

# Controllability under positivity constraints of a size-structured population model with delayed birth process

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## Abstract

In this paper, we investigate the controllability under positivity constraints from the birth of a size-structured population system with delayed birth process. Firstly, we establish the well-posedness and positivity property of the model. Secondly, and more importantly, we derive a sufficient condition for boundary approximate controllability under state and control positivity constraints. The method relies on the feedback theory of infinite-dimensional positive systems.

## ARTICLE TYPE

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## Summary

In this paper, we investigate the controllability under positivity constraints from the birth of a size-structured population system with delayed birth process. Firstly, we establish the well-posedness and positivity property of the model. Secondly, and more importantly, we derive a sufficient condition for boundary approximate controllability under state and control positivity constraints. The method relies on the feedback theory of infinite-dimensional positive systems.

## KEYWORDS:

Population dynamics, delay boundary condition, controllability under positive constraints

## 1 | INTRODUCTION

In recent years, there has been a significant focus on studying the controllability properties of age- or size-structured population models (see, for instance, <sup>1,2,3,4,5</sup> and related references). Among the pioneering works in this field are <sup>3,5</sup>, where the authors studied the null-controllability of age-structured population models with spatial diffusion. A key innovation in these works is the localization of control in both age and space. Moreover, in <sup>4,5</sup> it was shown that the controllability of Lotka-McKendrick type systems with age structuring can be obtained by preserving the positivity of the state, provided that the time horizon for the null-controllability is sufficiently large. Recently, the null-controllability of a degenerate population model structured by age, size, and spatial position was addressed in <sup>3</sup>, where Carleman estimates were used to obtain a local null-controllability result, except for small ages. To overcome this locality, a new (global) null-controllability result was generated in <sup>6</sup> using the approach developed in <sup>5</sup>, which combines final-state observability estimates with the characteristics of the semigroup generator.

In this paper, we investigate the constrained controllability from the birth of the following size-dependent population system in  $L^p$ -spaces ( $1 \leq p < \infty$ ):

$$\begin{cases} \frac{\partial z(t,s)}{\partial t} + \frac{\partial q(s)z(t,s)}{\partial s} = -\mu(s)z(t,s), & t \geq 0, s \in (0, s^*), \\ z(0,s) = f(s) \geq 0, \quad z(\theta,s) = \varphi(\theta,s) \geq 0, & \theta \in [-r, 0], s \in (0, s^*), \\ z(t,0) = \int_0^{s^*} \beta(s) \int_{-r}^0 d\eta(\theta) z(t+\theta,s) ds + bu(t), & t \geq 0, \end{cases} \quad (1)$$

where  $z(t,s)$  represents the population density of certain species of size  $s \in (0, s^*)$  at time  $t \geq 0$ , where  $s^* > 0$  is the maximum size of any individual in the population. The function  $q(s)$  is the growth rate of size over time and the size dependent functions  $\beta$  and  $\mu$  denote the fertility and mortality, respectively. Thus, the formula  $\int_0^{s^*} \int_{-r}^0 \beta(s) z(t+\theta,s) d\eta(\theta) ds$  determines the distribution of newborn individuals and takes into account the delay in the birth process, where the first integral is in the Lebesgue-Stieltjes

sense. The function  $u \geq 0$  represent the positive input control and  $b > 0$  denotes the boundary control operator at the birth. The function  $(f, \varphi) \geq 0$  are the initial distributions of our target population.

System (1) is a simple case of infinite-dimensional *structured population* systems. These types of problems model population dynamics by one or more partial differential equations defined on coupled domains and have been an active research subject since the 1925s (see e.g., <sup>1,2,7,8,5,6,9,10</sup> and references therein). Such research activity is motivated by the applications of age/size-structured models and interesting mathematical questions arising from their analysis (see e.g., <sup>11,12</sup>). In the studies of structured population equations, linear semigroup methods have been successfully developed to investigate the linear stability and bifurcation phenomena of solutions (see, e.g., <sup>13,8,9</sup>). In particular, the well-posedness and asymptotic behavior of such models have been investigated in several works <sup>14,15,13,8,16</sup>. Age-dependent models with delay in the birth process were investigated in <sup>15</sup>. In the latter, the author applied Perron-Frobenius techniques introduced in <sup>17</sup> and the theory of positive semigroups to establish stability criteria. Recently, using the feedback theory of  $L^p$ -well-posed and regular linear systems, the authors in <sup>16,8</sup> established the well-posedness and stability results of structured population systems with unbounded birth process. Unlike the asymptotic behavior, which is well understood, there are no works addressing the controllability properties of age/size-structured population models with a delayed birth process, to the best of our knowledge. This problem is important because it reflects our ability to control the evolution in time of our target population.

The study of system (1) has been prompted by various open problems on infinite-dimensional structured population systems. Indeed, studying the controllability under positivity constraints is a reasonable question for this class of systems since they are canonical examples where positivity is preserved for the free dynamics. Therefore, the question of whether the system can be controlled between two positive states by means of positive controls arises naturally. By exploiting the feedback theory of infinite-dimensional linear systems, we rewrite the size-structured population system as an abstract free-delay distributed control system on a suitable product of  $L^p$ -spaces. We then prove the existence of a positive mild solution of the system (1) by introducing suitable conditions on the coefficients, as discussed in Section 3. Specifically, we show that the corresponding differential operator generates a positive  $C_0$ -semigroup on an appropriate  $L^p$ -space (see Theorem 2). Using Laplace transform techniques and developing the solution into a variation of constant formula, we derive a sufficient condition for approximate controllability under state and control positivity constraints (see Theorem 4). Our argument is based on a general characterization of boundary approximate positive controllability developed in <sup>18</sup>, as well as on an approximation result for functions of two variables <sup>19</sup>.

The paper is structured as follows. In Section 2, we recall the concept of feedback theory of infinite-dimensional positive systems as well as a well-posedness result for non-homogeneous boundary value control problem (see Theorem 1). Section 3 introduces the setting and the main assumptions. We show that our system can be rewritten as a non-homogeneous boundary value control problem on a product space which leads to the well-posedness of (1). In Section 4, we characterize the boundary approximate positive controllability of (1). Specifically, in Theorem 4, we provide a sufficient condition for controllability under state and control positivity constraints of (1). Finally, we include a technical lemma in the Appendix that is required to prove the main controllability result of the paper.

## 2 | PRELIMINARIES

In this section, we shall briefly recall some background about infinite-dimensional positive systems. To this end, let us consider  $(X, \geq)$  a Banach lattice (see e.g., <sup>20</sup>) i.e., a partially ordered Banach space for which any given elements  $f, g$  of  $X$  have a supremum  $\sup(x, y)$  and for all  $x, y, z \in X$  and  $\alpha \geq 0$

$$\begin{cases} x \leq y & \Rightarrow (x + z \leq y + z \text{ and } \alpha x \leq \alpha y), \\ |x|_X \leq |y|_X & \Rightarrow \|x\|_X \leq \|y\|_X, \end{cases}$$

with, for all  $x \in X$ ,  $|x|_X = \sup(x, -x)$ . We will denote by  $X_+ = \{x \in X : x \geq 0\}$  the positive cone. We denote by  $\mathcal{L}(X, U)$  the Banach algebra of all linear bounded operators from  $X$  to  $U$ . An operator  $P \in \mathcal{L}(X, U)$  is positive if and only if  $PX_+ \subset U_+$  or, equivalently, if  $x \leq y$  implies  $Px \leq Py$ . The set of all positive operators, denoted by  $\mathcal{L}_+(E, F)$ , is a convex cone in  $\mathcal{L}(X, U)$ .

