big[|e'(t)|^2 + \\ \|\omega\|^2_{\mathbf{V}^1(\Omega)} \left(\nu^2 \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} + K_0^2 \int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)} \, ds \right) \Big] + \frac{1}{2k_0^2} \|\mathbf{g}\|^2_{2,\Omega} \|\mathbf{u}^n\|^2_{2,\Omega} . \end{aligned} \tag{7.11}$$

Plugging the inequalities (7.10),(7.11) into (7.8) we have

$$\begin{aligned} \frac{d}{dt} \left(1 + \|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \right) + \nu \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \leq C_1 \left(1 + \|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \right) + \\ C_2 \int_0^t \left(1 + \|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \right) ds + C_3 \left(1 + \|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \right)^{\frac{3}{2}} + 3 |e'(t)|^2 . \end{aligned} \tag{7.12}$$

where $C_1 := \frac{1}{k_0^2} \sup\limits_{t\in[0,T]} \|\mathbf{g}\|^2_{2,\Omega} + \frac{3\nu^2}{\varkappa} \|\omega\|^2_{\mathbf{V}^1(\Omega)}$; $C_2 := \frac{K_0^2}{\varkappa} \left(\frac{1}{\nu} + 3 \|\omega\|^2_{\mathbf{V}^1(\Omega)} \right)$;

$$C_3 := \frac{2C^3(\Omega)}{k_0 \varkappa^{\frac{3}{2}}} \sup_{t\in[0,T]} \|\mathbf{g}\|_{2,\Omega} \|\omega\|_{\mathbf{V}^1(\Omega)} .$$

Let us denote $y(t) := 1 + \|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)}$. After integrating (7.12) by s from 0 to t and simplifying we get the following nonlinear integral inequality

$$y(t) \leq C_5 + C_4 \int_0^t y^{\frac{3}{2}}(s) ds, \tag{7.13}$$

which by Grönwall's Lemma 1 yields

$$y(t) \leq \frac{C_1}{\left(1 - \frac{1}{2} C_4 C_5^{\frac{1}{2}} t \right)^2} \equiv K_3 < \infty \quad \text{for all } 0 \leq t \leq T_2 < T_\star := \frac{2}{C_4 C_5^{\frac{1}{2}}} \tag{7.14}$$

or according to the notation we have

$$\sup_{t\in(0,T_2]} \left(\|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \right) \leq K_3, \tag{7.15}$$

where $C_4 := \max\{C_3; C_1 + C_2 T\}$ and $C_5 := 3\|e'(t)\|^2_{L^2[0,T]} + 1 + \|\mathbf{u}_0\|^2_{2,\Omega} + \varkappa \|\mathbf{u}_0\|^2_{\mathbf{V}^1(\Omega)}$.

Applying the estimate (7.15) to the right hand side of (7.12) and taking the supremum by $t \in [0, T_2]$, we obtain the first enregy estimate

$$\sup_{t\in(0,T_2]} \left(\|\mathbf{u}^n\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \right) + \nu \|\mathbf{u}^n\|^2_{\mathbf{L}^2(0,T_2;\mathbf{V}^1(\Omega))} \leq M_0 < \infty, \tag{7.16}$$

where $M_0 = M_0(\nu, \varkappa, T_2, K_3)$. □

Lemma 4. *Assume that all conditions of the Lemma 1 are valid. Then the following a priori estimate is valid for all* $t \in (0, T_2]$

$$\sup_{t\in[0,T_2]} \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} + \|\mathbf{u}^n_t\|^2_{\mathbf{L}^2(Q_{T_2})} + \|\mathbf{u}^n_t\|^2_{L^2(0,T_2;\mathbf{V}^1(\Omega))} \le M_1 < \infty. \tag{7.17}$$

where T_2 is defined in (7.14) and M_1 is a positive constant depending on data of the problem.

Proof. Multiply the both sides of (7.6) by $\frac{d\,c^n_k}{d\,t}$ and summing with respect to k, from 1 to n, and integrating over Ω. Using formulas of integrating by parts, we obtain

$$\begin{aligned}
&\frac{\nu}{2}\frac{d}{dt}\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} + \|\mathbf{u}^n_t(t)\|^2_{2,\Omega} + \varkappa\,\|\mathbf{u}^n_t(t)\|^2_{\mathbf{V}^1(\Omega)} = \\
&-\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n_t(t)\right)ds + F(t)\left(\mathbf{g},\mathbf{u}^n_t\right)_{2,\Omega} + \left(\left(\mathbf{u}^n\cdot\nabla\right)\mathbf{u}^n_t,\mathbf{u}^n\right)_{2,\Omega}.
\end{aligned} \tag{7.18}$$

