Extreme values of solution of Caputo-Hadamard uncertain fractional differential equation and applications

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Abstract

Uncertain fractional differential equation (UFDE) is an useful tool for studying complex systems in uncertain environments. In this paper, we study the extreme value theorems of the solution to Caputo-Hadamard UFDEs and applications. A numerical algorithm for solving the numerical solution of a nonlinear Caputo-Hadamard UFDE is presented, the feasibility of the numerical algorithm is validated by numerical experiments. The extreme value theorems are applied to the financial markets, and the pricing formulas of the American option based on the new uncertain stock model are given. Considering the properties of the American option pricing, the algorithms for computing the expected value of the extreme values based on the Simpson's rule are designed. Finally, the price fluctuation of the American option is illustrated by numerical experiments.

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RESEARCH ARTICLE

Extreme values of solution of Caputo-Hadamard uncertain fractional differential equation and applications

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Summary

Uncertain fractional differential equation (UFDE) is an useful tool for studying complex systems in uncertain environments. In this paper, we study the extreme value theorems of the solution to Caputo-Hadamard UFDEs and applications. A numerical algorithm for solving the numerical solution of a nonlinear Caputo-Hadamard UFDE is presented, the feasibility of the numerical algorithm is validated by numerical experiments. The extreme value theorems are applied to the financial markets, and the pricing formulas of the American option based on the new uncertain stock model are given. Considering the properties of the American option pricing, the algorithms for computing the expected value of the extreme values based on the Simpson's rule are designed. Finally, the price fluctuation of the American option is illustrated by numerical experiments.

KEYWORDS:
Uncertain fractional differential equation; Extreme value theorems; Numerical algorithm; American option; Simpson's rule

1 | INTRODUCTION

In the world, when the events of emergencies such as wars, earthquakes, and financial crises are happen, there will be a lack of effective data to estimate the frequency. When quality of a thing is evaluated, such as scoring diving competition, it has to be done by judgement. In these cases, it is a common work to invite some experts in the relevant fields to assess the belief degree that an event happens. To deal more reasonably with the possibility of something happening, Liu [1] proposed the uncertainty theory as a powerful tool to handle the belief degree and refined it in 2010 [2]. To better describe the change of dynamic system in uncertain environment, Liu [3] introduced the definition of uncertain process. As an application of uncertain process, Liu [4] proposed the concept of uncertain calculus. Chen and Liu [5] proved the existence and uniqueness theorem for solution of uncertain differential equations. Yao and Chen [6] introduced the concept of α-path for uncertain differential equations, which is a class functions for solving the corresponding ordinary differential equations. Based on uncertainty theory, Liu [7] proposed an extreme value theorem involving special uncertain processes. Yao [8] gave the extreme value of the solution of uncertain differential equations.

Fractional calculus is well suited for describing processes with memory properties. Oldham and Spanier [9] and Samko et al. [10] had a preliminary study on fractional differential equation (FDE). Fractional calculus can be defined in different ways. In order to consider the memory properties in uncertain systems, Zhu [11] developed the concepts of UFDEs, and presented the analytic solution for some special Riemann-Liouville and Caputo UFDEs. Zhu [12] proved the existence and uniqueness theorems of solutions of UFDEs under the Lipschitz condition and the linear growth condition. Hadamard [13] introduced another special

[0] **Abbreviations:** UFDE, uncertain fractional differential equation; IUD, inverse uncertainty distribution; FDE, fractional differential equation

kind of differential operator $\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^p$, which is different from ordinary differential operator $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p$. Jarad et al.[14] and Gambo et al.[15] extended the fractional calculus of Hadamard type to the Caputo-Hadamard environment. He et al.[16] used the Caputo-Hadamard fractional derivative to discuss the stability of related systems. Gohar et al.[17] proposed the concept of Caputo-Hadamard FDE and gave the basic theorem for solution of Caputo-Hadamard FDE. Liu et al.[18] defined the Caputo-Hadamard UFDE, and deduced the explicit analytic solution for some special Caputo-Hadamard UFDEs.

After Liu[19] introduced uncertainty theory into uncertain financial markets, option pricing problem had gradually become a research hotspot. Peng and Yao[20] developed an uncertain stock model by using uncertain differential equation and gave the pricing formulas of the European option and the American option. Chen[21] derived the pricing formulas of American option in uncertain financial markets. Chen and Gao[22] studied the particular term structure equation and derived the valuation equation for zero-coupon bonds. Sun and Chen[23] gave an pricing formulas of the Asian option, and some corresponding characteristics were also studied. Lu et al.[24] gave the pricing formulas of the European option based on an uncertain stock model with mean-reverting process. Yao and Qin[25] investigated a class of barrier options in uncertain financial markets and presented the pricing formulas for the barrier options. Gao et al.[26] put forward a currency model of American barrier option under uncertain environment and gave four different pricing formulas of American barrier option.

However, the nonlinear differential equations are used to simulate the dynamic changes of stock price in financial markets, and the methods for solving analytic solutions of the linear differential equations are no longer applicable. Diethelm et al.[27] investigated the predictor-corrector method for solving the numerical solutions of FDEs. Subsequently, Diethelm et al.[28] designed a numerical solution for solving the numerical solutions of the nonlinear FDEs with initial conditions. Gu and Zhu[29] proposed a new Adams predictor-corrector method to solve uncertain differential equation, and used this method to get the extreme value of solution of uncertain differential equation. Lu and Zhu[30] presented a numerical approach for solving UFDEs with Caputo type derivatives. Jin et al.[31] designed a numerical algorithm to get the inverse uncertainty distributions (IUDs) for extreme values of solution of Caputo UFDEs. Gohar et al.[32] modified the predictor-corrector methods and applied it to solve the nolinear Caputo-Hadamard fractional ordinary differential equations. Liu et al.[33] introduced the numerical algorithm for solving the IUD of solution of Caputo-Hadamard UFDE.

This paper will develop the extreme values theorems for solution of Caputo-Hadamard UFDEs. Based on the modified predictor-corrector method, a new algorithm is designed to handle the nonlinear Caputo-Hadamard UFDEs. Furthermore, the numerical algorithms for calculating the price of the American option without explicit pricing formulas are presented. The rest of this paper will be arranged as the following ways: in Section 2, we mainly review some useful concepts and lemmas related to uncertainty theory and fractional calculus. In Section 3, the extreme values theorems of monotonic function for the solution of Caputo-Hadamard UFDEs are studied. In Section 4, a numerical algorithm are proposed, and accuracy and stability of the numerical algorithm are illustrated through numerical examples. In Section 5, the proved theorems are used to the American option pricing problem, and the corresponding pricing formulas are given. The last section gives the conclusion of this paper.

