Several novel inequalities associated with the Riesz-type fractional integral operator

Lu-Lu Geng[1] and Xiao-Jun Yang[1]

[1]China University of Mining and Technology School of Mathematics

February 7, 2023

Abstract

In this paper, we investigate some new inequalities based on the Riesz-type fractional integral operator for synchronous and bounded functions.

Several novel inequalities associated with the Riesz-type fractional integral operator

Lu-Lu Geng[1,2], Xiao-Jun Yang[1,2,3,4†]

1. School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, P. R. China.

2. State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, P. R. China.

3. Department of Mathematics Faculty of Science King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia.

4. Department of Mathematics, College of Science, Kyung Hee University, Seoul, Republic of Korea.

Abstract:

In this paper, we investigate some new inequalities based on the Riesz-type fractional integral operator for synchronous and bounded functions.

Key words: Riesz-type fractional integral; Chebychev functional; synchronous function; bounded function; Young's inequality.

Mathematics Subject Classification: 26A33; 26D10; 47B06.

1. Introduction

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of the natural, real, and complex numbers, respectively.

For $0 < \alpha < 1$, the Riesz-type fractional integral on a bounded domian is defined as (see [1], p.130)

$$_{Rz}I^{\alpha}_{[a,b]}f(t) = \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}\int_a^b \frac{f(\tau)}{|\tau - t|^{1-\alpha}}d\tau, \tag{1}$$

where

$$_{L}I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(\tau - t)^{1-\alpha}}d\tau, \tag{2}$$

and

$$_{L}I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(t - \tau)^{1-\alpha}}d\tau. \tag{3}$$

†Corresponding author E-mail: gengllcumt@163.com

The Chebychev's functional is expressed as [2–7]

$$\begin{aligned} T(f,g,j,k) = & \int_a^b k(x)dx \int_a^b j(x)f(x)g(x)dx \\ & + \int_a^b j(x)dx \int_a^b k(x)f(x)g(x)dx \\ & - [\int_a^b k(x)f(x)dx][\int_a^b j(x)g(x)dx] \\ & - [\int_a^b j(x)f(x)dx][\int_a^b k(x)g(x)dx], \end{aligned} \tag{4}$$

where $f, g : [a, b] \to \mathbb{R}$ are two integrable functions on $[a, b]$ and j, k are positive integrable functions on $[a, b]$.

It is well-known that (4) has wide applicability in numerical quadrature [8], transform theory [9], probability and statistical problems [10], and the bounding of special functions [11].

For $x, y \in [a, b]$, if f and g are synchronous on $[a, b]$, then [12, 13]

$$[f(x) - f(y)][g(x) - g(y)] \geq 0, \tag{5}$$

and

$$T(f, g, j, k) \geq 0. \tag{6}$$

What's more, if f and g are asynchronous on $[a, b]$, then [14]

$$[f(x) - f(y)][g(x) - g(y)] \leq 0. \tag{7}$$

The aim of this paper is to study of some inequalities by means of the Riesz-type fractional integral operator.

The structure of this paper is as follows. In section 2, based on the Riesz-type fractional integral operator, we introduce some inequalities about synchronous functions. In section 3, by the Riesz-type fractional integral operator, we present some inequalities about bounded functions.

2. Inequalities involving Riesz-type fractional integral operator for synchronous functions

In this section, we study some inequalities via the Riesz-type fractional integral operator for synchronous functions.

Let f and g be two continuous and synchronous functions on $[a, b]$, u, v, l, m and n be continuous functions on $[a, b]$. Then we have some theorems as follows:

Theorem 1. *If $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, then we have*

$$\begin{aligned} & {}_{Rz}I^{\alpha}_{[a,b]}\{v\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fgu\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{fgv\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{u\}(t) \\ & \geq {}_{Rz}I^{\alpha}_{[a,b]}\{gv\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fu\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{fv\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{gu\}(t). \end{aligned} \tag{8}$$

Proof. Based on (5), we have

$$[f(\tau) - f(\rho)][g(\tau) - g(\rho)] \geq 0, \tag{9}$$

where $\tau, \rho \in [a, b]$.

Multiplying (9) by

$$\frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{u(\tau)}{|\tau - t|^{1-\alpha}}, \tag{10}$$

we have

$$\begin{aligned} & \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} f(\tau)g(\tau)u(\tau) + f(\rho)g(\rho) \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} u(\tau) \\ & \geq g(\rho) \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} f(\tau)u(\tau) + f(\rho) \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} g(\tau)u(\tau). \end{aligned} \tag{11}$$

Integrating (11), we have

$$\begin{aligned} & {}_{Rz}I^{\alpha}_{[a,b]}\{fgu\}(t) + f(\rho)g(\rho)\,{}_{Rz}I^{\alpha}_{[a,b]}\{u\}(t) \\ & \geq g(\rho)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fu\}(t) + f(\rho)\,{}_{Rz}I^{\alpha}_{[a,b]}\{gu\}(t). \end{aligned} \tag{12}$$

Then, multiplying (12) by

$$\frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{v(\tau)}{|\tau - t|^{1-\alpha}}, \tag{13}$$

we can see

$$\begin{aligned} & \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} v(\rho)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fgu\}(t) \\ & + \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} f(\rho)g(\rho)v(\rho)\,{}_{Rz}I^{\alpha}_{[a,b]}\{u\}(t) \\ & \geq \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} g(\rho)v(\rho)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fu\}(t) \\ & + \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} f(\rho)v(\rho)\,{}_{Rz}I^{\alpha}_{[a,b]}\{gu\}(t). \end{aligned} \tag{14}$$