Analogical way as we did in (7.10) and (7.11), using the Hölder and Young inequalities together with the fist energy estimate (7.7), we obtain the following estimates for the terms on right hand side of (7.18)

$$\begin{aligned}
&\left|-\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s),\mathbf{u}^n_t(t)\right)ds\right| \le \frac{\varkappa}{4}\|\mathbf{u}^n_t\|^2_{\mathbf{V}^1(\Omega)} + \frac{K_0^2}{\varkappa}\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)}ds \le \\
&\frac{\varkappa}{4}\|\mathbf{u}^n_t\|^2_{\mathbf{V}^1(\Omega)} + \frac{K_0^2}{\varkappa}\sup_{t\in(0,T_2]}\|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)}T_2 \le \frac{\varkappa}{4}\|\mathbf{u}^n_t\|^2_{\mathbf{V}^1(\Omega)} + \frac{K_0^2}{\varkappa^2}M_0T_2
\end{aligned} \tag{7.19}$$

$$\begin{aligned}
&\left|F_2(\mathbf{u}^n,t)\left(\mathbf{g},\mathbf{u}^n_t\right)_{2,\Omega}\right| \le \frac{1}{2}\|\mathbf{u}^n_t\|^2_{2,\Omega} + \frac{2}{k_0^2}\|\mathbf{g}\|^2_{2,\Omega}\left[\left|e'(t)\right|^2 + \right. \\
&\left.\|\omega\|^2_{\mathbf{V}^1(\Omega)}\left(\nu^2\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} + C^4(\Omega)\|\mathbf{u}^n\|^4_{\mathbf{V}^1(\Omega)} + K_0^2\int_0^t\|\mathbf{u}^n(s)\|^2_{\mathbf{V}^1(\Omega)}ds\right)\right] \le \\
&\frac{1}{2}\|\mathbf{u}^n_t\|^2_{2,\Omega} + \frac{2}{k_0^2}\|\mathbf{g}\|^2_{2,\Omega}\left[\left|e'(t)\right|^2 + \|\omega\|^2_{\mathbf{V}^1(\Omega)}\left(\nu^2M_0 + C^4(\Omega)M_0^2 + K_0^2M_0T_2\right)\right].
\end{aligned} \tag{7.20}$$

$$\begin{aligned}
&\left|\left(\left(\mathbf{u}^n\cdot\nabla\right)\mathbf{u}^n_t,\mathbf{u}^n\right)_{2,\Omega}\right| \le \|\mathbf{u}^n_t\|_{\mathbf{V}^1(\Omega)}\|\mathbf{u}^n\|^2_{4,\Omega} \le C(\Omega)\|\mathbf{u}^n_t\|_{\mathbf{V}^1(\Omega)}\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} \le \\
&C(\Omega)M_0\|\mathbf{u}^n_t\|_{\mathbf{V}^1(\Omega)} \le \frac{\varkappa}{4}\|\mathbf{u}^n_t\|^2_{\mathbf{V}^1(\Omega)} + \frac{1}{\varkappa}C^2(\Omega)M_0^2.
\end{aligned} \tag{7.21}$$

Plugging (7.19)-(7.21) into (7.18) we have

$$\nu\frac{d}{dt}\|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} + \|\mathbf{u}^n_t(t)\|^2_{2,\Omega} + \varkappa\,\|\mathbf{u}^n_t(t)\|^2_{\mathbf{V}^1(\Omega)} \le K_4(t), \tag{7.22}$$

where

$$K_4(t) = \frac{4}{k_0^2}\|\mathbf{g}\|^2_{2,\Omega}\left[\left|e'(t)\right|^2 + K_5\right] + K_6.$$

$K_5 = \|\omega\|^2_{\mathbf{V}^1(\Omega)} (\nu^2 M_0 + C^4(\Omega) M_0^2 + K_0^2 M_0 T_2)$ and $K_6 = \frac{2K_0^2}{\varkappa^2} M_0 T_2 + \frac{2C^2(\Omega)M_0^2}{\varkappa}$. Integrating (7.22) from 0 to $t \in [0, T_2]$ then taking the supremum, we get the second energy estimate

$$\nu \sup_{t\in[0,T_2]} \|\mathbf{u}^n\|^2_{\mathbf{V}^1(\Omega)} + \int_0^{T_2} \left(\|\mathbf{u}^n_t(t)\|^2_{2,\Omega} + \varkappa \|\mathbf{u}^n_t(t)\|^2_{\mathbf{V}^1(\Omega)} \right) ds \le M_1, \tag{7.23}$$

where

$$M_1 = K_6 T_2 + \frac{4}{k_0^2} \sup_{t\in[0,T_2]} \|\mathbf{g}\|^2_{2,\Omega} \left[\|e'(t)\|^2_{L^2[0,T_2]} + K_5 T_2 \right] < \infty.$$

□

7.2. **Existence and uniqueness of strong solutions of the inverse problems (1.1)-(1.4), (1.8) and (1.1)-(1.3), (1.5), (1.9).** Analogical as Theorem 2, the following result holds for strong solutions of (1.1)-(1.4), (1.8) and (1.1)-(1.3), (1.5), (1.9).