2 | PRELIMINARY

In preparation for the next study, some useful lemmas and conclusions about uncertainty theory and fractional calculus will be introduced. More detailed knowledge, such as uncertain measure, uncertainty distribution, uncertain space, uncertain calculus, may refer to[1,2,14,18,33]. Without special statements in this paper, we always assume that $\delta = t\frac{\mathrm{d}}{\mathrm{dt}}$ and the positive number p satisfying that $0 \le n-1 < p \le n$, where n is usually considered as a integer. Besides, we always assume that $f, g : [a,\infty)\times \mathrm{R} \to \mathrm{R}$ are two continuous functions.

Liu[2] presented the monotonicity theorem of uncertain measure. For any given events Ω_1 and Ω_2, if $\Omega_1 \subset \Omega_2$, then $\mathcal{M}\left\{\Omega_1\right\} \le \mathcal{M}\left\{\Omega_2\right\}$. Subsequently, Liu[2] proved that if a function $\Psi^{-1} : (0,1) \to \mathfrak{R}$ is continuous and $\mathcal{M}\left\{\xi \le \Psi^{-1}(\alpha)\right\} = \alpha$ for all $\alpha \in (0,1)$, then the function is the IUD of an uncertain variable ξ.

Liu[2] gave a formula for calculating the expected value of uncertain variable. Suppose that ξ is an uncertain variable, then the expected value of ξ can be calculated by

$$E[\xi] = \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha,$$

where $\Psi^{-1}(\alpha)$ is the IUD of uncertain variable ξ.

Jarad et al.[14] introduced the definition of Caputo-Hadamard fractional order derivative. The Caputo-Hadamard fractional order derivative of function $g(t)$ is defined as follows:
(i) If $p \notin \mathrm{N}^+$, the fractional order derivative of Caputo-Hadamard type can be expressed as

$$^{CH}\mathcal{D}^p_{a^+}g(t) = \frac{1}{\Gamma(n-p)}\int_a^t \left(\log\frac{t}{s}\right)^{n-p-1} \delta^n g(s)\frac{\mathrm{d}s}{s}, \quad 0<a<t,$$

where $\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}\mathrm{d}t$ is a Gamma function.
(ii) If $p \in \mathrm{N}^+$, then

$$^{CH}\mathcal{D}^p_{a^+}g(t) = \delta^n g(t), \quad 0<a<t.$$

Liu et al.[18] defined the UFDE of Caputo-Hadamard type with initial conditions. Suppose that there are two continuous functions $F, G : [a,\infty)\times \mathrm{R} \to \mathrm{R}$, then a Caputo-Hadamard UFDE satisfies the following equation

$$\begin{cases} ^{CH}\mathcal{D}^p_{a+}X_t = F\left(t, X_t\right) + G\left(t, X_t\right)\frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad 0<a\le t, \\ \delta^k X_t\big|_{t=a} = x_k, \quad k=0,1,\cdots,n-1. \end{cases} \tag{1}$$

The α-path for the UFDE of Caputo-Hadamard type is defined by the solution X^α_t of the following equation

$$\begin{cases} ^{CH}\mathcal{D}^p_{a+}X^\alpha_t = F\left(t, X^\alpha_t\right) + \left|G\left(t, X^\alpha_t\right)\right|\Phi^{-1}(\alpha), \quad 0<a\le t, \\ \delta^k X^\alpha_t\big|_{t=a} = x_k, \quad k=0,1,\cdots,n-1, \end{cases} \tag{2}$$

where $\alpha \in (0,1)$, $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$ is the IUD of the standard normal uncertain variable.

Denote the $^{CH}\mathcal{D}^p_{1+}$ by the abbreviations $^{CH}\mathcal{D}^p$. Liu et al.[18] presented an analytic solution for a particular UFDE of Caputo-Hadamard type. Let $b(t)$ and $\sigma(t)$ be continuous functions on a given interval $[1,T]$ and a is a constant. Then the UFDE of Caputo-Hadamard type with initial conditions

$$\begin{cases} ^{CH}\mathcal{D}^p X_t = aX_t + b(t) + \sigma(t)\frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t\in[1,T], \\ \delta^k X_t\big|_{t=1} = x_k, \quad k=0,1,\cdots,n-1 \end{cases} \tag{3}$$

has an analytic solution

$$\begin{aligned} X_t = &\sum_{k=0}^{n-1} x_k(\log t)^k E_{p,(k+1)}\left(a(\log t)^p\right) + \int_1^t \left(\log\frac{t}{s}\right)^{p-1} E_{p,p}\left(a\left(\log\frac{t}{s}\right)^p\right) b(s)\frac{\mathrm{d}s}{s} \\ &+ \int_1^t \left(\log\frac{t}{s}\right)^{p-1} E_{p,p}\left(a\left(\log\frac{t}{s}\right)^p\right)\sigma(s)\frac{\mathrm{d}C_s}{s}, \end{aligned}$$

where $E_{p,q}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(pk+q)}$ is a Mittag-Leffler function.

The UFDE of Caputo-Hadamard type has a unique solution if the functions $F(t,x)$ and $G(t,x)$ in (1) satisfy the inequality

$$|F(s,x)-F(s,z)| + |G(s,x)-G(s,z)| \le L|x-z|, \quad \forall x,z\in \mathrm{R}, \quad 0<a\le s,$$

and the inequality

$$|F(s,x)| + |G(s,x)| \le L(1+|x|), \quad \forall x\in\mathrm{R}, \quad 0<a\le s,$$

where L is a positive constant.

Lemma 1 ([33])**.** Assume that X_t and X^α_t are the unique solution and α-path for the Caputo-Hadamard UFDE (1), respectively. Then

$$\begin{cases} \mathcal{M}\left\{X_t \le X^\alpha_t, \forall t\in(a,T]\right\} = \alpha, \\ \mathcal{M}\left\{X_t > X^\alpha_t, \forall t\in(a,T]\right\} = 1-\alpha. \end{cases} \tag{4}$$

Furthermore, the IUD of the solution X_t is $\Phi^{-1}_t(\alpha) = X^\alpha_t$.