For a Banach lattice  $U$ , we denote by  $L_{loc,+}^p(\mathbb{R}_+; U)$  the set of positive control functions  $u$  in  $L_{loc}^p(\mathbb{R}_+; U)$  such that  $u(t) \in U_+$  almost everywhere in  $\mathbb{R}_+$ , where we regard  $L_{loc}^p(\mathbb{R}_+; U)$  as a lattice ordered Fréchet space with the seminorms being the  $L^p$  norms on the intervals  $[0, n]$ ,  $n \in \mathbb{N}$ .

Let  $(A, D(A))$  be the generator of a positive  $C_0$ -semigroup  $\mathbb{T} := (T(t))_{t \geq 0}$  on  $X$ . We denote by  $\rho(A)$  the resolvent set of  $A$ , i.e., the set of all  $\lambda \in \mathbb{C}$  such that  $\lambda I_X - A$  is invertible with  $I_X$  denote the identity operator in  $X$ . By  $R(\lambda, A) := (\lambda I_X - A)^{-1}$

we denote the resolvent operator of  $A$ . The extrapolation space associated with  $X$  and  $A$ , denoted by  $X_{-1}$ , is the completion of  $X$  with respect to the norm  $\|x\|_{-1} := \|R(\lambda, A)x\|$  for  $x \in X$  and some  $\lambda \in \rho(A)$ . Note that the choice of  $\lambda$  is not important, since by the resolvent equation different choices lead to equivalent norms on  $X_{-1}$ . Note also that  $x \in X_{-1}$  is positive if  $x$  belongs to the closure of  $X_+$  in  $X_{-1}$ . In addition, we have  $X \subset X_{-1}$ . The unique extension of  $\mathbb{T}$  on  $X_{-1}$  is a  $C_0$ -semigroup which we denoted by  $\mathbb{T}_{-1}$  and whose generator is denoted by  $A_{-1}$ . For more details on positive semigroups, see e.g.<sup>21</sup>.

In what follows, let us consider the following non-homogeneous boundary value problem:

$$\begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \\ z(0) = x \in X_+, \\ (G - \Gamma)z(t) = Ku(t), & t > 0, \end{cases} \quad (2)$$

where the state variable  $z(\cdot)$  takes values in a Banach lattice  $X$  and the control function  $u(\cdot)$  is given in the Banach space  $L^p(\mathbb{R}_+; U)$ . The maximal (differential) operator  $A_m : D(A_m) \subset X \rightarrow X$  is closed and densely defined,  $K$  is a positive bounded linear operator from  $U$  into  $\partial X$  (both are Banach lattices), and  $G, \Gamma : D(A_m) \rightarrow \partial X$  are linear positive trace operators. Before going further in our exposition, we underline that:

*System (2) is positively well-posed if for every  $x \in X_+$  and  $u \in L^p_+(\mathbb{R}_+; U)$  there exists a unique positive solution  $z \in C(\mathbb{R}_+; X)$  that depends continuously on the initial data  $x$  and the control  $u$ .*

To make the above statement more clear, let us consider the operator  $\mathcal{A} : D(\mathcal{A}) \rightarrow X$  defined by

$$\mathcal{A} = A_m, \quad D(\mathcal{A}) = \{x \in D(A_m) : (G - \Gamma)x = 0\}.$$

Next, we shall claim that (2) is positively well-posed if  $\mathcal{A}$  generates a positive  $C_0$ -semigroup on  $X$ . To initiate this construction, let us rewrite (2) as the following boundary input-output system

$$\begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \quad z(0) = x, \\ Gz(t) - v(t) = Ku(t), & t > 0, \\ y(t) = \Gamma z(t), & t > 0, \end{cases} \quad (3)$$

with the feedback law

$$v = y. \quad (4)$$

In the sequel we make use of the following two assumptions:

**(H1)**  $A := A_m$  with domain  $D(A) := \ker(G)$  generates a  $C_0$ -semigroup  $\mathbb{T} := (T(t))_{t \geq 0}$  on  $X$ .

**(H2)**  $G : D(A_m) \rightarrow U$  is surjective.

It is shown in<sup>17</sup>, Lemmas 1.2 and 1.3 that, under the assumptions **(H1)**-**(H2)**, the Dirichlet operator

$$\mathbb{D}_\lambda := (G|_{\ker(\lambda I_X - A_m)})^{-1} : \partial X \rightarrow \ker(\lambda I_X - A_m) \subset X, \quad \lambda \in \rho(A),$$

exist and bounded. Thus, one can verify that the operator

$$B := (\lambda I_X - A_{-1})\mathbb{D}_\lambda \in \mathcal{L}(U, X_{-1}), \quad \lambda \in \rho(A),$$

is not depend on  $\lambda \in \rho(A)$  and that

$$(A_m - A_{-1})|_Z = BG.$$

The following result provided a very simple characterization of the well-posedness of the boundary input-output system (3), see<sup>18</sup>, Proposition 3.1 or<sup>22</sup>, Section 4.

**Lemma 1.** Let  $X, \partial X$  be Banach lattices,  $p \in [1, \infty)$  and let the assumptions **(H1)**-**(H2)** be satisfied. Furthermore, assume that  $\mathbb{T}$  is positive and  $D_\mu$  is positive for every  $\mu > s(A)$ . Then, the system (3) is positively well-posed if:

(i)  $B$  is an  $L^p$ -admissible positive control operator for  $A$ , i.e., for some (hence all)  $\tau > 0$ ,

$$\Phi_\tau^A v := \int_0^\tau T_{-1}(\tau - s)Bv(s)ds \in X_+,$$

for all  $v \in L^p_+(\mathbb{R}_+; \partial X)$ .

(ii)  $C := \Gamma|_{D(A)}$  is an  $L^p$ -admissible positive observation operator for  $A$ , i.e., for some (hence all)  $\alpha > 0$ ,

$$\int_0^\alpha \|CT(t)x\|^p dt \leq \gamma^p \|x\|^p,$$

for all  $0 \leq x \in D(A)$  and a constant  $\gamma := \gamma(\alpha) > 0$ .

(iii) For  $\tau > 0$ , there exists a constant  $\kappa := \kappa(\tau) > 0$  such that

$$\|\Gamma\Phi^A v\|_{L^p([0,\tau];\partial X)} \leq \eta \|v\|_{L^p([0,\tau];\partial X)},$$

for all  $v \in W_{0,+}^{1,p}([0,\tau];\partial X) := \{0 \leq v \in W^{1,p}([0,\tau];\partial X) : v(0) = 0\}$ .

In particular,  $(A, B, C)$  is a positive  $L^p$ -well-posed triplet on  $X, \partial X, \partial X$ .

