The Order of Finite Generation of SO(3) and Optimization of Rotation Sequences

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Abstract

We generalize an old result due to Lowenthal [1] and a more recent one due to Hamada [2] on the order of finite generation of the rotation group SO (3) both for fixed and arbitrary compound transformations. Unlike the above cited authors, we consider decompositions into factors with more than two invariant axes and provide rather intuitive geometric proofs. Thus, we derive a simple estimate for the number of factors in a decomposition and discuss possible means of optimization as well as particular examples of potential interest for the applications.

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We generalize an old result due to Lowenthal [1] and a more recent one due to Hamada [2] on the order of finite generation of the rotation group SO(3) both for fixed and arbitrary compound transformations. Unlike the above cited authors, we consider decompositions into factors with more than two invariant axes and provide rather intuitive geometric proofs. Thus, we derive a simple estimate for the number of factors in a decomposition and discuss possible means of optimization as well as particular examples of potential interest for the applications.

Introduction

The order of finite generation of a compact Lie group G with respect to a subset of elements $\Omega \subset G$ is defined as the minimal number N of elements $g_i \in \Omega$, in which an arbitrary $X \in G$ may be factorized. In 1971 Lowenthal [1] proved his famous result on the order of the rotation group in $\mathbb{R}^3$, namely

Theorem 1 *Each* SO(3) *transformation may be decomposed into*

$$N_\gamma = 1 + \left\lceil \frac{\pi}{\gamma} \right\rceil . \tag{1}$$

alternating rotations about two fixed axes with relative angle $\gamma \in \left(0, \frac{\pi}{2}\right]$.

Here and below we use the notation[1] $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid x \leq n\} = -\lfloor -x \rfloor$, where $\lfloor x \rfloor$ stands for the integer part of x. Note that the only way to ensure order three is to pick orthogonal axes that gives the classical Euler setting. For $\gamma = 60^\circ$, we may decompose into four factors, $\gamma = 45^\circ$ yields five etc. Below we consider the decomposition conditions for the cases of two and three factors and then generalize the corresponding results using induction.

[1] in literature $\lceil x \rceil$ and $\lfloor x \rfloor$ are often referred to as ceiling and floor function, respectively.

Low Order Cases

In the following we adopt the notation of [3]. Let $\mathbf{n}, \hat{\mathbf{c}}_i \in \mathbb{S}^2$ denote respectively the invariant unit vectors of the compound rotation $\mathcal{R}$ and those in the decomposition $\mathcal{R}_i$, with orientation chosen such that the relative angles

$$\gamma_{ij} = \arccos{(\hat{\mathbf{c}}_i \cdot \hat{\mathbf{c}}_j)}, \qquad \beta_i = \arccos{(\hat{\mathbf{c}}_i \cdot \mathbf{n})}$$

are (positive) acute or right. Similarly, we shall use the notation

$$\tilde{\gamma}_{ij} = \arccos{(\hat{\mathbf{c}}_i \cdot \mathcal{R}\, \hat{\mathbf{c}}_j)}$$

and let ϕ and ϕ_i denote the rotation angles with the so chosen orientation. Note that in this way $\mathbf{n}$ and $\hat{\mathbf{c}}_j$ are regarded as points on the closed unit semisphere $\bar{\mathbb{S}}^2_+$ and γ_{ij}, β_j, respectively as spherical distances. Below we shall relate lengths of broken geodesics in $\bar{\mathbb{S}}^2_+$ with decomposability conditions and thus derive an estimate for the order of SO(3) beginning with the following

Lemma 1 *With the above notation each $\mathcal{R}(\mathbf{n}, \phi) \in \mathrm{SO}(3)$ satisfies*

$$\gamma_{ij} - 2\beta_j \le \tilde{\gamma}_{ij} \le \gamma_{ij} + 2\beta_j, \qquad \gamma_{ij} - |\phi| \sin\beta_j \le \tilde{\gamma}_{ij} \le \gamma_{ij} + |\phi| \sin\beta_j.$$