3 | EXTREME VALUES OF SOLUTION

In the section, we will study the supremum and infimum of the solution of Caputo-Hadamard UFDE driven by Liu process, and give the extreme value theorems through the concept of α-path. The considered Caputo-Hadamard UFDE with initial conditions is as follows:

$$\begin{cases} {}^{CH}\mathcal{D}^{p} X_t = F\left(t, X_t\right) + G\left(t, X_t\right) \dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t \geq 1, \\ \delta^k X_t\big|_{t=1} = x_k, \quad k = 0, 1, \cdots, n-1. \end{cases} \tag{5}$$

3.1 | Extreme value theorems for increasing function $J(X_t)$

Theorem 1. Let X_t and X_t^α be the unique solution and α-path for the Caputo-Hadamard UFDE (5), respectively. Suppose that there is the function $J(x)$ is continuous and strictly increasing. For $\forall s > 1$, then $\sup_{1\leq t\leq s} J\left(X_t\right)$ has an IUD

$$\Psi_s^{-1}(\alpha) = \sup_{1\leq t\leq s} J\left(X_t^\alpha\right),$$

and $\inf_{1\leq t\leq s} J\left(X_t\right)$ has an IUD

$$\Psi_s^{-1}(\alpha) = \inf_{1\leq t\leq s} J\left(X_t^\alpha\right).$$

Proof. We will adopt a similar method in[8] to prove the result. For any time $s > 1$, let

$$\begin{aligned} \Omega_1^+ &= \left\{\gamma \mid X_t(\gamma) \leq X_t^\alpha, \forall t\right\}, \\ \Omega_1^- &= \left\{\gamma \mid X_t(\gamma) > X_t^\alpha, \forall t\right\}, \\ \Omega_2^+ &= \left\{\gamma \mid \sup_{1\leq t\leq s} J\left(X_t(\gamma)\right) \leq \sup_{1\leq t\leq s} J\left(X_t^\alpha\right)\right\}, \\ \Omega_2^- &= \left\{\gamma \mid \sup_{1\leq t\leq s} J\left(X_t(\gamma)\right) > \sup_{1\leq t\leq s} J\left(X_t^\alpha\right)\right\}. \end{aligned}$$

Since the function $J(x)$ is continuous and strictly increasing, then we have $\Omega_1^+ \subset \Omega_2^+$ and $\Omega_1^- \subset \Omega_2^-$. It follows from the monotonicity theorem of uncertain measure and Lemma 1 that

$$\alpha = \mathcal{M}\left\{\Omega_1^+\right\} \leq \mathcal{M}\left\{\Omega_2^+\right\}$$

and

$$1-\alpha = \mathcal{M}\left\{\Omega_1^-\right\} \leq \mathcal{M}\left\{\Omega_2^-\right\}.$$

According to the duality axiom in uncertainty theory, we have

$$\mathcal{M}\left\{\Omega_2^+\right\} + \mathcal{M}\left\{\Omega_2^-\right\} = 1.$$

Therefore, we can get that

$$\mathcal{M}\left\{\Omega_2^+\right\} = \alpha.$$

Then $\sup_{1\leq t\leq s} J\left(X_t\right)$ has an IUD

$$\Psi_s^{-1}(\alpha) = \sup_{1\leq t\leq s} J\left(X_t^\alpha\right).$$

Similarly, let

$$\begin{aligned} \Omega_3^+ &= \left\{\gamma \mid \inf_{1\leq t\leq s} J\left(X_t(\gamma)\right) \leq \inf_{1\leq t\leq s} J\left(X_t^\alpha\right)\right\}, \\ \Omega_3^- &= \left\{\gamma \mid \inf_{1\leq t\leq s} J\left(X_t(\gamma)\right) > \inf_{1\leq t\leq s} J\left(X_t^\alpha\right)\right\}. \end{aligned}$$

It is obviously that we have $\Omega_1^+ \subset \Omega_3^+$ and $\Omega_1^- \subset \Omega_3^-$. It follows from the monotonicity theorem of uncertain measure and Lemma 1 that

$$\alpha = \mathcal{M}\left\{\Omega_1^+\right\} \leq \mathcal{M}\left\{\Omega_3^+\right\}$$

and

$$1-\alpha = \mathcal{M}\left\{\Omega_1^-\right\} \leq \mathcal{M}\left\{\Omega_3^-\right\}.$$

According to the duality axiom in uncertainty theory, we have

$$\mathcal{M}\left\{\Omega_3^+\right\} + \mathcal{M}\left\{\Omega_3^-\right\} = 1.$$

Therefore, we can get that

$$\mathcal{M}\left\{\Omega_3^+\right\} = \alpha.$$

Then $\inf_{1\le t\le s} J\left(X_t\right)$ has an IUD

$$\Psi_s^{-1}(\alpha) = \inf_{1\le t\le s} J\left(X_t^\alpha\right).$$

The proof ends. □

Theorem 2. Under the assumption of Theorem 1, X_t has an uncertainty distribution $\Phi_t(x)$ at time t. For $\forall s > 1$, then supremum $\sup_{1\le t\le s} J\left(X_t\right)$ has an uncertainty distribution

$$\Psi_s(x) = \inf_{1\le t\le s} \Phi_t\left(J^{-1}(x)\right),$$

and the infimum $\inf_{1\le t\le s} J\left(X_t\right)$ has an uncertainty distribution

$$\Psi_s(x) = \sup_{1\le t\le s} \Phi_t\left(J^{-1}(x)\right).$$

Proof. Since $X_t^\alpha = \Phi_t^{-1}(\alpha)$, it follows from Theorem 1 that we have

$$\mathcal{M}\left\{\sup_{1\le t\le s} J\left(X_t\right) \le \sup_{1\le t\le s} J\left(\Phi_t^{-1}(\alpha)\right)\right\} = \alpha.$$

Let $x = \sup_{1\le t\le s} J\left(\Phi_t^{-1}(\alpha)\right)$. Then $\alpha = \inf_{1\le t\le s} \Phi_t\left(J^{-1}(x)\right)$. Thus

$$\mathcal{M}\left\{\sup_{1\le t\le s} J\left(X_t\right) \le x\right\} = \inf_{1\le t\le s} \Phi_t\left(J^{-1}(x)\right).$$

That is to say, the supremum $\sup_{1\le t\le s} J\left(X_t\right)$ has an uncertainty distribution

$$\Psi_s(x) = \inf_{1\le t\le s} \Phi_t\left(J^{-1}(x)\right).$$

Similarly, we have

$$\mathcal{M}\left\{\inf_{1\le t\le s} J\left(X_t\right) \le \inf_{1\le t\le s} J\left(\Phi_t^{-1}(\alpha)\right)\right\} = \alpha.$$

Let $y = \inf_{1\le t\le s} J\left(\Phi_t^{-1}(\alpha)\right)$. Then $\alpha = \sup_{1\le t\le s} \Phi_t\left(J^{-1}(y)\right)$. Thus

$$\mathcal{M}\left\{\inf_{1\le t\le s} J\left(X_t\right) \le y\right\} = \sup_{1\le t\le s} \Phi_t\left(J^{-1}(y)\right).$$