*Remark 1.* For  $\tau > 0$ , let us denote by  $\mathbb{F}$  the operator defined by

$$(\mathbb{F}v)(t) := \Gamma\Phi_t^A v, \quad \forall t \in [0, \tau], \quad 0 \leq v \in W_0^{1,p}([0, \tau]; \partial X),$$

Then, according to Lemma 1,  $\mathbb{F}$  is extendable to  $\mathbb{F} \in \mathcal{L}(L_+^p([0, \tau]; \partial X))$  for each  $\tau \geq 0$ . The operator  $\mathbb{F}$  is called the extended input-output control operator of  $(A, B, C)$ . If, in addition, for every  $v \in \partial X$ ,

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t (\mathbb{F}(\mathbb{1}_{\mathbb{R}_+} v))(\sigma) d\sigma = 0, \quad (\text{in } \partial X),$$

then  $(A, B, C)$  is a positive  $L^p$ -well-posed regular triplet with feedthrough zero, see<sup>22, Section 4</sup>.

We end this section by recalling a perturbation result from<sup>22, Theorem 4.3</sup> (see also<sup>18, Theorem 3.1</sup>) which prove the existence of a positive mild solution of (2).

**Theorem 1.** Let the assumptions **(H1)** and **(H2)** be satisfied and let  $A, B, C$  be the operators defined from the operators  $A_m, G, \Gamma$ . Furthermore, assume that

1.  $(A, B, C)$  is a positive  $L^p$ -well-posed regular triplet with feedthrough zero.
2.  $I_X - \mathbb{F}$  has uniformly positive bounded inverse.

Then, the operator  $(\mathcal{A}, D(\mathcal{A}))$  generates a positive  $C_0$ -semigroup  $\mathcal{T}$  on  $X$ . Additionally, the non-homogeneous boundary value control problem (2) has a unique mild solution  $z : \mathbb{R}_+ \rightarrow X_+$  given by

$$z(t) = \mathcal{T}(t)x + \int_0^t \mathcal{T}_{-1}(t-s)BKu(s)ds,$$

for all  $t \geq 0$ ,  $x \in X_+$  and  $u \in L_+^p(\mathbb{R}_+; U)$ .

### 3 | WELL-POSEDNESS

In this section, we propose to use the feedback theory of of infinite-dimensional positive linear systems to prove existence of a positive mild solution of (1). In a first step, we reformulate the size-structured population system (1) as a non-homogeneous boundary value control problem. For this purpose, we introduce the following Banach spaces

$$X := L^p([0, s^*]), \quad Y := L^p([-r, 0]; X).$$

The space  $Y$  is endowed with the norm

$$\|\varphi\|_Y := \int_{-r}^0 \|\varphi(\theta, \cdot)\|_X d\theta.$$

On the space  $X$ , we introduce the operator

$$(A_m f)(s) = -q(s)\partial_s f - \xi(s)f, \quad \text{for a.e. } s > 0,$$

with domain

$$D(A_m) := W^{1,p}([0, s^*]).$$

On the space  $Y$ , we introduce the operator  $Q_m : D(Q_m) \subset Y \rightarrow Y$  defined by

$$Q_m \varphi = \partial_\theta \varphi, \quad \varphi \in D(Q_m) = W^{1,p}([-r, 0]; X).$$

Define the operator  $\mathbb{L} \in \mathcal{L}(W^{1,p}([-r, 0]; X), \mathbb{R})$  by

$$\mathbb{L} \varphi := \int_0^{s^*} \beta(s) \int_{-r}^0 \varphi(\theta, s) d\eta(\theta) ds, \quad \varphi \in W^{1,p}([-r, 0]; X).$$

With these spaces and operators, the structured population model (1) becomes

$$\begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \\ (I_X \otimes \delta_0) z(t) = \mathbb{L} z_t + bu(t), & t > 0, \\ z(0) = f, \quad z_0 = \varphi, \end{cases}$$

where  $z : [0, +\infty[ \rightarrow X$  is defined as  $z(t) = z(t, \cdot)$  and  $z_t : [-r, 0] \rightarrow X$  is the history function defined by

$$z_t(\theta) = z(t + \theta), \quad \theta \in [-r, 0].$$

It is well known that the function  $t \mapsto z_t$  is the solution of the following boundary equation:

$$\begin{cases} \dot{v}(t) = Q_m v(t), & t > 0, \theta \in [-r, 0], \\ (I_Y \otimes \delta_0) v = z(t), & t > 0 \\ v(0) = \varphi, \end{cases} \quad (5)$$

Finally, we define the product space

$$\mathcal{X} := X \times Y \quad \text{with norm} \quad \|(\begin{smallmatrix} f \\ \varphi \end{smallmatrix})\| := \|f\|_X + \|\varphi\|_Y,$$

on which we define the following operators matrices:

$$\begin{aligned} \mathcal{A}_m &:= \begin{pmatrix} A_m & 0 \\ 0 & Q_m \end{pmatrix}, \quad D(\mathcal{A}_m) := D(A_m) \times D(Q_m), \\ \mathcal{G} &:= \begin{pmatrix} I_X \otimes \delta_0 & 0 \\ 0 & I_Y \otimes \delta_0 \end{pmatrix} : D(\mathcal{A}_m) \rightarrow \mathbb{R} \times X, \\ \mathcal{M} &:= \begin{pmatrix} 0 & \mathbb{L} \\ I_X & 0 \end{pmatrix} : D(\mathcal{A}_m) \rightarrow \mathbb{R} \times X. \end{aligned}$$

Now, by selecting the new state

$$\zeta(t) = \begin{pmatrix} z(t) \\ z_t \end{pmatrix}, \quad t \geq 0,$$

and using the equation (5), the system (1) is reformulated in  $\mathcal{X}$  as the following free-delay perturbed boundary value system

$$\begin{cases} \dot{\zeta}(t) = \mathcal{A}_m \zeta(t), & t > 0, \\ \mathcal{G} \zeta(t) = \mathcal{M} \zeta(t) + \begin{pmatrix} bu(t) \\ 0 \end{pmatrix}, & t > 0, \\ \zeta(0) = \begin{pmatrix} f \\ \varphi \end{pmatrix} \geq 0. \end{cases} \quad (6)$$

In what follows, we make use of the following assumptions:

### Main Assumptions 3.1.

(A1)  $\beta \in L^\infty((0, s^*))$  and  $\beta(s) \geq 0$  for a.e.  $s \in (0, s^*)$ ,

(A2)  $\mu \in L^\infty((0, s^*))$ ,  $\mu(s) \geq 0$  for a.e.  $s \in (0, s^*)$  and  $\int_0^{s^*} \mu(s) ds = +\infty$ ,

(A3)  $q \in W^{1,\infty}([0, s^*])$ ,  $q_* \leq q(s)$  and  $\xi(s) := \partial_s q + \mu(s) \geq 0$  for a.e.  $s \in [0, s^*]$  and a constant  $q_* > 0$ ,

(A4)  $\eta$  is an increasing function of bounded variation with total variation  $|\eta|$  satisfying

$$\lim_{\varepsilon \rightarrow 0} |\eta|([-\varepsilon, 0]) = 0.$$

We select the following definition.