Theorem 5. *Let the conditions (3.3)-(3.5), (3.7), (7.4) be fulfilled. Assume that also*

$$\mathbf{u}_0(\mathbf{x}) \in \mathbf{V}^1(\Omega) \cap \mathbf{V}^2(\Omega). \tag{7.24}$$

Then the direct problem problems (3.9), (1.3)-(1.4) and (3.9), (1.3), (1.5) have unique strong solution $\mathbf{u}(\mathbf{x}, t)$ in Q_{T_2}. Therefore, corresponding inverse problems (1.1)-(1.4), (1.8) and (1.1)-(1.3), (1.5), (1.9) have a unique strong solution and for them the following estimate is hold

$$\|\mathbf{u}\|^2_{\mathbf{L}^\infty(0,T_2;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))} + \|\mathbf{u}_t\|^2_{\mathbf{L}^2(0,T_2;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))} \le M_3 < \infty. \tag{7.25}$$

where M_3 is positive constant depending on data of the problem.

8. Some special cases of inverse problems I-IV allowing global in time unique solvability and removing the restriction (4.1)

In this section, we consider the inverse problems (1.1)-(1.4), (1.6) (Inverse problem I); and (1.1)-(1.3), (1.5), (1.6) (Inverse problem II); (1.1)-(1.4), (1.8), (Inverse problem III); (1.1)-(1.3), (1.5), (1.9), (Inverse problem IV) in some special cases which the existence and uniqueness of weak and a strong solutions of (1.1)-(1.4), (1.8), (Inverse problem III); (1.1)-(1.3), (1.5), (1.9), (Inverse problem IV) can be established global in time, and the restriction (4.1) on data which also means local solvability, can be removed for problems (1.1)-(1.4), (1.6) (Inverse problem I); and (1.1)-(1.3), (1.5), (1.6) (Inverse problem II). The main difficulty to prove this is obtaining global in time first a priory estimate like (7.5), which arises from the presence of a nonlinear convective member $(\mathbf{u} \cdot \nabla)\,\mathbf{u}$ in the functional $F(\mathbf{u}, t)$ in (3.9) and given by (3.8). Global solvability of these inverse problems without the convective term, i.e. linear cases, were studied in [27]. However, the global solvability can be established under some additional restrictions on the given functions for the nonlinear problems.

8.1. **Inverse problems III-IV: Global existence and uniqueness in the case of special source term.** Let us consider the problem (1.1)-(1.4), (1.8) ((1.1)-(1.3), (1.5), (1.9) is similar) with the special right-hand side $f(t)\mathbf{g}(\mathbf{x},t) := f(t)\sigma(\mathbf{x})$, i.e. with the same function $\sigma(\mathbf{x}) = \omega(\mathbf{x}) - \varkappa\mathbf{\Delta}\omega(\mathbf{x})$ a included in the integral overdetermination conditions (1.7):

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p - \int\limits_0^t K(t-s)\Delta\mathbf{u}(s)ds = f(t)\sigma(\mathbf{x}), \quad (x,t)\in Q_T, \tag{8.1}$$

$$\mathbf{div}\ \mathbf{u}(\mathbf{x},t) = 0, \quad (x,t)\in Q_T, \tag{8.2}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0\left(\mathbf{x}\right),\ \ \mathbf{x}\in\Omega, \tag{8.3}$$

$$\mathbf{u}(\mathbf{x},t) = 0, \quad (\mathbf{x},t)\in\Gamma_T, \tag{8.4}$$

or

$$\mathbf{u}_{\mathbf{n}}(\mathbf{x},t) = \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{curl}\,\mathbf{u}\times\mathbf{n} = 0, \quad (\mathbf{x},t)\in\Gamma_T \tag{8.5}$$

and

$$(\mathbf{u}(t),\omega)_{2,\Omega} + \mathbf{a}\left(\mathbf{u}(t),\omega\right) = e(t), \ \ t\in[0,T]. \tag{8.6}$$

Let us assume that in addition to (3.5)-(3.6) the following conditions are fulfilled

$$\omega \neq 0, \quad \forall\mathbf{x}\in\Omega \ \ (\text{or } \ \|\omega\|_{2,\Omega}^2 + \varkappa\,\|\omega\|_{\mathbf{V}^1(\Omega)}^2 \neq 0). \tag{8.7}$$