That is to say, the infimum $\inf_{1\le t\le s} J\left(X_t\right)$ has an uncertainty distribution

$$\Psi_s(y) = \sup_{1\le t\le s} \Phi_t\left(J^{-1}(y)\right).$$

The proof ends. □

3.2 | Extreme value theorems for decreasing function $J(X_t)$

Theorem 3. Let X_t and X_t^α be the unique solution and α-path for the Caputo-Hadamard UFDE (5), respectively. Suppose that there is the function $J(x)$ is continuous and strictly decreasing. For $\forall s > 1$, then $\sup_{1\le t\le s} J\left(X_t\right)$ has an IUD

$$\Psi_s^{-1}(\alpha) = \sup_{1\le t\le s} J\left(X_t^{1-\alpha}\right),$$

and $\inf_{1\le t\le s} J\left(X_t\right)$ has an IUD

$$\Psi_s^{-1}(\alpha) = \inf_{1\le t\le s} J\left(X_t^{1-\alpha}\right).$$

Proof. The proof is similar to that of Theorem 1. □

Theorem 4. Under the assumption of Theorem 3, X_t has an uncertainty distribution $\Phi_t(x)$ at time t. For $\forall s > 1$, then supremum $\sup_{1\le t\le s} J\left(X_t\right)$ has an uncertainty distribution

$$\Psi_s(x) = 1 - \sup_{1\le t\le s} \Phi_t\left(J^{-1}(x)\right),$$

and the infimum $\inf_{1\le t\le s} J\left(X_t\right)$ has an uncertainty distribution

$$\Psi_s(x) = 1 - \inf_{1\le t\le s} \Phi_t\left(J^{-1}(x)\right).$$

Proof. The proof is similar to that of Theorem 2. □

4 | NUMERICAL ALGORITHM FOR SOLVING THE NOLINEAR CAPUTO-HADAMARD UFDE

When we encounter some linear ordinary differential equations, it is easy to obtain their analytic solutions. However, the practical problems that we encounter are usually nonlinear, and the method of getting the linear differential equations analytic solution is no longer applicable. Yao and Chen[6] designed a numerical method for solving uncertain differential equations, which obtained an IUD for solution of uncertain differential equations by solving each α-path. Jin et al.[31] proposed the numerical algorithm based on the general predictor-corrector method to solve the IUD for solution of Caputo UFDE. However, the proposed numerical algorithm is only suitable for solving the corresponding differential equation. Therefore, we need to design a new algorithm to deal with the problems of the nonlinear Caputo-Hadamard UFDEs.

On the basis of the modified predictor-corrector method[32], we develop a new numerical algorithm for solving the IUD of extreme values of solution to Caputo-Hadamard UFDEs. According to Theorem 1 and Theorem 3, the numerical algorithm for solving the IUD of extreme values of solution to Caputo-Hadamard UFDE are provided by Algorithm 1.

Algorithm 1 (IUD of the supremum or infimum)

Step 1. Fixed a positive integer number N which is based on the approximation need, and divide the interval $[1, T]$ into N parts. Let $h = (T-1)/N$ be a step length.
Step 2. Set $\alpha = 0$, and the step length $\Delta\alpha$.
Step 3. Update $\alpha \leftarrow \alpha + \Delta\alpha$. Let $L = J(X_1^\alpha)$ and $i = 1$.
Step 4. Based on the modified predictor-corrector method[32] at the grid $t_i = 1 + ih$, solve the corresponding FDE with initial conditions

$$\begin{cases} {}^{CH}\mathcal{D}^p X_t^\alpha = F\left(t, X_t^\alpha\right) + \left|G\left(t, X_t^\alpha\right)\right| \Phi^{-1}(\alpha), \quad t \ge 1, \\ \delta^k X_t^\alpha\big|_{t=1} = x_k, \quad k = 0, 1, \cdots, n-1, \end{cases}$$

and calculate $J\left(X_{t_i}^\alpha\right)$.
Step 5. Update $L \leftarrow \max\left(L, J\left(X_{t_i}^\alpha\right)\right)$ or $\min\left(L, J\left(X_{t_i}^\alpha\right)\right)$, and $i \leftarrow i + 1$.
Step 6. If $i \le N$, go to back Step 4.
Step 7. Obtain $\sup\limits_{1\le t\le T} J\left(X_t^\alpha\right) = L$ or $\inf\limits_{1\le t\le T} J\left(X_t^\alpha\right) = L$.
Step 8. If $\alpha + \Delta\alpha < 1$, go to back Step 3.
Step 9. Obtain the results. Thus, we can get the IUD $\Psi_T^{-1}(\alpha) = \sup\limits_{1\le t\le T} J\left(X_t^\alpha\right)$ or $\inf\limits_{1\le t\le T} J\left(X_t^\alpha\right)$.

Subsequently, the Caputo-Hadamard UFDEs with analytic solution is considered, and the IUD for extreme values of solution is calculated by Algorithm 1. The absolute error is presented to illustrate the accuracy of the proposed Algorithm 1.

Example 4.1. The considered Caputo-Hadamard UFDE with initial conditions is as follows:

$$\begin{cases} {}^{CH}\mathcal{D}^p X_t = eX_t + b\,(\log t)^v \frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad 1 \le t \le T, \\ \delta^k X_t\big|_{t=1} = x_k, \quad k = 0, 1, \ldots n-1, \end{cases} \tag{6}$$

where the Caputo-Hadamard UFDE (6) has an α-path X_t^α by

$$X_t^\alpha = \sum_{k=0}^{n-1} x_k (\log t)^k E_{p,(k+1)}\left(e(\log t)^p\right) + |b|\,\Gamma(v+1)(\log t)^{p+v} E_{p,p+v+1}\left(e\,(\log t)^p\right)\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Let $J(x) = x$. According to Theorem 1, the supremum $\sup_{1\le t\le T} J\left(X_t\right)$ has an IUD

$$\begin{aligned}\Psi_T^{-1}(\alpha) &= \sup_{1\le t\le T} X_t^\alpha \\ &= \sup_{1\le t\le T}\left(\sum_{k=0}^{n-1} x_k (\log t)^k E_{p,(k+1)}\left(e(\log t)^p\right)\right. \\ &\quad \left. + |b|\,\Gamma(v+1)(\log t)^{p+v} E_{p,p+v+1}\left(e\,(\log t)^p\right)\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).\end{aligned} \tag{7}$$

Set the parameters as follows: $p = 1.8$, $e = 0.6$, $b = 0.7$, $v = 1$, $T = 2$, $N = 100$, $\Delta\alpha = 0.01$, $x_0 = 2$, $x_1 = 0.1$. Numerical solution by Algorithm 1 and analytic solution by (7) of the IUD of $\sup_{1\le t\le T} J\left(X_t\right)$ are calculated and shown in Figure 1. To demonstrate the accuracy of Algorithm 1, the absolute errors between numerical solutions and analytic solutions are shown in Figure 2. The maximum absolute error that can be obtained from Figure 2 is less than 4.5×10^{-3}. Therefore, the proposed Algorithm 1 has high accuracy.