**Definition 1.** Let  $s_1, s_2 \in [0, s^*]$  and set

$$\tau(s_1, s_2) = \int_{s_1}^{s_2} \frac{1}{q(s)} ds, \quad \Xi(s_1, s_2) = \int_{s_1}^{s_2} \frac{\mu(s)}{q(s)} ds. \quad (7)$$

Here, for an individual of size  $s_1$ ,  $\tau(s_1, s_2)$  determine the time it takes the individual to reach the size  $s_2$  with the growth rate of size  $q(s)$  for  $s \in [s_1, s_2]$ , while  $\Xi(s_1, s_2)$  is the loss of individuals on this journey resulting from the factor  $\mu(s)$ .

*Remark 2.* In view of the assumptions (A2)-(A3) the functions  $\tau(\cdot, \cdot)$  and  $\Xi(\cdot, \cdot)$  are well-defined. Moreover, if we set  $h(s) = \tau(0, s)$  for every  $s \in [0, s^*]$ , then the assumption (A3) further yields that the function  $h$  is strictly increasing and continuous and hence invertible. So, we select the following function

$$\tilde{s}(t) := h^{-1}(h(s_0) - t), \quad t \in [0, h(s_0)],$$

where  $h^{-1}$  denote the inverse of  $h$ .  $\tilde{s}(t)$  presents the size of individuals at time  $t$ , who were of size  $s_0$  at time 0. Note that  $\tilde{s} \in C([0, h(s_0)])$  and satisfies the initial value problem

$$\partial_t \tilde{s}(t) = q(\tilde{s}(t)), \quad \tilde{s}(0) = s_0,$$

Moreover,  $\tau(\tilde{s}(t), s_0) = t$ .

**Lemma 2.** Let the assumptions (A2) and (A3) be satisfied. Then, the operator

$$A := A_m, \quad D(A) = \ker\{I_X \otimes \delta_0\},$$

generates a positive  $C_0$ -semigroup  $\mathbb{T} := (T(t))_{t \geq 0}$  on  $X$  given by

$$(T(t)f)(s) = \begin{cases} \frac{q(\tilde{s}(t))}{q(s)} e^{-\Xi(\tilde{s}(t), s)} f(\tilde{s}(t)), & \text{if } t \leq h(s), \\ 0, & \text{if not,} \end{cases} \quad (8)$$

for all  $t \geq 0$ ,  $f \in X$  and  $s \in [0, s^*]$ .

*Proof. First step.* Let  $\lambda \in \mathbb{C}$ ,  $f \in X$ . For  $s \in [0, s^*]$ , we set

$$g(s) = \int_0^s e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} \frac{1}{q(\sigma)} f(\sigma) d\sigma = \frac{1}{q(s)} \int_0^s e^{-\int_\sigma^s \frac{\lambda + \mu(a)}{q(a)} da} f(\sigma) d\sigma \in D(A).$$

Then, we have

$$\begin{aligned} ((\lambda - A)g)(s) &= \lambda g(s) + q(s) \partial_s g(s) + \xi(s) g(s) \\ &= \lambda g(s) - \lambda g(s) - \xi(s) g(s) + f(s) + \xi(s) g(s) = f(s). \end{aligned}$$

On the other hand, for  $f \in D(A)$ , we have

$$\begin{aligned}
 & \int_0^s e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} \frac{1}{q(\sigma)} ((\lambda - A)f)(\sigma) d\sigma \\
 &= \int_0^s e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} \frac{1}{q(\sigma)} (\lambda f(\sigma) + q(\sigma) \partial_\sigma f(\sigma) + \xi(\sigma) f(\sigma)) d\sigma \\
 &= \int_0^s e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} \frac{1}{q(\sigma)} (\lambda + \xi(\sigma)) f(\sigma) d\sigma + \int_0^s e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} \partial_\sigma f(\sigma) d\sigma \\
 &= \int_0^s e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} \partial_\sigma f(\sigma) d\sigma - \int_0^s \partial_\sigma \left( e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} \right) f(\sigma) d\sigma \\
 &= \left[ e^{-\int_\sigma^s \frac{\lambda + \xi(a)}{q(a)} da} f(\sigma) \right]_{\sigma=0}^{\sigma=s} = f(s).
 \end{aligned}$$

Thus, for every  $\lambda > -\|\mu\|_\infty$ ,  $s \in [0, s^*]$  and  $f \in X$ , the resolvent of  $A$  is given by

$$(R(\lambda, A)f)(s) = \frac{1}{q(s)} \int_0^s e^{-\int_\sigma^s \frac{\lambda + \mu(a)}{q(a)} da} f(\sigma) d\sigma.$$

*Second step.* Let  $\mathbb{V}$  be the family of bounded linear operators  $(\mathbb{V}(t))_{t \geq 0}$  on  $X$  defined by

$$(\mathbb{V}(t)f)(s) = \begin{cases} \frac{q(\tilde{s}(t))}{q(s)} e^{-\Xi(\tilde{s}(t), s)} f(\tilde{s}(t)), & \text{if } t \leq h(s), \\ 0, & \text{if not,} \end{cases}$$

for all  $t \geq 0$ ,  $f \in X$  and  $s \in [0, s^*]$ . First, we shall prove that  $\mathbb{V}$  is a positive  $C_0$ -semigroup on  $X$ . Indeed, we clearly have  $\mathbb{V}(0)f = f$  for every  $f \in X$ , since  $\tilde{s}(0) = s$ . Moreover, the continuity of  $\tilde{s}$  (see Remark 2) yields that  $\mathbb{V}$  is strongly continuous. Furthermore, for  $t_1 + t_2 \leq h(s)$ , we have

$$\begin{aligned}
 (\mathbb{V}(t_1 + t_2)f)(s) &= \frac{q(\tilde{s}(t_1 + t_2))}{q(s)} e^{-\Xi(\tilde{s}(t_1 + t_2), s)} f(\tilde{s}(t_1 + t_2)) \\
 &= \frac{q(\tilde{s}(t_1)(t_2))}{q(s)} e^{-\Xi(\tilde{s}(t_1)(t_2), \tilde{s}(t_1))} f(\tilde{s}(t_1)) \\
 &= (\mathbb{V}(t_2)(\mathbb{V}(t_1)f))(s),
 \end{aligned}$$

since  $t_1 \leq h(s)$  and  $t_2 \leq \tau(\tilde{s}(t_1), s)$ , where

$$\tilde{s}(t_1)(t_2) = h^{-1}(h(\tilde{s}(t_1)) - t_2) = \tilde{s}(t_1 + t_2).$$

On the other hand, if  $t_1 + t_2 \geq h(s)$ , then either  $t_1 \geq h(s)$  or  $t_2 \geq h(\tilde{s}(t_1))$ , hence  $(\mathbb{V}(t_1 + t_2)f)(s) = (\mathbb{V}(t_2)(\mathbb{V}(t_1)f))(s) = 0$ . Therefore,  $\mathbb{V}$  is a  $C_0$ -semigroup on  $X$ . Let  $P$  denote its generator.