Proof. The first estimate says that $2\beta_i$ is the maximal angle, at which $\mathcal{R}$ shifts vectors on the unit sphere (the case of a half-turn). For the second one we introduce spherical coordinates with polar and azimuthal angles: respectively ϕ and β_j and point out that the $\tilde{\gamma}_{jj} \le |\phi| \sin\beta_j$ since the geodesic distance $\tilde{\gamma}_{jj}$ cannot exceed the length of the corresponding meridian arc. The triangle inequality completes the proof as $\tilde{\gamma}_{ij} \in [\gamma_{ij} - \tilde{\gamma}_{jj}, \gamma_{ij} + \tilde{\gamma}_{jj}]$. $\square$

Next, we considered the decomposition problem beginning with two factors:

Lemma 2 *A transformation $\mathcal{R} \in \mathrm{SO}(3)$ is decomposable into a pair of consecutive rotations about $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$ (in this order) if and only if $\tilde{\gamma}_{21} = \gamma_{21}$.*

Proof. Necessity is easier to prove since the invariant axis theorem yields

$$(\hat{\mathbf{c}}_2, \mathcal{R}\, \hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_2, \mathcal{R}_2 \mathcal{R}_1\, \hat{\mathbf{c}}_1) = (\mathcal{R}^t_2\, \hat{\mathbf{c}}_2, \mathcal{R}_1\, \hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_2,\, \hat{\mathbf{c}}_1)$$

that is seen as an equality for the cosines of the positive acute or right angles $\tilde{\gamma}_{21}$ and γ_{21}. Next, we note that SO(3) is compact and connected, acting freely on itself via left shifts, so the map $\tilde{\mathcal{R}}_\lambda = \mathcal{R}\mathcal{R}^t_1(\lambda)$ with $\lambda \in \mathbb{S}^1$ satisfies

$$(\hat{\mathbf{c}}_2, \tilde{\mathcal{R}}_\lambda \hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_2, \hat{\mathbf{c}}_1)$$

and thus, the λ-orbit of $\hat{\mathbf{c}}_1$ is a rotation about $\hat{\mathbf{c}}_2$. But then, $\tilde{\mathcal{R}}_\lambda$, and hence $\mathcal{R}$, can be decomposed into a pair of successive rotations about $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$. $\square$

With this in mind, it is not hard to prove the following

Lemma 3 *The decomposition* $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_1$ *exists*

- *for an arbitrary angle* ϕ *if and only if* $\beta_1 \leq \gamma_{12}$*;*
- *for an arbitrary axis* $\mathbf{n}$ *if and only if* $|\phi| \leq 2\gamma_{12}$*.*

Proof. We use the notation $\gamma = \gamma_{12}$ for convenience and conjugate obtaining

$$\mathcal{R}(\mathbf{n}', \phi) = \mathcal{R}_2(\phi_2)\mathcal{R}_1(\phi_1 + \phi_3), \qquad \mathbf{n}' = \mathcal{R}_1(-\phi_3)\,\mathbf{n}$$

with $\beta_1' = \measuredangle(\hat{\mathbf{c}}_1, \mathbf{n}') = \beta_1$ for an arbitrary angle ϕ_3. The locus of $\mathcal{R}(\mathbf{n}', \phi)\,\hat{\mathbf{c}}_1$ for any fixed angle $\phi \in \mathbb{S}^1$ is a circle centered at $\hat{\mathbf{c}}_1$ and parameterized with ϕ_3, whose radius obviously does not exceed $2\beta_1$. Therefore, if $\beta_1 \leq \gamma$ this orbit has at least one common point with the γ-orbit of $\hat{\mathbf{c}}_1$ about $\hat{\mathbf{c}}_2$, i.e., one can set the value of ϕ_3 in such a way that the angle $\tilde{\gamma}_{21}'$ between $\mathcal{R}(\mathbf{n}', \phi)\,\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$ equals γ and the above decomposition is guaranteed by Lemma 2. The exact same argument leads to the conclusion that the above ϕ_3-orbit has a common point with the γ-orbit of $\hat{\mathbf{c}}_1$ about $\hat{\mathbf{c}}_2$ as long as $|\phi| \leq 2\gamma$ that proves necessity and sufficiency is implied by the invertibility of Lemma 2. $\square$

Next, we discuss a more general result obtained also in [3] in a different way.