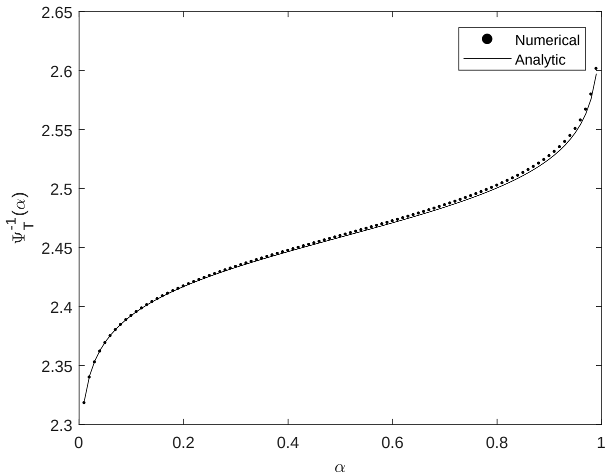

Figure 1 Numerical and analytic solutions for IUD of $\sup\limits_{1\le t\le T} J\left(X_t\right)$.

The stability of Algorithm 1 can be demonstrated by calculating the maximum absolute errors of analytic and numerical solutions of supremum $\sup_{1\le t\le T} J\left(X_t\right)$ under different fractional order p. The remaining parameters are set to $e = 0.6$, $b = 0.7$, $v = 1$, $T = 2$, $N = 100$, $\Delta\alpha = 0.01$, $x_0 = 2$, $x_1 = 0.1$. The results of the maximum absolute error are shown in Table 1. The maximum absolute error of analytic and numerical solutions of $\sup_{1\le t\le T} J\left(X_t\right)$ is 0.0066. Thus, the proposed Algorithm 1 can be shown to have high stability. Again according to Theorem 1, the infimum $\inf_{1\le t\le T} J\left(X_t\right)$ has an IUD

Table 1 The maximum absolute error for analytic and numerical solutions of $\sup_{1\le t\le T} J\left(X_t\right)$ with different order p.

p	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Error	0.0042	0.0061	0.0066	0.0064	0.0059	0.0053	0.0047	0.0041	0.0036	0.0032

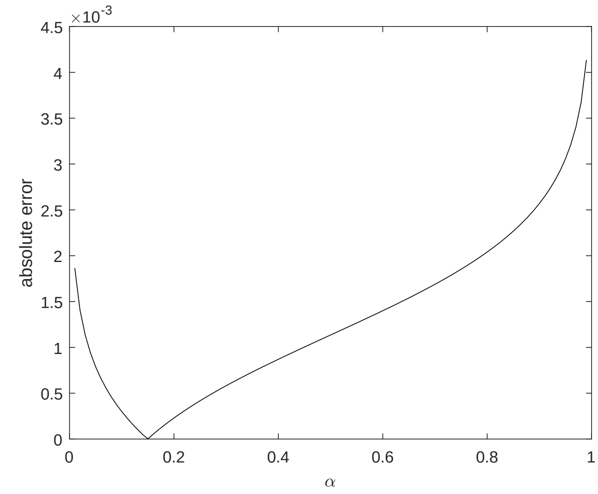

Figure 2 Absolute error between numerical and analytic solutions of $\sup_{1\leq t\leq T} J\left(X_t\right)$.

$$\begin{aligned}\Psi_T^{-1}(\alpha) &= \inf_{1\leq t\leq T} X_t^{\alpha} \\ &= \inf_{1\leq t\leq T}\left(\sum_{k=0}^{n-1} x_k(\log t)^k E_{p,(k+1)}\left(e(\log t)^p\right)\right. \\ &\quad \left. + |b|\,\Gamma(v+1)(\log t)^{p+v} E_{p,p+v+1}\left(e\,(\log t)^p\right)\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).\end{aligned} \tag{8}$$

Set the parameters as follows: $p = 0.9$, $e = -3.5$, $b = 0.7$, $v = 1$, $T = 2$, $N = 100$, $\Delta\alpha = 0.01$, $x_0 = 2$. Numerical solution by Algorithm 1 and analytic solution by (8) of the IUD of $\inf_{1\leq t\leq T} J\left(X_t\right)$ are calculated and shown in Figure 3. To demonstrate the accuracy of Algorithm 1, the absolute errors between numerical solutions and analytic solutions are shown in Figure 4. The maximum absolute error that can be obtained from Figure 4 is less than 3×10^{-3}. Therefore, the proposed Algorithm 1 has high accuracy.

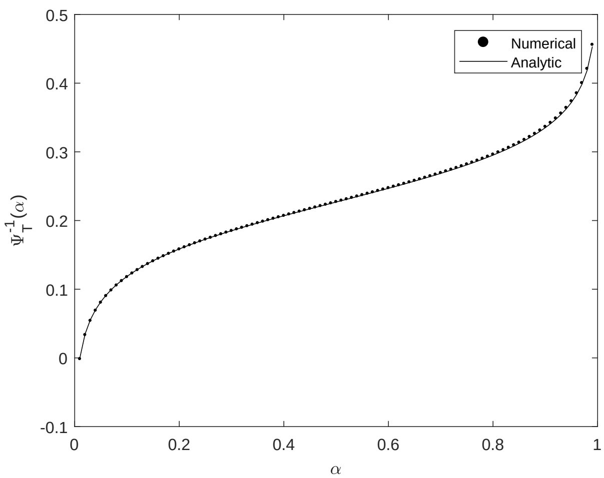

Figure 3 Numerical and analytic solutions for IUD of $\inf_{1\leq t\leq T} J\left(X_t\right)$.

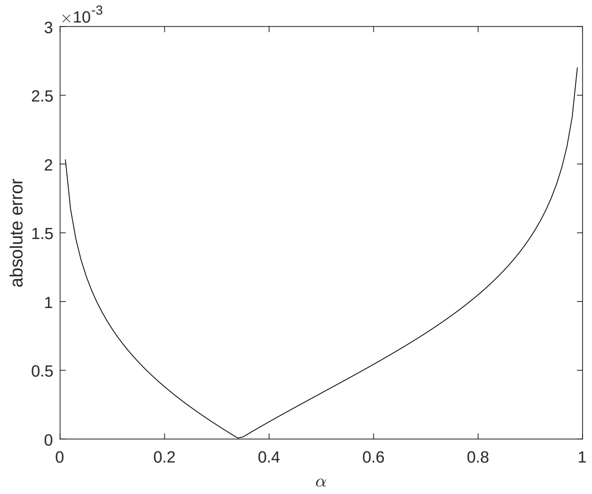

Figure 4 Absolute error between numerical and analytic solutions of $\inf\limits_{1\le t\le T} J\left(X_t\right)$.