In this case, an equivalent direct problem corresponding to (8.1)-(8.4), (8.6)((8.1)-(8.3), (8.5), (8.6)) is the following initial-boundary value problem, which need to define the function $\mathbf{u}$ from (8.3), (8.4) ((8.3), (8.5)) and

$$\begin{gathered}\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} - \int\limits_0^t K(t-s)\Delta\mathbf{u}(s)ds + \nabla p = F(\mathbf{u},t)\sigma(\mathbf{x}), (x,t)\in Q_T,\\ \mathbf{div}\,\mathbf{u}(\mathbf{x},t) = 0, \quad (x,t)\in Q_T,\end{gathered} \tag{8.8}$$

with the nonlocal functionals

$$F(\mathbf{u},t) := \frac{1}{\omega_0}\left(e'(t) - \left((\mathbf{u}\cdot\nabla)\,\omega,\mathbf{u}\right)_{2,\Omega} + \nu a\left(\mathbf{u},\omega\right) - \int\limits_0^t K(t-s)\left(\mathbf{u},\omega\right)ds\right), \tag{8.9}$$

where $\omega_0 := \|\omega\|_{2,\Omega}^2 + \varkappa\,\|\omega\|_{\mathbf{V}^1(\Omega)}^2 > 0$ is strictly positive number. For this problem the following assertion is hold.

Theorem 6. *Assume that the conditions (3.2), (3.5), (3.7), (7.4) and (8.7) are fulfilled. Then the direct problem* (8.8),(8.9), *(8.3), (8.4)* ((8.8),(8.9),*(8.3),* (8.5)) *has global in time a unique weak solution* $\mathbf{u}(\mathbf{x},t)$ *in* Q_T, *and for a weak solution the estimate* (7.5) *is hold for all* $t\in(0,T]$.

Lemma 5. *Assume that the conditions (3.2), (3.5), (3.7), (7.4) and (8.7) are fulfilled. Then the following a priori estimate is valid for all* $t\in(0,T]$

$$\|\mathbf{u}^n\|_{\mathbf{L}^\infty(0,T;\mathbf{V}^1(\Omega))}^2 + \|\mathbf{u}^n\|_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega))}^2 \leq M_3 < \infty, \tag{8.10}$$

where M_3 *is a positive constant depending on data of the problem*

Proof. As we note above, in order to prove this, it is sufficient to establish the first a priori estimate (7.7) for any $t \in (0, T]$ for solution of (8.8)-(8.9). Then repeat the next steps of the proof of Theorem 4 and 6. In this case, the energy equality (7.8) and (7.18) have the form

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}^n\|_{2,\Omega}^2 + \varkappa \|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2\right) + \nu \|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 = \\ -\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s), \mathbf{u}^n(t)\right) ds + F(\mathbf{u}^n, t)e(t),\end{aligned} \tag{8.11}$$

$$\begin{aligned}&\frac{\nu}{2}\frac{d}{dt}\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + \|\mathbf{u}_t^n(t)\|_{2,\Omega}^2 + \varkappa \|\mathbf{u}_t^n(t)\|_{\mathbf{V}^1(\Omega)}^2 = \\ &-\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s), \mathbf{u}_t^n(t)\right) ds + F(\mathbf{u}^n, t)e'(t) - \left(\left(\mathbf{u}^n \cdot \nabla\right)\mathbf{u}_t^n, \mathbf{u}^n\right)_{2,\Omega}.\end{aligned} \tag{8.12}$$

where the functional $F(\mathbf{u}^n, t)$ is defined by (8.9), and for them hold the estimate (7.9) with $k_0 := \omega_0$, respectively. Next, estimate the term on the right-hand side of (8.11) as (7.10), (7.11)

$$\left|-\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s), \mathbf{u}^n(t)\right) ds\right| \le \frac{\nu}{4}\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + \frac{K_0^2}{\nu}\int_0^t \|\mathbf{u}^n(s)\|_{\mathbf{V}^1(\Omega)}^2 ds \tag{8.13}$$

$$\begin{aligned}&|F(\mathbf{u}^n, t)e(t)| \le \frac{|e(t)|}{\omega_0}\Big[|e'(t)| + \nu \|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)} \|\omega\|_{\mathbf{V}^1(\Omega)} + \\ &C^2(\Omega)\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 \|\omega\|_{\mathbf{V}^1(\Omega)} + \|\omega\|_{\mathbf{V}^1(\Omega)} K_0 \left(\int_0^t \|\mathbf{u}^n(s)\|_{\mathbf{V}^1(\Omega)}^2 ds\right)^{\frac{1}{2}}\Big] \le \\ &\frac{1}{2\omega_0}\left(|e(t)|^2 + |e'(t)|^2\right) + \frac{\nu}{4}\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\nu}{\omega_0^2}\|\omega\|_{\mathbf{V}^1(\Omega)}^2 |e(t)|^2 + \\ &\frac{C^2(\Omega)}{\omega_0}|e(t)|\,\|\omega\|_{\mathbf{V}^1(\Omega)} \|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + \frac{K_0^2}{2\varkappa\omega_0^2}|e(t)|^2 \|\omega\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\varkappa}{2}\int_0^t \|\mathbf{u}^n(s)\|_{\mathbf{V}^1(\Omega)}^2 ds.\end{aligned} \tag{8.14}$$