Proposition 1 *The decomposition* $\mathcal{R} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1$ *exists if and only if*

$$|\gamma_{12} - \gamma_{23}| \leq \tilde{\gamma}_{31} \leq \gamma_{12} + \gamma_{23}. \tag{2}$$

Proof. Let us consider a dual system of axes $\{\hat{\mathbf{c}}_k'\}$ attached to the rotating object called the *rotating* or *body frame*, while the stationary one $\{\hat{\mathbf{c}}_k\}$ is usually referred to as the *spacial frame*. Obviously, the first rotation axis in the decomposition is the same in the two frames, i.e., $\hat{\mathbf{c}}_1' = \hat{\mathbf{c}}_1$, while the other pairs are related respectively as $\hat{\mathbf{c}}_2' = \mathcal{R}_1'\hat{\mathbf{c}}_2$ and $\hat{\mathbf{c}}_3' = \mathcal{R}_2'\mathcal{R}_1'\,\hat{\mathbf{c}}_3$, where we denote $\mathcal{R}_k' = \mathcal{R}(\mathbf{c}_k')$. Moreover, suppose that $\mathcal{R}$ can be decomposed in the body frame as $\mathcal{R} = \mathcal{R}_3'\mathcal{R}_2'\mathcal{R}_1'$. Since $\hat{\mathbf{c}}_3'$ is an invariant vector for $\mathcal{R}_3'$, this yields $\hat{\mathbf{c}}_3' = \mathcal{R}\,\hat{\mathbf{c}}_3$. Then, the matrix entries g_{ij}' and r_{ij}' in the rotating (body) frame are naturally expressed in terms of those in the spacial one as

$$g_{12}' = g_{12}, \qquad g_{23}' = g_{23}, \qquad g_{13}' = r_{13}$$

and the corresponding Gram determinant is given by the expression

$$\mathrm{G}(\hat{\mathbf{c}}_1', \hat{\mathbf{c}}_2', \hat{\mathbf{c}}_3') = 1 + 2g_{12}\, g_{23}\, r_{13} - g_{12}^2 - g_{23}^2 - r_{13}^2 \geq 0.$$

Next, we claim that the decompositions in the two dual systems of axes coexist (each one implies the other) and are related by the following formula

$$\mathcal{R}_1\mathcal{R}_2 \ldots \mathcal{R}_n = \mathcal{R}_n'\mathcal{R}_{n-1}' \ldots \mathcal{R}_1', \qquad \mathcal{R}_k' = \mathcal{R}_1\mathcal{R}_2 \ldots \mathcal{R}_{k-1}\mathcal{R}_k\mathcal{R}_{k-1}^{-1} \ldots \mathcal{R}_2^{-1}\mathcal{R}_1^{-1}$$

that is easy to prove by induction starting with

$$\mathcal{R}_2'\mathcal{R}_1' = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_1^{-1}\mathcal{R}_1 = \mathcal{R}_1\mathcal{R}_2$$

since we obviously have $\tilde{\mathcal{R}}\,\mathcal{R}(\mathbf{c})\,\tilde{\mathcal{R}}^{-1} = \mathcal{R}(\tilde{\mathcal{R}}\,\mathbf{c})$ and $\mathcal{R}_1' = \mathcal{R}_1$ by construction. Then, the decomposition $\mathcal{R} = \mathcal{R}_3'\mathcal{R}_2'\mathcal{R}_1'$ is equivalent to $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$, so we need to reorder the vectors in the above Gram determinant, which is the same as replacing r_{13} with r_{31}. Moreover, since $g_{ij} = \cos\gamma_{ij}$ and $r_{31} = \cos\tilde{\gamma}_{31}$, the quadratic inequality $\Delta = \mathrm{G}(\hat{\mathbf{c}}_3', \hat{\mathbf{c}}_2', \hat{\mathbf{c}}_1') \geq 0$ is equivalent to

$$\cos(\gamma_{12} + \gamma_{23}) \leq \cos\tilde{\gamma}_{31} \leq \cos(\gamma_{12} - \gamma_{23}). \tag{3}$$