The stability of Algorithm 1 can be demonstrated by calculating the maximum absolute errors of analytic and numerical solutions of infimum $\inf_{1\le t\le T} J\left(X_t\right)$ under different fractional order p. The remaining parameters are set to $e = -3.5$, $b = 0.7$, $\upsilon = 1$, $T = 2$, $N = 100$, $\Delta\alpha = 0.01$, $x_0 = 2$. The results of the maximum absolute error are shown in Table 2. The maximum absolute error of analytic and numerical solutions of $\inf_{1\le t\le T} J\left(X_t\right)$ is 0.0046. Thus, the proposed Algorithm 1 can be shown to have high stability.

Table 2 The maximum absolute error for analytic and numerical solutions of $\inf_{1\le t\le T} J\left(X_t\right)$ with different order p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Error	0.0024	0.0036	0.0043	0.0046	0.0045	0.0041	0.0037	0.0032	0.0027	0.0023

Since some differential equations have classical properties, these differential equations can be solved the analytic solution. However, not all differential equations can be solved the analytic solution. In this case, the numerical solution obtained by the numerical algorithm can be used to simulate the analytical solution of the differential equation.

Example 4.2. The considered Caputo-Hadamard UFDE with initial conditions is as follows:

$$\begin{cases} {}^{CH}\mathcal{D}^p X_t = \sqrt{X_t - 1} + (1-(\log t))\frac{\mathrm{d}C_t}{\mathrm{d}t}, & 1 \le t \le T, \\ \delta^k X_t\big|_{t=1} = x_k, & k = 0, 1, \dots n-1. \end{cases} \tag{9}$$

The α-path of (9) can be given by solving the following Caputo-Hadamard FDE

$$\begin{cases} {}^{CH}\mathcal{D}^p X_t = \sqrt{X_t - 1} + |1-(\log t)|\,\Phi^{-1}(\alpha), & 1 \le t \le T, \\ \delta^k X_t\big|_{t=1} = x_k, & k = 0, 1, \dots n-1. \end{cases}$$

Set the parameters as follows: $p = 1.2$, $T = 2$, $N = 100$, $\Delta\alpha = 0.01$, $x_0 = 2$, $x_1 = 1$ and $J(x) = x$. The numerical solutions for IUD of $\sup_{1\le t\le T} J\left(X_t\right)$ and $\inf_{1\le t\le T} J\left(X_t\right)$ are shown in Figure 5 and Figure 6, respectively. Therefore, if the extreme value of the solution of Caputo-Hadamard UFDE does not have an explicit IUD, the proposed Algorithm 1 can be used to obtain the corresponding IUD.

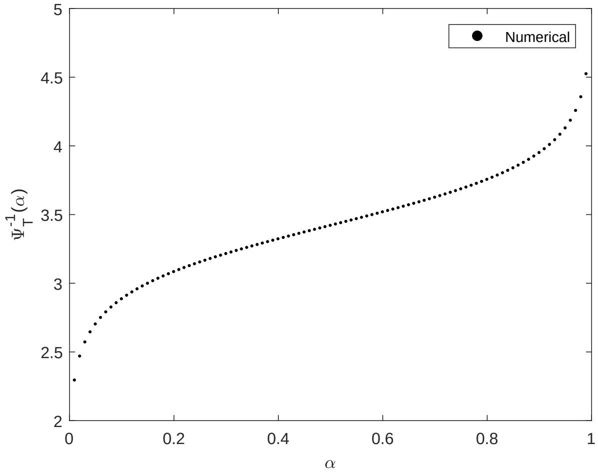

Figure 5 Numerical solutions for IUD of $\sup_{1\leq t\leq T} J\left(X_t\right)$.

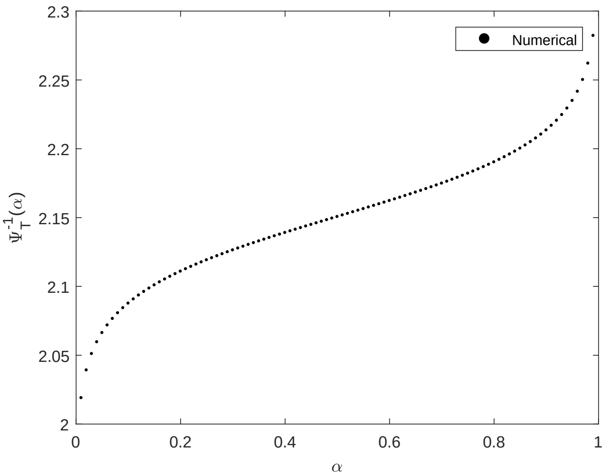

Figure 6 Numerical solutions for IUD of $\inf_{1\leq t\leq T} J\left(X_t\right)$.

5 | UNCERTAIN STOCK MODEL

This section present an uncertain stock model as an application of the extreme value theorems for solution of Caputo-Hadamard UFDEs. Subsequently, we price the American call option and American put option determined by the uncertain stock models in uncertain financial markets. Suppose that X_t and Y_t are the bond price and the stock price at time t, respectively. The stock price Y_t is simulated by the UFDE of Caputo-Hadamard type, an uncertain stock model to satisfy the following differential equation.

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t, \\ {}^{CH}\mathcal{D}^p Y_t = f\left(t, Y_t\right) + g\left(t, Y_t\right)\frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t \in [1, T], \\ \delta^k Y_t|_{t=1} = y_k, \quad k = 0, 1, \ldots n-1, \end{cases} \tag{10}$$

where r represents the riskless interest rate.

5.1 American call option

As a derivative of uncertain financial market, American option is a contract that can be exercised on any day within the validity period after the transaction. An American call option is a type of contract that endows the holder the right to buy the stock at strike price K before expiration time T. Let f_c denote the price of the American call option and Y_t is the stock price at time t. Taking into account the time value of money generated by the bond, the present value of the profit is $\sup_{1\le t\le T}\exp(-rt)\left(Y_t-K\right)^+$. Then the net return for the investor is $\left(-f_c+\sup_{1\le t\le T}\exp(-rt)\left(Y_t-K\right)^+\right)$, and the profit for the bank is opposite. For the buyer and seller to have the same return, then the price of the American call option can be defined as follows.

Definition 1. Assume that an American call option with the strike price K and expiration time T. Then the American call option price based on model (10) is defined by

$$f_c = E\left[\sup_{1\le t\le T}\exp(-rt)\left(Y_t-K\right)^+\right],$$

where Y_t is the stock price.

Theorem 5. The American call option price based on model (10) is

$$f_c = \int_0^1 \left[\sup_{1\le t\le T}\exp(-rt)\left(Y_t^\alpha-K\right)^+\right] \mathrm{d}\alpha. \tag{11}$$

Proof. The conclusion follows from the formula for the expected value and Theorem 1. □

The price of the American call option is mainly calculated based on the expected value of the supremum. When the function $J\left(X_t\right)$ increases strictly with respect to X_t, then we have

$$E\left[\sup_{1\le t\le T} J\left(X_t\right)\right] = \int_0^1 \sup_{1\le t\le T} J\left(X_t^\alpha\right) d\alpha.$$

Subsequently, this integral may be calculated by applying Simpson's rule. There are two improper points 0 and 1 in the above improper integral, we choose the small enough positive number ϵ such that $\int_0^1 \sup_{1\le t\le T} J\left(X_t^\alpha\right) d\alpha = \lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} \sup_{1\le t\le T} J\left(X_t^\alpha\right) d\alpha$. The specific algorithm is shown in Algorithm 2.