*Third step.* We show that  $A = P$  by proving that the resolvents of them coincide. Indeed, for  $f \in X$ , we have

$$\begin{aligned}
 (R(\lambda, P)f)(s) &= \int_0^{+\infty} e^{-\lambda t} (\mathbb{V}(t)f)(s) dt \\
 &= \int_0^{h(s)} e^{-\lambda t} \frac{q(\tilde{s}(t))}{q(s)} e^{-\Xi(\tilde{s}(t), s)} f(\tilde{s}(t)) dt.
 \end{aligned}$$

If we set  $\sigma = \tilde{s}(t)$ , then  $d\sigma = -q(\tilde{s}(t))dt$ ,  $\sigma = 0$  if  $t = h(s)$  and  $\sigma = s$  if  $t = 0$ . Therefore

$$(R(\lambda, P)f)(s) = \frac{1}{q(s)} \int_0^s e^{-\int_\sigma^s \frac{\lambda + \mu(a)}{q(a)} da} f(\sigma) d\sigma,$$



for all  $f \in X$  and  $s \in [0, s^*]$ . It follows from the first step, that  $R(\lambda, P) = R(\lambda, A)$  for all  $\lambda > -\|\mu\|_\infty$  and hence  $A = P$ . This completes the proof.  $\square$

It is clear that the operator  $I_X \otimes \delta_0$  is surjective. By a simple calculations, the Dirichlet operator  $\mathbb{D}_\lambda : \mathbb{R} \rightarrow D(A_m)$  associated with  $A_m$  and  $I_X \otimes \delta_0$  is given by

$$(\mathbb{D}_\lambda x)(s) = \frac{q(0)}{q(s)} e^{-\lambda h(s) - \Xi(0,s)} x, \quad x \in \mathbb{R}, s \in [0, s^*], \lambda > -\|\mu\|_\infty. \quad (9)$$

We define also the control operator associated with  $A_m$  and  $I_X \otimes \delta_0$  by

$$B := A_{-1} \mathbb{D}_0 \in \mathcal{L}(\mathbb{R}, X_{-1}). \quad (10)$$

**Lemma 3.** Let the assumptions (A1)-(A3) be satisfied. Then,  $B$  is an  $L^p$ -admissible positive control operator for  $A$  and the associated input-map is given by

$$(\Phi_t^{A,B} v)(s) = \begin{cases} \frac{q(0)}{q(s)} e^{-\Xi(0,s)} v(t - h(s)), & t \geq h(s), \\ 0, & \text{if not,} \end{cases} \quad (11)$$

for all  $t \geq 0$ ,  $s \in [0, s^*]$  and  $v \in L^p(\mathbb{R}^+, \mathbb{R})$ .

*Proof.* By tacking Laplace transform in both side of (11), one can see that  $\widehat{\Phi_t^{A,B} v}(\lambda) = D_\lambda \hat{v}(\lambda)$  for a large  $\lambda > 0$ , where  $\hat{v}$  denotes the Laplace transform of a function  $v$ . According to the injectivity of the Laplace transform, we get that for any  $v \in L^p(\mathbb{R}^+, \mathbb{R})$

$$\int_0^t T_{-1}(t-s) B v(s) ds = \Phi_t^{A,B} v, \quad t \geq 0.$$

Moreover, it follows from (11) that the input-maps  $\Phi_t^{A,B}$  are positive for any  $t \geq 0$ .  $\square$

Now, we consider the operator

$$Q := Q_m, \quad D(Q) := \ker(I_Y \otimes \delta_0) = \{\phi \in W^{1,p}([-r, 0], Y), \quad \phi(0) = 0\}.$$

It is well known that the operator  $Q$  generates the left shift semigroup  $\mathbb{S} := (S(t))_{t \geq 0}$  on  $Y$  defined by

$$(S(t)\phi)(\theta) = \begin{cases} 0, & -t \leq \theta \leq 0, \\ \phi(t + \theta), & -r \leq \theta \leq -t. \end{cases} \quad (12)$$

The Dirichlet operator associated with  $Q_m$  and  $I_Y \otimes \delta_0$  is  $d_\lambda : X \rightarrow Y$  given by

$$(d_\lambda \phi)(\theta, s) = e^{\lambda \theta} g(s), \quad (13)$$

for any  $\theta \in [-r, 0]$ ,  $s \in [0, s^*]$  and  $g \in X$ . We select the control operator

$$\Theta := (-Q_{-1})d_0 \in \mathcal{L}(X, Y_{-1}). \quad (14)$$

**Lemma 4.** Let the assumptions (A1) and (A4) be satisfied. Then,  $(Q, \Theta, \mathbb{I}_{D(Q)})$  is a positive  $L^p$ -well-posed regular triplet on  $Y, X, X$  with feedthrough zero. Moreover, the input-maps of  $(Q, \Theta)$  are given by

$$(\Phi_t^{Q,\Theta} g)(\theta, s) = \begin{cases} g(t + \theta, s), & -t \leq \theta \leq 0, \\ 0, & -r \leq \theta \leq -t, \end{cases} \quad (15)$$

for any  $t \geq 0$  and  $s \in [0, s^*]$ , and  $g \in L^p(\mathbb{R}_+, X)$ .

*Proof.* It suffices to verify the condition of Lemma 1. We omit further details.  $\square$

The well-posedness of the size-structured population system (1) follows from the following result.

**Theorem 2.** Let the assumptions (A1)-(A4) be satisfied. Then the operator  $(\mathfrak{A}, D(\mathfrak{A}))$  defined by

$$\begin{aligned} \mathfrak{A} &:= \mathcal{A}_m, \\ D(\mathfrak{A}) &:= \left\{ \begin{pmatrix} f \\ \varphi \end{pmatrix} \in D(A_m) \times D(Q_m) : f(0) = \int_0^{s^*} \beta(s) \int_{-r}^0 d\eta(\theta) \varphi(\theta, s) ds, \varphi(0, \cdot) = f(\cdot) \right\} \end{aligned} \quad (16)$$

generates a positive  $C_0$ -semigroup  $\mathfrak{T}$  on  $\mathcal{X}$ . Thus, the size-structured population system (6) has a unique positive mild solution.

*Proof.* First, we rewrite the domain of  $\mathfrak{A}$  as

$$D(\mathfrak{A}) := \{ \xi \in D(\mathcal{A}_m), \quad (\mathcal{G} - \mathcal{M})\xi = 0 \}.$$

Thus we will use Theorem 1 to prove our claims. In fact, define the operator

$$\mathcal{A} := \mathcal{A}_m, \quad D(\mathcal{A}) := \ker \mathcal{G}.$$

It is clear that  $\mathcal{A}$  generates a positive  $C_0$ -semigroup  $\mathcal{T} := (\mathcal{T}(t))_{t \geq 0}$  on  $\mathcal{X}$ , given by

$$\mathcal{T} = \begin{pmatrix} \mathbb{T} & 0 \\ 0 & \mathbb{S} \end{pmatrix}, \quad (17)$$

where the semigroups  $\mathbb{T}$  and  $\mathbb{S}$  are given by (8) and (12), respectively.

Obviously, the operator  $\mathcal{G}$  is surjective and positive. By a standard argument, the Dirichlet operator associated to  $\mathcal{A}_m$  and  $\mathcal{G}$  is given by

$$\mathcal{D}_\lambda = \begin{pmatrix} \mathbb{D}_\lambda & 0 \\ 0 & d_\lambda \end{pmatrix}, \quad \lambda > -\|\mu\|_\infty,$$

where  $\mathbb{D}_\lambda$  and  $d_\lambda$  are given by (9) and (13), respectively. It follows from the expressions of  $\mathbb{D}_\lambda$  and  $d_\lambda$ , that  $\mathcal{D}_\lambda$  is positive.