Finally, one may always choose the orientation of $\hat{\mathbf{c}}_i$ in such a way that $\gamma_{12}, \gamma_{23} \in \left(0, \frac{\pi}{2}\right]$ so that the solution is given namely by formula (2). This proves the necessity of (2). Then, one needs to show that $\Delta \geq 0$ is sufficient for the existence of the corresponding rotating frame $\{\hat{\mathbf{c}}_k'\}$, or simply point out that the solutions obtained in [4] rely only on the definiteness of Δ. □

One straightforward consequence is the Davenport universality condition

$$\mathcal{R} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1 \quad \forall\,\mathcal{R} \in \mathrm{SO}(3) \quad \Leftrightarrow \quad \gamma_{12} = \gamma_{23} = \frac{\pi}{2}\cdot \tag{4}$$

Another one is certainly Lemma 3, which follows directly in the case of coincident first and third axis with the aid of Lemma 1. Note that the non-orthogonal Euler setting $\gamma_{12} = \gamma_{23} = \gamma$ and $\gamma_{13} = 0$ is less restrictive on β_1 compared to the Bryan case, in which all relative angles are equal $\gamma_{ij} \equiv \gamma$. More precisely, the former yields the estimate $\beta_1 \leq \gamma$, while for the latter we have $2\beta_1 \leq \gamma$. We shall see it is a common property of rotational sequences. Note also that with the aid of the famous Rodrigues' rotation formula

$$\mathcal{R}(\mathbf{n}, \phi) = \cos\phi\,\mathcal{I} + (1 - \cos\phi)\,\mathbf{n}\mathbf{n}^t + \sin\phi\,\mathbf{n}^\times \tag{5}$$

we obtain in the case $\gamma_{31} = 0$ from the inequality of Proposition 1

$$\cos\tilde{\gamma}_{11} = \cos^2\beta_1 + \cos\phi\sin^2\beta_1 = (\cos\phi - 1)\sin^2\beta_1 + 1 \geq \cos 2\gamma_{12}$$

and hence, the necessary and sufficient condition takes the form

$$\sin\frac{|\,\phi\,|}{2}\sin\beta_1 \leq \sin\gamma_{12}. \tag{6}$$

The Induction Step

We shall use the notation $\Sigma_k = \gamma_{12} + \gamma_{23} + \ldots + \gamma_{k-1,k}$ and $\bar{\Sigma}_k = \Sigma_k + \gamma_{k,1}$ respectively for the lengths of the open and closed spherical paths connecting the points $\hat{\mathbf{c}}_i \in \bar{\mathbb{S}}^2_+$ associated with the rotation axes in the given order. Moreover, we let $\Delta_k^{ij} = \bar{\Sigma}_k - 2\gamma_{ij} \geq 0$ $(k \geq 3)$ represent the path defect given by the triangle inequality on $\mathbb{S}^2$ and omit the subscript if possible. Next, we shall use induction to generalize Proposition 1 to $N = k+1$ factors. The case $N = 4$ has been studied in [5] in the context of optimal sequences.

Lemma 4 *The existence of the decomposition $\mathcal{R} = \mathcal{R}_k \ldots \mathcal{R}_2 \mathcal{R}_1$, such that $\mathcal{R}_i \in \mathrm{SO}(3)$, implies either the estimate $2\beta_1 \leq \Delta_k^{k,1}$ and/or $|\phi| \sin\beta_1 \leq \Delta_k^{k,1}$.*

Proof. For $k = 3$ the result is implied by Lemma 1 and Proposition 1 and for $k > 3$ we proceed by induction noting that $\tilde{\gamma}^{(k+1)}_{k+1,k} = \tilde{\gamma}^{(k)}_{k+1,k}$ due to the invariant axis theorem, while the triangle inequality on the sphere yields $\tilde{\gamma}_{k,1} - \gamma_{k,k+1} \leq \tilde{\gamma}_{k+1,1} \leq \tilde{\gamma}_{k,1} + \gamma_{k,k+1}$, so the result follows by induction. □

Note that typically no $\gamma_{i,i+1}$ exceeds the sum of the rest, e.g. in the case of a closed path, and this condition is both necessary and sufficient as the lower bound for $\tilde{\gamma}_{k,1}$ becomes trivial and the triangle inequality is minimal. Otherwise the precise estimate involves the minimum of Δ_k^{12}, Δ_k^{23} and $\Delta_k^{k,1}$.