Algorithm 2 (Expected value of supremum for the increasing function $J\left(X_t\right)$.)

Step 1. Set $\epsilon > 0$ small enough. Fix a positive even number M. Set $\Delta\alpha = \frac{1}{M}$ and $i = 1$.
Step 2. Compute $\sup\limits_{1\le t\le T} J\left(X_t^\epsilon\right)$ and $\sup\limits_{1\le t\le T} J\left(X_t^{1-\epsilon}\right)$ by Algorithm 1.
Step 3. $\alpha_i = \Delta\alpha \cdot i$.
Step 4. Compute $\sup\limits_{1\le t\le T} J\left(X_t^{\alpha_i}\right)$ by Algorithm 1.
Step 5. If $i < M-1$, $i \leftarrow i+1$ and go to back Step 3.
Step 6. The expected value is provided by

$$E\left[\sup_{1\le t\le T} J\left(X_t\right)\right] = \frac{\Delta\alpha}{3}\left[\sup_{1\le t\le T} J\left(X_t^\epsilon\right) + 2\sum_{i=1}^{M-1}\sup_{1\le t\le T} J\left(X_t^{\alpha_i}\right) + 2\sum_{i=1}^{M/2}\sup_{1\le t\le T} J\left(X_t^{\alpha_{2i-1}}\right) + \sup_{1\le t\le T} J\left(X_t^{1-\epsilon}\right)\right].$$

Next, a special model is considered, let X_t and Y_t satisfy the following equations

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t, \\ {}^{CH}\mathcal{D}^p Y_t = \left(m - aY_t\right) + \sigma Y_t^l \frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t \in [1, T], \\ \delta^k Y_t|_{t=1} = y_k, \quad k = 0, 1, \dots n-1, \end{cases} \tag{12}$$

where r, m, a, l, σ are some positive constants.

When $l \neq 1$, we cannot get an explicit pricing formula of the American call option. Now, we consider the special case with $l = 1$.

Theorem 6. Assume that an American call option for the uncertain stock model (12) with $l = 1$ has the strike price K and expiration time T. Then we have

$$\begin{aligned} f_c = \int_0^1 \sup_{1\le t\le T} \exp(-rt) \Bigg[& \sum_{k=0}^{n-1} y_k (\log t)^k E_{p,(k+1)}\left(\left(-a + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)(\log t)^p\right) \\ & + m(\log t)^p E_{p,p+1}\left(\left(-a + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)(\log t)^p\right) - K \Bigg]^+ \mathrm{d}\alpha. \end{aligned} \tag{13}$$

Proof. The Caputo-Hadamard UFDE (12) has an α-path Y_t^α is

$$\begin{aligned} Y_t^\alpha = & \sum_{k=0}^{n-1} y_k (\log t)^k E_{p,(k+1)}\left(\left(-a + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)(\log t)^p\right) \\ & + m(\log t)^p E_{p,p+1}\left(\left(-a + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)(\log t)^p\right). \end{aligned}$$

Thus, the price of the American call option can be given by Theorem 5, and we have

$$\begin{aligned} f_c = \int_0^1 \sup_{1\le t\le T} \exp(-rt) \Bigg[& \sum_{k=0}^{n-1} y_k (\log t)^k E_{p,(k+1)}\left(\left(-a + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)(\log t)^p\right) \\ & + m(\log t)^p E_{p,p+1}\left(\left(-a + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)(\log t)^p\right) - K \Bigg]^+ d\alpha. \end{aligned}$$

The proof ends. □

Remark 1. The American call option price (13) is calculated with an integral which may be approximated by Simpson's rule.

Example 5.1. Suppose that there is a stock follows the uncertain stock model (12), which has the initial stock price $y_0 = 30$, the initial change rate $y_1 = 2$, and the riskless interest rate $r = 2.68\%$ per annum. Make the other parameters as follows: $m = 3.4$, $a = 0.06$, $\sigma = 0.26$, $K = 31$, $T = 2$. We choose different parameters $l = 0.5$ and $l = 1$, and denote the corresponding prices of the American call option as f_{1c} and f_{2c}, respectively. The price f_{1c} and f_{2c} with different fractional order p $(0 < p \le 2)$ can be effectively computed by Algorithm 2 and Eq. (13) in Theorem 6, respectively, as shown in Table 3.

Table 3 The price of the American call option with different order p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_{1c}	0.8386	0.8604	0.8365	0.7906	0.7334	0.6697	0.6016	0.5310	0.4595	0.3888
f_{2c}	3.6029	3.7156	3.6736	3.5422	3.3550	3.1323	2.8900	2.6385	2.3868	2.1406
p	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
f_{1c}	1.3260	1.2454	1.1670	1.0919	1.0203	0.9529	0.8900	0.8319	0.7786	0.7301
f_{2c}	2.6714	2.4400	2.2222	2.0192	1.8309	1.6574	1.4984	1.3537	1.2225	1.1044

It is easy to see from Table 3, the price f_{1c} and f_{2c} is slowly decreasing in the interval $p \in (0.2, 1]$ and $p \in (1, 2]$. In addition, the price f_{1c} and f_{2c} of the American call option has a small increase in the interval $p \in (0, 0.2]$, and the fluctuation of f_{1c} is smaller than that of f_{2c}. Due to the influence of the initial condition y_1, the price f_{1c} and f_{2c} of the American call option has a large surge when p increases from 1 to 1.1.

5.2 American put option

An American put option is considered a type of contract, which endows the holder the right to sell the stock at strike price K before expiration time T. Let f_p denote the price of the American put option and Y_t is the stock price at time t. Taking into account the time value of money generated by the bond, the present value of the profit is $\sup_{1\le t\le T} \exp(-rt)\left(K - Y_t\right)^+$. Then the net return of the investor is $\left(-f_p + \sup_{1\le t\le T} \exp(-rt)\left(K - Y_t\right)^+\right)$, and the profit of the bank is opposite. For the buyer and seller to have the same return, then the price of the American call option can be defined as follows.

Definition 2. Assume that an American put option with the strike price K and expiration time T. Then the American put option price based on model (10) is defined by

$$f_p = E\left[\sup_{1\le t\le T} \exp(-rt)\left(K - Y_t\right)^+\right],$$

where Y_t is the stock price.