Define the control operator  $\mathcal{B}$  associated to  $\mathcal{A}_m$  and  $\mathcal{G}$  and its input-maps

$$\mathcal{B} = \begin{pmatrix} B & 0 \\ 0 & \Theta \end{pmatrix} \in \mathcal{L}(\mathbb{R} \times X, \mathcal{X}_{-1}), \quad \Phi_t^{\mathcal{A}, \mathcal{B}} = \begin{pmatrix} \Phi_t^{A,B} & 0 \\ 0 & \Phi_t^{Q,\Theta} \end{pmatrix}, \quad t \geq 0,$$

where  $B$ ,  $\Theta$ ,  $\Phi_t^{A,B}$  and  $\Phi_t^{Q,\Theta}$  are given by (10), (14), (11) and (15), respectively. Thus, according to Lemmas 3 and 4,  $\mathcal{B}$  is an  $L^p$ -admissible positive control operator for  $\mathcal{A}$ . On the other hand, it follows from Lemma 4 that the operator

$$\mathcal{C} := \mathcal{M}|_{D(\mathcal{A})} = \begin{pmatrix} 0 & \mathbb{L}|_{D(\mathcal{Q})} \\ I_X & 0 \end{pmatrix},$$

is an  $L^p$ -admissible positive observation operator for  $\mathcal{A}$ .

Now, for any  $\begin{pmatrix} v \\ f \end{pmatrix} \in W_{0,+}^{1,p}([0, \tau]; \mathbb{R} \times X)$  the input-output control operator associated with  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  is given by

$$\mathbb{F}^{\mathcal{A}, \mathcal{B}, \mathcal{C}} \begin{pmatrix} v \\ f \end{pmatrix} = \begin{pmatrix} \mathbb{F}^{Q,\Theta,\mathbb{L}} f \\ \Phi_t^{A,B} v \end{pmatrix},$$

where  $\mathbb{F}^{Q,\Theta,\mathbb{L}}$  is the input-output control operator associated with  $(Q, \Theta, \mathbb{L})$ . In view of (11),

$$(\mathbb{F}^{\mathcal{A}, \mathcal{B}, \mathcal{C}} \begin{pmatrix} x \\ f \end{pmatrix})(t) = \begin{pmatrix} 0 & (\mathbb{F}^{Q,\Theta,\mathbb{L}} f)(t) \\ 0 & 0 \end{pmatrix},$$

for all  $\begin{pmatrix} v \\ f \end{pmatrix} \in \mathbb{R} \times X$  and  $t < h(s)$  with  $s \in [0, s^*]$ . Thus, it follows from Lemmas 3 and 4 that  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  is a positive  $L^p$ -well-posed regular triplet. In addition, we have

$$I_{\mathbb{R} \times X} - \mathbb{F}^{\mathcal{A}, \mathcal{B}, \mathcal{C}} = \begin{pmatrix} I_{\mathbb{R}} & -\mathbb{F}^{Q,\Theta,\mathbb{L}} \\ 0 & I_X \end{pmatrix}, \quad \text{on } L^p([0, \tau]; \mathbb{R} \times X),$$

for any  $\tau < h(s)$  and  $s \in [0, s^*]$ . Thus,  $I_{\mathbb{R} \times X} - \mathbb{F}^{\mathcal{A}, \mathcal{B}, \mathcal{C}}$  is invertible and

$$(I_{\mathbb{R} \times X} - \mathbb{F}^{\mathcal{A}, \mathcal{B}, \mathcal{C}})^{-1} = \begin{pmatrix} I_{\mathbb{R}} & \mathbb{F}^{Q,\Theta,\mathbb{L}} \\ 0 & I_X \end{pmatrix}, \quad \text{on } L^p([0, \tau]; \mathbb{R} \times X),$$

is uniformly bounded and positive, since  $\mathbb{F}^{Q,\Theta,\mathbb{L}}$  is bounded positive. Hence, according to Theorem 1, the operator  $\mathfrak{A}$  generates a positive  $C_0$ -semigroup on  $\mathcal{X}$  and therefore the size-structured population system (6) has a unique positive mild solution. This ends the proof.  $\square$

#### 4 | APPROXIMATE CONTROLLABILITY CRITERIA

Here, we investigate the controllability under positivity constraints of the size-structured population model (1). We recall that (1) is reformulated in  $\mathcal{X}$  as the following non-homogeneous boundary value control problem

$$\begin{cases} \dot{\zeta}(t) = \mathcal{A}_m \zeta(t), & t > 0, \\ \mathcal{G}\zeta(t) = \mathcal{M}\zeta(t) + \begin{pmatrix} bu \\ 0 \end{pmatrix}, & t > 0, \\ \zeta(0) = \begin{pmatrix} f \\ \varphi \end{pmatrix} \geq 0. \end{cases} \quad (18)$$

Consideration similar to <sup>18, Theorem 3.1</sup> shows that (18) is equivalent to

$$\begin{cases} \dot{\zeta}(t) = \mathfrak{A}_{-1}\zeta(t) + \mathcal{B} \begin{pmatrix} bu(t) \\ 0 \end{pmatrix}, & t \geq 0, \\ \zeta(0) = \begin{pmatrix} f \\ \varphi \end{pmatrix} \geq 0. \end{cases}$$

Furthermore, Theorem 1 implies that for the initial condition  $\zeta_0 := \begin{pmatrix} f \\ \varphi \end{pmatrix} \in X_+ \times Y_+$  and control  $u \in L_+^p(\mathbb{R}^+; \mathbb{R})$ , the size-structured (1) has unique mild solution  $\zeta(\cdot)$  satisfying  $\zeta(t) \in X_+$  and for all  $t \geq 0$

$$\begin{aligned} \zeta(t) &= \mathfrak{T}(t)\zeta_0 + \int_0^t \mathfrak{T}_{-1}(t-s)\mathcal{B} \begin{pmatrix} bu(s) \\ 0 \end{pmatrix} ds \\ &= \mathfrak{T}(t)\zeta_0 + \int_0^t \mathfrak{T}_{-1}(t-s)\mathfrak{B}u(s)ds, \end{aligned}$$

where  $\mathfrak{B} : \mathbb{R}_+ \rightarrow X_{-1} \times \{0\}$  is given by

$$\mathfrak{B}u = \begin{pmatrix} A_{-1} \frac{q(0)}{q(\cdot)} e^{-\Xi(0,\cdot)} bu \\ 0 \end{pmatrix}, \quad u \in \mathbb{R}_+.$$

Let  $\Phi_t^{\mathfrak{A}, \mathfrak{B}}$  be the input-map associated with  $\mathfrak{A}$  and  $\mathfrak{B}$ . Then the solution  $\zeta$  is given by

$$\zeta(t) = \mathfrak{T}(t) \begin{pmatrix} f \\ \varphi \end{pmatrix} + \Phi_t^{\mathfrak{A}, \mathfrak{B}} u, \quad t \geq 0, \quad u \in L^p(\mathbb{R}_+; \mathbb{R}).$$

Let us now precise the framework of approximate controllability under state and control positivity constraints (also called approximate positive controllability) associated with the size-structured population system (1). To this end, we introduce the following set of reachable positive states from the origin in time  $\tau$ :

$$\text{Ran } \Phi_{\tau,+}^{\mathfrak{A}, \mathfrak{B}} = \left\{ \int_0^\tau \mathfrak{T}_{-1}(\tau-s)\mathfrak{B}u(s)ds, \quad u \in L_+^p([0, \tau]; \mathbb{R}) \right\}.$$

Then the concept of approximate positive controllability in finite time is defined as follows.