Corollary 1 *The existence of the decompositions*

$$\mathcal{R} = \mathcal{R}_1(\phi_1')\mathcal{R}_k(\phi_k) \ldots \mathcal{R}_1(\phi_1), \qquad \mathcal{R} = \mathcal{R}_k(\phi_k)\mathcal{R}_1(\phi_1')\mathcal{R}_{k-1} \ldots \mathcal{R}_1(\phi_1)$$

for an arbitrary $\mathcal{R} \in \mathrm{SO}(3)$ implies that $2\beta_1$ does not exceed the length of the geodesic path connecting the points $\hat{\mathbf{c}}_i$ on $\mathbb{S}^2_+$ in the corresponding order.

Proof. For the first decomposition we simply apply Lemma (4) taking into account that $\Delta_k^{11} = \bar{\Sigma}_k$ and the lower bound for $\tilde{\gamma}_{11}$ is trivial. To show the second one we express $\mathcal{R} = \mathcal{R}_1 \mathcal{R}_k' \mathcal{R}_{k-1} \ldots \mathcal{R}_1$ where $\mathcal{R}_k'$ is an appropriate conjugation of $\mathcal{R}_k$ with $\mathcal{R}_1$, for which the above result finally yields

$$2\beta_1 \leq \gamma_{12} + \gamma_{23} + \ldots + \gamma_{k-1,k}' + \gamma_{k,1}'$$

with $\gamma_{k,1}' = \gamma_{k,1}$ and by the triangle inequality $\gamma_{k-1,k}' \leq \gamma_{k-1,1} + \gamma_{k,1}$. □

Corollary 2 *With the above notation let $\bar{\gamma} = \frac{1}{k}\bar{\Sigma}_k$ be the mean spherical distance of the path. Then, an arbitrary $\mathcal{R} \in \mathrm{SO}(3)$ may be decomposed into*

$$N_{\bar{\gamma}}(\beta) \leq 1 + \left\lceil \frac{2\beta}{\bar{\gamma}} \right\rceil \tag{7}$$

rotations about the $\hat{\mathbf{c}}_i$'s where $\beta = \min \beta_i$. Axes can be reordered optimally.

Restriction on the Angle

Now, let us consider the case, in which the angle of the compound rotation is under control, but we have no information about its axis $\mathbf{n}$ starting with

$$\mathcal{R} = \mathcal{R}_1(\phi_1')\mathcal{R}_k \ldots \mathcal{R}_2(\phi_2)\mathcal{R}_1(\phi_1)$$

where the condition for the decomposition given by Lemma 1 yields

$$\cos\tilde{\gamma}_{11} = (\hat{\mathbf{c}}_1, \mathcal{R}\hat{\mathbf{c}}_1) \geq \cos\bar{\Sigma}_k. \tag{8}$$

Using Rodrigues' rotation formula (5) like in equation (6), from the above scalar product we obtain with the optimal choice of a first axis the condition

$$\sin\frac{|\,\phi\,|}{2}\,\sin\beta \leq \sin\frac{\bar{\Sigma}_k}{2}, \qquad \beta = \min\beta_i. \tag{9}$$