Theorem 7. The American put option price based on model (10) is

$$f_p = \int_0^1 \left[\sup_{1\le t\le T} \exp(-rt)\left(K - Y_t^{1-\alpha}\right)^+\right] \mathrm{d}\alpha. \tag{14}$$

Proof. The conclusion follows from the formula for the expected value and Theorem 1. □

The price of the American put option is mainly calculated based on the expected value of the supremum. When the function $J\left(X_t\right)$ decreases strictly with respect to X_t, then we have

$$E\left[\sup_{1\le t\le T} J\left(X_t\right)\right] = \int_0^1 \sup_{1\le t\le T} J\left(X_t^{1-\alpha}\right) \mathrm{d}\alpha.$$

Subsequently, this integral is calculated by applying Simpson's rule. There are two improper points 0 and 1 in the above improper integral, we choose the small enough positive number ϵ such that $\int_0^1 \sup_{1\le t\le T} J\left(X_t^{1-\alpha}\right) d\alpha = \lim_{\epsilon\to 0} \int_\epsilon^{1-\epsilon} \sup_{1\le t\le T} J\left(X_t^{1-\alpha}\right) \mathrm{d}\alpha$. The specific algorithm is shown in Algorithm 3.

Algorithm 3 (Expected value of the supremum for decreasing function $J\left(X_t\right)$.)

Step 1. Set $\epsilon > 0$ small enough. Fix a positive even number M. Set $\Delta\alpha = \frac{1}{M}$ and $i = 1$.
Step 2. Compute $\sup_{1\le t\le T} J\left(X_t^{\epsilon}\right)$ and $\sup_{1\le t\le T} J\left(X_t^{1-\epsilon}\right)$ by Algorithm 1.
Step 3. $\alpha_i = \Delta\alpha \cdot i$.
Step 4. Compute $\sup_{1\le t\le T} J\left(X_t^{\alpha_i}\right)$ by Algorithm 1.
Step 5. If $i < M - 1$, $i \leftarrow i + 1$ and go to back Step 3.
Step 6. The expected value is provided by

$$\begin{aligned} E\left[\sup_{1\le t\le T} J\left(X_t\right)\right] = & \frac{\Delta\alpha}{3}\left[\sup_{1\le t\le T} J\left(X_t^{1-\epsilon}\right) + 2\sum_{i=1}^{M-1} \sup_{1\le t\le T} J\left(X_t^{1-\alpha_i}\right)\right. \\ & \left. + 2\sum_{i=1}^{M/2} \sup_{1\le t\le T} J\left(X_t^{1-\alpha_{2i-1}}\right) + \sup_{1\le t\le T} J\left(X_t^{\epsilon}\right)\right]. \end{aligned}$$

Theorem 8. Assume that an American put option for the uncertain stock model (12) with $l = 1$ has the strike price K and expiration time T. Then we have

$$f_{\overline{p}}=\int_0^1 \sup_{1\le t\le T} \exp(-rt) \left[K - \sum_{k=0}^{n-1} y_k(\log t)^k E_{p,(k+1)}\left(\left(\left(-a+|\sigma|\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)(\log t)^p\right)\right.\right.$$
$$\left.\left. -m(\log t)^p E_{p,p+1}\left(\left(-a+|\sigma|\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)(\log t)^p\right)\right]^+ \mathrm{d}\alpha. \tag{15}$$

Proof. The proof is similar to that of Theorem 6. □

Remark 2. The American put option price (15) is calculated with an integral which may be approximated by Simpson's rule.

Example 5.2. Suppose that there is a stock follows the uncertain stock model (12), which has the initial stock price $y_0 = 30$, the initial change rate $y_1 = -2$, and the riskless interest rate $r = 2.68\%$ per annum. Make the other parameters as follows: $m = 3.4$, $a = 0.06$, $\sigma = 0.26$, $K = 31$, $T = 2$. We choose different parameter $l = 0.5$ and $l = 1$, and denote the corresponding prices of the American put option as f_{1p} and f_{2p}, respectively. The price f_{1p} and f_{2p} with different fractional order p ($0 < p \le 2$) can be effectively computed by Algorithm 3 and Eq. (15) in Theorem 8, respectively, as shown in Table 4.

Table 4 The price of the American put option with different order p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_{1p}	0.3220	0.3423	0.3517	0.3537	0.3515	0.3476	0.3437	0.3410	0.3405	0.3427
f_{2p}	2.2910	2.6692	2.7706	2.7489	2.6670	2.5540	2.4249	2.2883	2.1496	2.0131
p	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
f_{1p}	1.1081	1.2032	1.2966	1.3870	1.4735	1.5552	1.6317	1.7028	1.7682	1.8280
f_{2p}	2.5995	2.5660	2.5352	2.5070	2.4813	2.4580	2.4396	2.4177	2.4004	2.3849

It is easy to see from Table 4, the price f_{1p} is slowly increasing in the interval $p \in (0, 0.4]$ and $p \in (1, 2]$, and it grows faster in the interval $p \in (1, 2]$ than in the interval $p \in (0, 0.4]$. Furthermore, the price f_{1p} decreases smoothly in the interval $p \in (0.4, 1]$. The price f_{2p} of the American put option is slowly decreasing in the interval $p \in (0.3, 1]$ and $p \in (1, 2]$, and it fall faster in the interval $p \in (0.3, 1]$ than in the interval $p \in (1, 2]$. In addition, the price f_{2p} has a small increase in the interval $p \in (0, 0.3]$. Due to the influence of the initial condition y_1, the price f_{1p} and f_{2p} of the American put option has a large surge when p increases from 1 to 1.1.

6 | CONCLUSION

For strictly increasing and strictly decreasing functions, the IUDs of the extreme values (supremum and infimum) of solutions to Caputo-Hadamard UFDEs are given, respectively. Considering that the methods for solving the analytic solution of the linear Caputo-Hadamard UFDEs are no longer applicable for the nonlinear problems, we present the numerical algorithm 1 to obtain the numerical solutions of the nonlinear Caputo-Hadamard UFDEs. Through the verification of numerical experiments, the numerical algorithm 1 has high accuracy and effectiveness. Considering that the calculation of the American option pricing is mainly based on computing the expected value of the supremum, the algorithms 2 and 3 for calculating the expected value of extreme value of monotonic function through the Simpson's rule are designed, respectively. Subsequently, the new uncertain stock model is presented based on the Caputo-Hadamard UFDE, and the pricing formulas of the American option are given. Finally, the price fluctuation of American option with different order p is explained by numerical experiments. For the next work, we will focus on investigate the option pricing formula based on other appropriate models.

Author contributions

Hanjie Liu: Writing-original draft, Validation. Yuanguo Zhu: Methodology, Supervision. Liu He: Resources.

Financial disclosure

None reported.

Conflict of interest

The authors declare no potential conflict of interests.

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