**Definition 2.** Let the assumptions (A1)-(A4) be satisfied. We say that the system (1) is boundary approximately positive controllable if the reachable set from the origin in finite time

$$\bigcup_{\tau>0} \text{Ran } \Phi_{\tau,+}^{\mathfrak{A}, \mathfrak{B}} \text{ is dense in } \mathcal{X}_+.$$

*Remark 3.* The approximate positive controllability in finite time of (1) is equivalent to the following: For any  $\zeta_0, \zeta_1 \in \mathcal{X}_+$  and any  $\varepsilon > 0$ , there exist  $\tau_0 > 0$  and  $u \in L_+^p([0, \tau_0]; \mathbb{R})$  such that the solution  $\zeta$  of (1) satisfies  $\|\zeta(\tau_0) - \zeta_1\| < \varepsilon$ .

We have the following characterization of the approximate positive controllability of the system (1).

**Theorem 3.** Let the assumptions (A1)-(A4) be satisfied and let  $\lambda_1 > -\|\mu\|_\infty$  such that  $\mathbb{A}_{\lambda_1} := \mathbb{L}d_{\lambda_1}\mathbb{D}_{\lambda_1} < 1$ . Then the size-structured population system (1) is boundary approximately positive controllable if and only if the following implication holds for all  $f^* \in X'$  and  $\varphi^* \in Y'$ :

$$\langle \mathbb{A}_{\lambda}^n \mathbb{D}_{\lambda} bu, f^* \rangle + \langle d_{\lambda} \mathbb{A}_{\lambda}^n \mathbb{D}_{\lambda} bu, \varphi^* \rangle \leq 0, \quad \forall u \in \mathbb{R}_+, n \in \mathbb{N}, \lambda \geq \lambda_1 \Rightarrow f^* \leq 0, \varphi^* \leq 0. \quad (19)$$

*Proof.* We first note that as  $L^1$  satisfies the Radon-Nikodým property, then  $Y' = L^\infty([-r, 0], X')$  with  $X' = L^\infty([0, s^*])$ . In view of <sup>18, Theorem 2.2</sup>, the system (1) is boundary approximately positive controllable if and only if the following implication holds

for all  $\begin{pmatrix} f^* \\ \varphi^* \end{pmatrix} \in \mathcal{X}'$ :

$$\left\langle \mathcal{D}_\lambda(I_{\mathbb{R} \times X} - \mathcal{M}\mathcal{D}_\lambda)^{-1} \begin{pmatrix} bu \\ 0 \end{pmatrix}, \begin{pmatrix} f^* \\ \varphi^* \end{pmatrix} \right\rangle \leq 0, \quad \forall u \in \mathbb{R}_+, n \in \mathbb{N}, \lambda \geq \lambda_1 \Rightarrow \begin{pmatrix} f^* \\ \varphi^* \end{pmatrix} \leq 0. \quad (20)$$

By a simple computation we obtain

$$\mathcal{D}_\lambda(I_{\mathbb{R} \times X} - \mathcal{M}\mathcal{D}_\lambda)^{-1} \begin{pmatrix} bu \\ 0 \end{pmatrix} = \begin{pmatrix} (I_{\mathbb{R}} - \mathbb{A}_\lambda)^{-1} \mathbb{D}_\lambda bu \\ d_\lambda(I_{\mathbb{R}} - \mathbb{A}_\lambda)^{-1} \mathbb{D}_\lambda bu \end{pmatrix} = \begin{pmatrix} \sum_{n \in \mathbb{N}} \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu \\ \sum_{n \in \mathbb{N}} d_\lambda \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu \end{pmatrix},$$

for all  $u \in \mathbb{R}$  and  $\lambda \geq \lambda_1$ . By Replacing the above explicit expression of  $\mathcal{D}_\lambda(I_{\mathbb{R} \times X} - \mathcal{M}\mathcal{D}_\lambda)^{-1}$  in (20) and using the fact that the family  $(\mathbb{A}_\lambda)_{\lambda \geq \lambda_1}$  is positive and monotonically decreasing, we get (19).  $\square$

**Corollary 1.** Let the assumptions of Theorem 3 be satisfied. Then the system (1) is boundary approximately positively controllable if and only if the following implication holds for all  $\varphi^* \in Y'$ :

$$\langle d_\lambda \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu, \varphi^* \rangle \leq 0, \quad \forall u \in \mathbb{R}_+, n \in \mathbb{N}, \lambda \geq \lambda_1 \Rightarrow \varphi^* \leq 0. \quad (21)$$

*Proof.* To prove our claim, we will show that (21) implies for all  $f^* \in X'$ :

$$\langle \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu, f^* \rangle \leq 0, \quad \forall u \in \mathbb{R}_+, n \in \mathbb{N}, \lambda \geq \lambda_1 \Rightarrow f^* \leq 0. \quad (22)$$

To this end, let assume that (21) holds and let  $f^* \in X'$  such that  $\langle \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu, f^* \rangle \leq 0$  for all  $u \in \mathbb{R}_+, n \in \mathbb{N}$  and  $\lambda \geq \lambda_1$ . Set  $\varphi^* := d_{-\lambda} f^*$ , then  $\varphi^* \in Y'$  and  $\varphi^*(\theta) = e^{-\lambda\theta} f^*$  for all  $\theta \in [-r, 0]$ . Moreover, we have

$$\begin{aligned} \langle d_\lambda \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu, \varphi^* \rangle &= \int_{-r}^0 \langle (d_\lambda \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu)(\theta), \varphi^*(\theta) \rangle d\theta \\ &= \int_{-r}^0 \langle e^{\lambda\theta} \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu, e^{-\lambda\theta} f^* \rangle d\theta = \langle \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu, f^* \rangle r \leq 0, \end{aligned}$$

for all  $u \in \mathbb{R}_+, n \in \mathbb{N}$  and  $\lambda \geq \lambda_1$ . Thus,  $\varphi^* \leq 0$  and hence  $f^* \leq 0$ . Thus, (21) implies (22) and hence the system (1) is boundary approximately positively controllable if and only if the condition (21) for all  $\varphi^* \in Y'$ . This completes the proof.  $\square$

We end this section by the following useful characterization of the boundary approximate positive controllability of the population model (1).

**Theorem 4.** Let the assumptions (A1)-(A4) be satisfied. Assume that there exists  $\lambda_1 > 0$  such that

$$0 < \mathbb{A}_{\lambda_1} < 1. \quad (23)$$

Then, the size-structured population system (1) is boundary approximately positive controllable.

*Proof.* Since  $L^1$  has the Radon-Nikodym property, then using Fubini's theorem we can identify  $Y$  with  $L^p([-r, 0] \times [0, s^*])$  for all  $p \in [1, \infty)$ . So, for  $q \in [1, \infty)$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $Y' = L^q([-r, 0] \times [0, s^*])$ .