Note that in the latter estimate we assume $2\beta > \bar{\Sigma}_k \leq \pi$, since otherwise no restriction on the angle is necessary. Moreover, it is not hard to show that

$$\tilde{\gamma}_{11} = 2\arcsin\left|\sin\frac{\phi}{2}\,\sin\beta\right| \leq |\,\phi\,|\sin\beta$$

which is also a good approximation for small values of $|\,\phi\,|$, so the condition

$$|\,\phi\,|\sin\beta \leq \bar{\Sigma}_k$$

is sufficient for the approximate estimate

$$N_{\bar{\gamma}}(\beta,\phi) \leq 1 + \left\lceil\frac{|\,\phi\,|\sin\beta}{\bar{\gamma}}\right\rceil. \tag{10}$$

In particular, if there is no information about the invariant axis $\mathbf{n}$, we assume the highest possible value for β and the above formula is reduced to

$$N_{\bar{\gamma}}(\phi) \leq 1 + \left\lceil\frac{|\,\phi\,|}{\bar{\gamma}}\right\rceil. \tag{11}$$

Combining these results with Corollary 2, we obtain the following

Proposition 2 *With β, $\bar{\gamma}$ and ϕ as before, we may decompose $\mathcal{R}(\mathbf{n},\phi)$ into*

$$N_{\bar{\gamma}}(\beta,\phi) \leq 1 + \left\lceil\frac{\min\left(|\,\phi\,|\sin\beta,\ 2\beta\right)}{\bar{\gamma}}\right\rceil \tag{12}$$

rotations about the $\hat{\mathbf{c}}_i$'s where the second option is considered only if $|\,\phi\,| > 2$.

Relations to Classical Results

In this section we see how the problems we focus on correspond to classical facts about spherical trigonometry, the theorems of Rodrigues-Hamilton and Donkin etc. To begin with, if the rotation axes are associated with points z_i on the unit sphere, then they form a polygon with side lengths $\gamma_{jk} = \arccos g_{jk}$ and let $\tilde{\gamma}_{jk} = \arccos \tilde{g}_{jk}$ with $\tilde{g}_{jk}$ denoting the co-factor of g_{jk} in the Gram determinant $g = \omega^2$. Then, the spherical cosine theorem determines the vertex angles α_k in the case of three axes as

$$\cos\alpha_i = -\frac{\tilde{g}_{jk}}{\sqrt{\omega^2 + \tilde{g}_{jk}^2}}, \qquad \sin\alpha_i = \frac{\omega}{\sqrt{\omega^2 + \tilde{g}_{jk}^2}} \tag{13}$$

and the decomposition of unity $\mathcal{I} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1$ derived in [4] with scalar parameters $\tau_k = \dfrac{\omega}{\tilde{g}_{ij}}$ is actually a verification of the famous Rodrigues-Hamilton theorem (see [6]). Moreover, we have the dual statement, known as Donkin's theorem, where the axes of rotation are related to the poles of the given triangle, which has α_k as its side lengths and g_{ij} as the corresponding vertex angles. Note also that the two-axes decompositions considered in [4] may be considered similarly as a manifestation of Rodrigues-Hamilton theorem, with the $\mathbf{a}_3$ replaced by the compound rotation invariant vector $\mathbf{n}$

$$\tau_1 = \frac{\tilde{\zeta}_3}{g_{12}\zeta_1 - \zeta_2}, \qquad \tau_2 = \frac{\tilde{\zeta}_3}{g_{12}\zeta_2 - \zeta_1}, \qquad \tau = \frac{\tilde{\zeta}_3}{g_{12} - \zeta_1\zeta_2} \tag{14}$$

where we denote $\zeta_i = \mathbf{n}\cdot\mathbf{a}_i$ and $\tilde{\zeta}_i = \mathbf{n}\cdot\tilde{\mathbf{a}}_i$, respectively. Note that the first two equalities above provide the scalar parameters for the decomposition

$$\mathcal{R}(\tau_2\mathbf{a}_2)\mathcal{R}(\tau_1\mathbf{a}_1) = \mathcal{R}(\tau\mathbf{n})$$

while the third one can be interpreted as a necessary and sufficient condition. Similarly, the classical Euler decomposition (for non-orthogonal axes)