Plugging (8.13), (8.14) into (8.11), then integrating by s from 0 to t we have

$$y(t) + \nu \|\mathbf{v}^n\|_{L^2(0,T;\mathbf{V}^1(\Omega))}^2 \le C_1 \int_0^t y(s)ds + C_2, \tag{8.15}$$

where $y(t) = \|\mathbf{u}^n\|_{2,\Omega}^2 + \varkappa \|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2$, $\quad C_1 = \frac{2C^2(\Omega)}{\omega_0\varkappa}\sup\limits_{t\in[0,T]}|e(t)|\,\|\omega\|_{\mathbf{V}^1(\Omega)} + T + \frac{2TK_0^2}{\nu\varkappa}$,

$$C_2 = \frac{\|e(t)\|_{L^2[0,T]}^2 + \|e'(t)\|_{L^2[0,T]}^2}{\omega_0^2} + \frac{2\nu\varkappa + K_0^2}{\varkappa\omega_0^2}\|\omega\|_{\mathbf{V}^1(\Omega)}^2 \|e(t)\|_{L^2[0,T]}^2 + \|\mathbf{u}_0^n\|_{2,\Omega}^2 + \varkappa\|\mathbf{u}_0^n\|_{\mathbf{V}^1(\Omega)}^2.$$

Apply classical Grönwall's lemma to (8.15), we have

$$\|\mathbf{u}^n\|_{2,\Omega}^2 + \varkappa \|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 \le C_2 e^{C_1 T}. \tag{8.16}$$

Taking the supremum from both sides of (8.15) by $t \in [0,T]$ and using the estimate (8.16) we have

$$\sup_{t\in[0,T]}\left(\|\mathbf{u}^n\|_{2,\Omega}^2 + \varkappa\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2\right) + \nu\|\mathbf{u}^n\|_{L^2(0,T;\mathbf{V}^1(\Omega))}^2 \le M_3 = M_3(C_1, C_2, T, \varkappa). \tag{8.17}$$

□

Lemma 6. *Assume that all conditions of the Lemma 5 are valid. Then for* $\mathbf{u}^n$ *the following a priori estimate is valid for all* $t \in (0,T]$

$$\sup_{t\in[0,T]}\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + \|\mathbf{u}_t^n\|_{\mathbf{L}^2(Q_T)}^2 + \|\mathbf{u}_t^n\|_{L^2(0,T;\mathbf{V}^1(\Omega))}^2 \le M_4 < \infty. \tag{8.18}$$

where M_6 is a positive constant depending only on data of the problem.

Proof. Estimate the term on the right-hand side of (8.12) as (7.18)

$$\begin{gathered}\left|-\int_0^t K(t-s)\mathbf{a}\left(\mathbf{u}^n(s), \mathbf{u}_t^n(t)\right)ds\right| \le \frac{\varkappa}{4}\|\mathbf{u}_t^n\|_{\mathbf{V}^1(\Omega)}^2 + \frac{K_0^2}{\varkappa}\int_0^t \|\mathbf{u}^n(s)\|_{\mathbf{V}^1(\Omega)}^2 ds \le \\ \frac{\varkappa}{4}\|\mathbf{u}_t^n\|_{\mathbf{V}^1(\Omega)}^2 + \frac{K_0^2}{\varkappa}M_3,\end{gathered} \tag{8.19}$$

$$\begin{gathered}|F(\mathbf{u}^n,t)e'(t)| \le \frac{|e'(t)|}{\omega_0}\Big[|e'(t)| + \|\omega\|_{\mathbf{V}^1(\Omega)}\Big(\nu\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)} + C^2(\Omega)\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + \\ K_0\left(\int_0^t\|\mathbf{u}^n(s)\|_{\mathbf{V}^1(\Omega)}^2 ds\right)^{\frac{1}{2}}\Big)\Big] \le \frac{|e'(t)|^2}{2\omega_0^2} + \frac{1}{2}\Big[|e'(t)|^2 + \\ \|\omega\|_{\mathbf{V}^1(\Omega)}^2\left(\nu^2\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + C^4(\Omega)\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^4 + K_0^2\int_0^t\|\mathbf{u}^n(s)\|_{\mathbf{V}^1(\Omega)}^2 ds\right)\Big] \le \\ \frac{|e'(t)|^2}{2\omega_0^2} + \frac{1}{2}\left[|e'(t)|^2 + \|\omega\|_{\mathbf{V}^1(\Omega)}^2 M_3\left(\nu^2 + C^4(\Omega)M_3 + K_0^2\right)\right].\end{gathered} \tag{8.20}$$