It follows from Corollary 1 that the system (1) is boundary approximately positive controllable if and only if the following implication holds for all  $\varphi^* \in L^q([-r, 0] \times [0, s^*])$ :

$$\langle d_\lambda \mathbb{A}_\lambda^n \mathbb{D}_\lambda bu, \varphi^* \rangle \leq 0, \quad \forall u \in \mathbb{R}_+, n \in \mathbb{N}, \lambda \geq \lambda_1 \Rightarrow \varphi^* \leq 0,$$

or, equivalently,

$$\mathbb{A}_\lambda^n \langle d_\lambda \mathbb{D}_\lambda bu, \varphi^* \rangle \leq 0, \quad \forall u \in \mathbb{R}_+, n \in \mathbb{N}, \lambda \geq \lambda_1 \Rightarrow \varphi^* \leq 0.$$

The condition (23) further yields that the system (1) is boundary approximately positive controllable if and only if for all  $\varphi^* \in L^q([-r, 0] \times [0, s^*])$ :

$$\langle d_\lambda \mathbb{D}_\lambda, \varphi^* \rangle \leq 0, \quad \forall \lambda \geq \lambda_1 \Rightarrow \varphi^* \leq 0. \quad (24)$$

Now, let  $n \geq \lambda_1$  and  $\varphi^* \in L^q([-r, 0] \times [0, s^*])$ . The explicit expression of  $d_\lambda$  and  $\mathbb{D}_\lambda$  further yield that

$$\langle d_n \mathbb{D}_n, \varphi^* \rangle = \int_{-r}^0 \int_0^{s^*} e^{-\Xi(0,s)} \ell_{n,n}(\theta, s) \varphi^*(\theta, s) ds d\theta,$$

where  $(\ell_{n,m})_{n,m \in \mathbb{N}}$  is defined in (26). By virtue of Lemma 6 the fact that for all  $\varphi^* \in L^q([-r, 0] \times [0, s^*])$

$$\langle d_n \mathbb{D}_n, \varphi^* \rangle \leq 0,$$

implies that  $\varphi^* \leq 0$ . Hence, according to Corollary 1, the size-structured population system (1) is boundary approximately positive controllable.  $\square$

## 5 | CONCLUSION

In this work, we studied the controllability under positivity constraints of a size-structured population model with a delayed birth process. We exploited the feedback theory of infinite-dimensional positive linear systems to rewrite the aforementioned system as an abstract free-delay distributed control system on a suitable product of  $L^p$ -spaces. We proved the existence and uniqueness of a positive mild solution in this product space. By developing the solution into a variation of the constant formula and applying Laplace transform techniques, we derived a sufficient condition for boundary approximate controllability under state and control positivity constraints.

## 6 | APPENDIX

In this appendix we prove an approximation lemma needed to prove the main controllability result of the paper. Let us first recall the following approximation result for functions of two variables by the so-called Szász-Mirakjan operators<sup>19</sup>.

**Lemma 5.** Let  $J := [0, +\infty) \times [0, +\infty)$  and let  $C(J)$  denote the space of all continuous real valued functions on  $J$ . Set

$$C_b(J) := \{f \in C(J) : \exists \alpha, \delta, \kappa \geq 0 \text{ such that } |f(x, y)| \leq \delta e^{\alpha x + \kappa y}\},$$

and for every  $f \in C_b(J)$ ,  $n, m \geq 1$ , and  $x, y \geq 0$ , define

$$\mathcal{J}_{n,m}(f; x, y) := e^{-nx} e^{-my} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f\left(\frac{n}{k}, \frac{m}{j}\right) \frac{(nx)^k}{k!} \frac{(my)^j}{j!}. \quad (25)$$

Then, for every  $f \in C_b(J)$  we have

$$\lim_{n \rightarrow +\infty} \mathcal{J}_n(f) = f, \quad \text{uniformly on compact subsets of } J.$$

With the help of the above approximation lemma, one can deduce the following density result.

**Lemma 6.** Let  $n, m \in \mathbb{N}$ ,  $s^*, r > 0$  and  $p, q \geq 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Let us consider the family of functions  $(\ell_{n,m})_{n,m \in \mathbb{N}^2}$  on  $[-r, 0] \times [0, s^*]$  defined by

$$\ell_{n,m}(\theta, s) = e^{n\theta} e^{-mh(s)}, \quad (26)$$

where  $h(s) := \int_0^s \frac{1}{q(\tau)} d\tau$ . Then for all  $\varphi \in L^q([-r, 0] \times [0, s^*])$

$$\int_{-r}^0 \int_0^{s^*} \ell_{n,n}(\theta, s) \varphi^*(\theta, s) ds d\theta \leq 0, \quad \forall n \in \mathbb{N} \Rightarrow \varphi^* \leq 0. \quad (27)$$

*Proof.* Let  $\varphi \in L^\infty([-r, 0] \times [0, s^*])$  such that  $\int_{-r}^0 \int_0^{s^*} \ell_{n,n}(\theta, s) \varphi^*(\theta, s) ds d\theta \leq 0$  for all  $n \in \mathbb{N}$ . Let  $0 \leq f \in C([-r, 0] \times [0, s^*])$  and define the function

$$\tilde{f}(\sigma, s) := \begin{cases} f(-\sigma, s), & (\sigma, s) \in [0, r] \times [0, s^*], \\ f(-r, s), & \sigma \geq r, s \geq s^*. \end{cases}$$

Then  $0 \leq \tilde{f} \in C_b(J)$  and

$$\int_0^r \int_0^{s^*} \ell_{n,n}(-\sigma, s) \tilde{f}\left(\frac{n}{k}, \frac{n}{j}\right) \frac{(n\sigma)^k}{k!} \frac{(ns)^j}{j!} \varphi^*(-\sigma, s) d\sigma ds \leq 0, \quad \forall n \geq 1, k, j \in \mathbb{N}.$$

It follows that

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \int_0^r \int_0^{s^*} \ell_{n,n}(-\sigma, s) \tilde{f}\left(\frac{n}{k}, \frac{n}{j}\right) \frac{(n\sigma)^k}{k!} \frac{(ns)^j}{j!} \varphi^*(-\sigma, s) d\sigma ds \leq 0, \quad \forall n \geq 1.$$

Therefore, by the continuity of  $f$ , Lemma 5 and the dominated convergence theorem we obtain

$$\int_0^r \int_0^{s^*} f(-\sigma, s) \varphi(-\sigma, s) d\sigma ds \leq 0, \quad \forall 0 \leq f \in C([-r, 0] \times [0, s^*]).$$

The density of the positive cone of  $C([-r, 0] \times [0, s^*])$  in  $L_+^p([-r, 0] \times [0, s^*])$  further yields that

$$\int_{-r}^0 \int_0^{s^*} f(\theta, s) \varphi(\theta, s) d\theta ds \leq 0, \quad \forall f \in L_+^p([-r, 0] \times [0, s^*]).$$

and hence  $\varphi \leq 0$ . □

## DECLARATIONS

**Ethics approval** Not applicable.

**Conflict of interest statement** The authors declare that there is no potential conflict of interest.

**Authors' contributions** All authors contributed to the manuscript.

**Availability of data and materials** Not applicable.

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