$$\mathcal{R}(\phi,\mathbf{n}) = \mathcal{R}(\phi_3,\mathbf{a}_1)\mathcal{R}(\phi_2,\mathbf{a}_2)\mathcal{R}(\phi_1,\mathbf{a}_1) = \mathcal{R}(\phi_2,\mathbf{a}_2')\mathcal{R}(\phi_1+\phi_3,\mathbf{a}_1)$$

with $\mathbf{a}_2' = \mathcal{R}(\phi_3,\mathbf{a}_1)\,\mathbf{a}_2$ may be obtained in this way as

$$\phi_1 + \phi_3 = 2\arctan\frac{\tilde{\zeta}_3'}{g_{12}\zeta_1 - \zeta_2'}, \qquad \tau_2 = \frac{\tilde{\zeta}_3'}{g_{12}\zeta_2' - \zeta_1}, \qquad \tau = \frac{\tilde{\zeta}_3'}{g_{12} - \zeta_1\zeta_2'}$$

with the notation $\zeta_i' = \mathbf{n}\cdot\mathbf{a}_i'$ and $\tilde{\zeta}_i' = \mathbf{n}\cdot\tilde{\mathbf{a}}_i'$ where $\tilde{\mathbf{a}}_3' = \mathbf{a}_1 \times \mathbf{a}_2'$. This problem, however, allows for a simpler treatment [4]. Note that in the case of gimbal lock $\mathbf{a}_3 = \mathcal{R}\mathbf{a}_1$ the above holds for the unprimed quantities ζ_i, $\tilde{\zeta}_i$.

Final Remarks

Let us note that in the case of two axes considered by Lowenthal and Hamada the above estimate for the order becomes exact and in particular, if there is no additional information about $\mathbf{n}$ or ϕ, they all reduce to formula (1). More generally, for equal relative angles γ_{ij} the estimate is independent of the choice of path. However, this is possible only for two or three axes with $g_{ij} \geq 0$. In other cases one can minimize the length of the rotation sequence by maximizing the one of the corresponding spherical path $\bar{\Sigma}_k$ connecting the $\hat{\mathbf{c}}_i$'s. One straightforward way to do so is by choosing $\gamma = \max \gamma_{ij}$ and proceeding with only two axes, but in some cases this maximum may not be unique, for instance if the axes determine a proper spherical polygon, we can also choose the maximal billiard orbit with fixed number of reflections. The first axis $\hat{\mathbf{c}}_1$, on the other hand, should be chosen closest to $\mathbf{n}$ so that β_1 is minimal. Typically one may need to make a compromise between minimizing β_1 and maximizing γ as formula (12) suggests. Similar arguments clearly hold for the spin cover SU(2) as well and may be applied to spin systems and in particular q-bits used in quantum computation with the proper definition of relative angles in that case. It is not quite clear, however, how much of that refers to the non-compact group $\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1)$ playing also a major role in physics, despite its similar decomposition properties [4].

Our last remark concerns rigid motions in $\mathbb{E}^3$ represented via screws (see [7] for details) modeled using unit dual extension to the underlying algebra $\mathbb{R} \to \mathbb{R}[\varepsilon]$, incorporating translations as nilpotent elements ($\varepsilon^2 = 0$). We introduce the dual angle $\underline{\varphi} = \varphi + \varepsilon d$ and axis vector $\underline{\mathbf{n}} = \mathbf{n} + \varepsilon \mathbf{m} \in \mathbb{S}^2[\varepsilon]$ (i.e., $\mathbf{n}^2 = 1$ and $\mathbf{m} \perp \mathbf{n}$) using the screw displacement $d = \mathbf{n} \cdot \mathbf{p}$ (with $\mathbf{p}$ denoting the translation vector) and moment $\mathbf{m}$, which provide the Plücker coordinates of the screw axis $\underline{\mathbf{n}}$ given by Mozzi-Chasles theorem stating that every rigid motion in $\mathbb{E}^3$ is a screw motion, i.e., rotation and translation with a common axis. Dual extensions exhibit the transfer principle allowing us to extend the above results for decompositions of screw motions by considering analogous conditions on the unit dual sphere $\mathbb{S}^2[\varepsilon]$. However, this goes beyond the scope of the present work so we leave it for future detailed study.

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