$$\left|-\left(\left(\mathbf{u}^n\cdot\nabla\right)\mathbf{u}_t^n, \mathbf{u}^n\right)_{2,\Omega}\right| \le \frac{\varkappa}{4}\|\mathbf{u}_t^n\|_{\mathbf{V}^1(\Omega)}^2 + \frac{C^2}{\varkappa}M_3^2. \tag{8.21}$$

Plugging (8.19)-(8.21) into (8.12), then taking supremum and integrating by s from 0 to t, we get

$$\nu\sup_{t\in[0,T]}\|\mathbf{u}^n\|_{\mathbf{V}^1(\Omega)}^2 + \int_0^t\left(\|\mathbf{u}_s^n(s)\|_{2,\Omega}^2 + \varkappa\|\mathbf{u}_s^n(s)\|_{\mathbf{V}^1(\Omega)}^2\right)ds \le M_4, \tag{8.22}$$

where $M_4 = M_4(\nu, \varkappa, K_0, M_3)$. Therefore, the estimates (8.17) and (8.22) give (7.5). □

Theorem 7. *Assume that the conditions (3.2), (3.5)-(3.7), (8.7), and (7.24) are fulfilled. Then the inverse problem (8.8), (8.9), (8.3), (8.4) ((8.8), (8.9),(8.3), (8.5)) has a unique strong solution for all* $t \in (0,T]$ *and the estimate (7.25) is valid.*

8.2. **Uniqueness and existence in the case of without restriction (4.1) and special source terms: Inverse problems I-II.** Let us consider the problems (1.1)-(1.4), (1.6) and (1.1)-(1.3), (1.5), (1.6) with the right-hand side $f(t)\mathbf{g}(\mathbf{x},t) := f(t)\omega(\mathbf{x})$, where $\omega(\mathbf{x})$ is the same function included in the integral overdetermination condition (1.7):

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p - \int\limits_0^t K(t-s)\Delta\mathbf{u}(s)ds = f(t)\omega(\mathbf{x}), \quad (x,t)\in Q_T, \tag{8.23}$$

$$\mathbf{div}\ \mathbf{u}(\mathbf{x},t) = 0, \quad (x,t)\in Q_T, \tag{8.24}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0\left(\mathbf{x}\right), \ \mathbf{x}\in\Omega, \tag{8.25}$$

$$\mathbf{u}(\mathbf{x},t) = 0, \quad (\mathbf{x},t)\in\Gamma_T, \tag{8.26}$$

or

$$\mathbf{u}_\mathbf{n}(\mathbf{x},t) = \mathbf{u}\cdot\mathbf{n} = 0, \quad (\mathbf{D}(\mathbf{u})\cdot\mathbf{n})\times\mathbf{n} = 0, \quad (\mathbf{x},t)\in\Gamma_T \tag{8.27}$$

and

$$\int\limits_\Omega \mathbf{u}\omega(\mathbf{x})d\mathbf{x} = e(t), \ t\geq 0. \tag{8.28}$$

Let (3.4)-(3.6) be fulfilled and $\omega\neq 0, \quad \forall\mathbf{x}\in\Omega$. In this case, an equivalent direct problem corresponding to (8.23)-(8.26), (8.28) ((8.23)-(8.25), (8.27), (8.28)) is the following initial-boundary value problem, which need to define the function $\mathbf{u}$ from (8.25), (8.26) ((8.25), (8.27)) and

$$\begin{gathered}\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} - \int\limits_0^t K(t-s)\Delta\mathbf{u}(s)ds + \nabla p = \Phi(\mathbf{u},t)\omega(\mathbf{x}), (x,t)\in Q_T,\\ \mathbf{div}\,\mathbf{u}(\mathbf{x},t) = 0, \quad (x,t)\in Q_T,\end{gathered} \tag{8.29}$$

with the nonlocal functionals

$$\Phi(\mathbf{u},t) := \frac{1}{\omega_0}\left(e'(t) - \left((\mathbf{u}\cdot\nabla)\,\omega,\mathbf{u}\right)_{2,\Omega} + \varkappa a\left(\mathbf{u}_t,\omega\right) + \nu a\left(\mathbf{u},\omega\right) - \int\limits_0^t K(t-s)\left(\mathbf{u},\omega\right)ds\right), \tag{8.30}$$

where $\omega_0 := \|\omega\|^2_{2,\Omega} > 0$ is strictly positive number. For this problem the following assertion is hold without any restriction like (4.1).

Theorem 8. *Assume that the conditions (3.2), (3.4)-(3.7) are fulfilled with $\omega\neq 0, \quad \forall\mathbf{x}\in\Omega$. Then the nonlocal problem (8.29), (8.30), (8.25), (8.26) ((8.29), (8.30), (8.25), (8.27)) has local in time a unique weak solution $\mathbf{u}(\mathbf{x},t)$ in Q_T, and for a weak solution the estimate (4.2) is hold for all $t\in(0,T]$.*

Acknowledgment

All authors were fully supported by the Grant No. AP19676624 of the Ministry of Education and Science of the Republic of Kazakhstan.

References

[1] U.U. Abylkairov, S.E. Aitzhanov, L.K. Zhapsarbayeva, Solvability of the inverse problem for a heat convection system with integral condition of overdetermination, Applied Mathematical Sciences. 45(9) (2015) 2403-2421. 1

[2] U.U. Abylkairov, Kh. Khompysh, An inverse problem of identifying the coefficient in Kelvin-Voigt equations, Applied Mathematical Sciences. 104(9) (2015) 5079–5088. 1

[3] S.N. Antontsev, H.B. de Oliveira, Kh. Khompysh, The classical Kelvin-Voigt problem for nonhomogeneous and incompressible fluids: existence, uniqueness and regularity. Nonlinearity. 34 (2021) 3083–3111. 1

[4] S.N. Antontsev, J.I. Diaz, S. Shmarev, Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics, Progress in Nonlinear (Differential Equations and their Applications 48), Birkhäuser, Basel, 2002. 2

[5] A.N. Antontsev, Kh. Khompysh, An inverse problem for generalized Kelvin-Voigt equation with p-Laplacian and damping term, Inverse Problems. 37 (2021) 085012. 1, 1

[6] A.N. Antontsev, Kh. Khompysh, Inverse problems for a Boussinesq system for incompressible viscoelastic fluids, Math. Meth. Appl. Sci. (2023), 1–27. DOI 10.1002/mma.9172 1

[7] H.A. Barnes, Handbook Of Elementary Rheology, Cambrian Printers, Aberystwyth, 2000. 1

[8] C. I. Carstea, J.N. Wang, Propagation of smallness for an elliptic PDE with piecewise Lipschitz coefficients, Journal of Differential Equations. 268(12) (2000) 7609–7628. 1

[9] A.Y. Chebotarev, Determination of the right-hand side of the Navier-Stokes system of equations and inverse problems for the thermal convection equations, Computational Mathematics and Mathematical Physics. 51(12) (2011) 2146–2154. 1

[10] A.Y. Chebotarev, Inverse problems for stationary Navier-Stokes systems, Computational Mathematics and Mathematical Physics. 54(3) (2014) 537–545.

[11] F. Di Plinio, A. Giorgini, V. Pata, R. Temam, Navier-Stokes-Voigt Equations with Memory in 3D Lacking Instantaneous Kinematic Viscosity, J. Nonlinear Sci. 2018.

[12] M. Di Cristo, E. Francini, C.L. Lin, S. Vessella, J.N. Wang, Carleman estimate for second order elliptic equations with Lipschitz leading coefficients and jumps at an interface, Journal de Mathematiques Pures et Appliquees. 108(2) 2017 163-206.

[13] J. Fan, M. Di Cristo, Y. Jiang, G. Nakamura, Inverse viscosity problem for the Navier-Stokes equation, Journal of Mathematical Analysis and Applications. 365(2) (2010) 750–757. 1

[14] J. Fan, G. Nakamura, Well-posedness of an inverse problem of Navier-Stokes equations with the final overdetermination, Journal of Inverse and Ill-Posed Problems. 17(6) (2009) 565–584.

[15] V.E. Fedorov, N.D. Ivanova, Inverse problem for Oskolkov's system of equations, Mathematical Methods in the Applied Sciences. 40(17) (2017) 6123–6126. 1, 1

[16] J.M. Ghidaglia, Regularite des solutions des certain problemes aux limites lineaires lies aux equations d'Euler, Comm. Part. Diff. Equations. 9 (1984) 1265-1298. 2

[17] H. Heck, L. Xiaosheng, J.N. Wang, Identification of viscosity in an incompressible fluid, Indiana Univ. Math. J., 56 (2007), 2489-2510. 1

[18] O.Y. Imanuvilov, M. Yamamoto, Global uniqueness in inverse boundary value problems for the Navier-Stokes equation and Lame system in two dimensions, Inverse Problems. 31 (2015) 035004. 1

[19] V. Isakov, R.Y. Lai, J.N. Wang, Increasing Stability for the Conductivity and Attenuation Coefficients, SIAM Journal on Mathematical Analysis. 48(1) (2016) 569-594. 1

[20] V. Isakov, J.N. Wang, Uniqueness and increasing stability in electromagnetic inverse source problems, Journal of Differential Equations. 283 (2021) 110-135. 1

[21] Y. Jiang, J. Fan, S.Nagayasu, and G. Nakamura, Local solvability of an inverse problem to the Navier-Stokes equation with memory term, Inverse Problems, (2020). Article in press. https://doi.org/10.1088/1361-6420/ab7e05 1

[22] D.D. Joseph, Fluid dynamics of viscoelastic liquids, Springer-Verlag, New York, 1990. 1

[23] N.A. Karazeeva, Solvability of initial boundary value problems for equations describing motions of linear viscoelastic fluids, Journal of Applied Mathematics. 25(8) (2005) 59-80. 1

[24] S.I. Kabanikhin, Inverse and Ill-posed Problems, De Gruyter, Berlin, 2011. 1

[25] Kh. Khompysh, Kh. Kenzhebai, An inverse problem for Kelvin-Voigt equation perturbed by isotropic diffusion and damping, Mathematical Methods in Applied Sciences. 45(7) (2022) 3817–3842. 1
[26] Kh. Khompysh, N.K. Nugymanova, Inverse Problem For Integro-Differential Kelvin-Voigt Equation, Journal of Inverse and Ill-Posed Problems. 2023. https://doi.org/10.1515/jiip-2020-0157 Accepted. 1, 1
[27] Kh. Khompysh, A.G. Shakir, Inverse problems for Kelvin-Voigt system with memory: global existence and uniqueness. Lobachevskii journal of mathematics. Submitted. 1, 8
[28] A.A. Kotsiolis, A.P. Oskolkov, The initial boundary value problem with a free surface condition for the ε-approximations of the Navier–Stokes equations and some of their regularizations, 80(3) (1996) 1773–1801. 1, 2, 2, 2
[29] P.Z. Kow, J.N. Wang, Reconstruction of an impenetrable obstacle in anisotropic inhomogeneous background, IMA J. Appl. Math., 80(2) (2021) 100–128. 1
[30] P. Kumar, K. Kinra, M.T. Mohan, A local in time existence and uniqueness result of an inverse problem for the Kelvin-Voigt fluids. arXiv:2103.14448 1
[31] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow II, Nauka, Moscow, 1970. 2, 5
[32] O.A. Ladyzhenskaya, On the global unique solvability of some two-dimensional problems for the water solutions of polymers, Journal of Mathematical Sciences. 99 (2000) 888–897. 2, 2, 2, 2, 5
[33] J.L. Lions, Quelques methodes de resolution des problemes aux limites non liniaires, Dunod, Paris, 1969.
[34] A.P. Oskolkov, Theory of nonstationary flows of Kelvin-Voigt fluids, J. Soviet Math. 1 (1985) 751-758. 1
[35] Oskolkov AP. Initial-boundary value problems for equations of motion of Kelvin–Voigt fluids and Oldroyd fluids. Proceedings of the Steklov Institute of Mathematics. 179 (1989) 137-182. 1
[36] A.P. Oskolkov, Initial boundary-value problems with a free surface condition for the modified Navier-Stokes equations, J. Math. Sci. 84 (1997) 873–887. 1
[37] A.P. Oskolkov, Nonlocal problems for the equations of Kelvin-Voight fluids and their ε-approximations in classes of smooth functions, Journal of Mathematical Sciences, 91(2) (1998), 2840-2859. 1
[38] A.P. Oskolkov, Nonlocal problems for the equations of motion of the Kelvin-Voight fluids, Zap. Nauchn. Sem. LOMI. 197 (1992) 120–158.
[39] A.I. Prilepko, D.G. Orlovsky, I.A. Vasin, Method for Solving Inverse Problems in Mathematical Physics, Monograths and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, 2000. 1, 1
[40] X. Wang, Y. Qin, Three-dimensional Navier-Stokes-Voight equation with a memory and the Brinkman-Forchheimer damping term, Math. Meth. Appl. Sci. (2019) 1–17. 1
[41] E.V. Yushkov, On the blow-up of a solution of a non-local system of equations of hydrodynamic type, Izv. Math. 76(1) (2012) 190–213. 1
[42] V.G. Zvyagin, M.V. Turbin, The study of initial-boundary value problems for mathematical models of the motion of Kelvin-Voigt fluids, J. Math. Sci. 168 (2010) 157–308. 1