Integrating (14), we obtain

$$\begin{aligned} & {}_{Rz}I^{\alpha}_{[a,b]}\{v\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fgu\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{fgv\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{u\}(t) \\ & \geq {}_{Rz}I^{\alpha}_{[a,b]}\{gv\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fu\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{fv\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{gu\}(t). \end{aligned} \tag{15}$$

Therefore, (8) is true. □

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, $\rho \in [a,t]$, and $t \in [a,b]$. Then*

$$\begin{aligned}
&{}_{Rz}I_{[a,b]}^{\beta}\{v\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fgu\}(t) + {}_{Rz}I_{[a,b]}^{\beta}\{fgv\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{u\}(t) \\
&\geq {}_{Rz}I_{[a,b]}^{\beta}\{gv\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fu\}(t) + {}_{Rz}I_{[a,b]}^{\beta}\{fv\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{gu\}(t).
\end{aligned} \tag{16}$$

Proof. Multiplying (12) by

$$\frac{1}{2\Gamma(\beta)\cos(\frac{\pi\beta}{2})}\frac{v(\rho)}{|\rho - t|^{1-\beta}}, \tag{17}$$

we have

$$\begin{aligned}
&\frac{1}{2\Gamma(\beta)\cos(\frac{\pi\beta}{2})}\frac{1}{|\rho - t|^{1-\beta}}v(\rho)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fgu\}(t) \\
&\quad + \frac{1}{2\Gamma(\beta)\cos(\frac{\pi\beta}{2})}\frac{1}{|\rho - t|^{1-\beta}}f(\rho)g(\rho)v(\rho)\,{}_{Rz}I_{[a,b]}^{\alpha}\{u\}(t) \\
&\geq \frac{1}{2\Gamma(\beta)\cos(\frac{\pi\beta}{2})}\frac{1}{|\rho - t|^{1-\beta}}g(\rho)v(\rho)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fu\}(t) \\
&\quad + \frac{1}{2\Gamma(\beta)\cos(\frac{\pi\beta}{2})}\frac{1}{|\rho - t|^{1-\beta}}f(\rho)v(\rho)\,{}_{Rz}I_{[a,b]}^{\alpha}\{gu\}(t).
\end{aligned} \tag{18}$$

Integrating (18), we get

$$\begin{aligned}
&{}_{Rz}I_{[a,b]}^{\beta}\{v\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fgu\}(t) + {}_{Rz}I_{[a,b]}^{\beta}\{fgv\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{u\}(t) \\
&\geq {}_{Rz}I_{[a,b]}^{\beta}\{gv\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fu\}(t) + {}_{Rz}I_{[a,b]}^{\beta}\{fv\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{gu\}(t).
\end{aligned} \tag{19}$$

Therefore, the proof of (16) can be completed. □

Theorem 3. *Suppose $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, and $t \in [a,b]$. Then*

$$\begin{aligned}
&{}_{Rz}I_{[a,b]}^{\alpha}\{l\}(t)[{}_{Rz}I_{[a,b]}^{\alpha}\{fgn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{m\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{n\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fgm\}(t)] \\
&\quad + 2\,{}_{Rz}I_{[a,b]}^{\alpha}\{m\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{n\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fgl\}(t) \\
&\geq {}_{Rz}I_{[a,b]}^{\alpha}\{l\}(t)[{}_{Rz}I_{[a,b]}^{\alpha}\{gn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fm\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{fn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{gm\}(t)] \\
&\quad + {}_{Rz}I_{[a,b]}^{\alpha}\{m\}(t)[{}_{Rz}I_{[a,b]}^{\alpha}\{gn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fl\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{fn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{gl\}(t)] \\
&\quad + {}_{Rz}I_{[a,b]}^{\alpha}\{n\}(t)[{}_{Rz}I_{[a,b]}^{\alpha}\{gm\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fl\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{fm\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{gl\}(t)],
\end{aligned} \tag{20}$$

is true.

Proof. For (8), consider $u = m$ and $v = n$, we have

$$\begin{aligned}
&{}_{Rz}I_{[a,b]}^{\alpha}\{n\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fgm\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{fgn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{m\}(t) \\
&\geq {}_{Rz}I_{[a,b]}^{\alpha}\{gn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{fm\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{fn\}(t)\,{}_{Rz}I_{[a,b]}^{\alpha}\{gm\}(t).
\end{aligned} \tag{21}$$

According to ${}_{Rz}I_R^\alpha\{l\}(t) \geq 0$, and multiplying (21) by ${}_{Rz}I_R^\alpha\{l\}(t) \geq 0$, we have

$$\begin{aligned}&{}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t)]\\ &\geq\ {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fm\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gm\}(t)].\end{aligned}\tag{22}$$

What's more, for (8), we try to use l, n instead of u, v and l, m instead of u, v, then multiplying by ${}_{Rz}I_{[a,b]}^\alpha\{m\}(t)$ and ${}_{Rz}I_{[a,b]}^\alpha\{n\}(t)$, we have

$$\begin{aligned}&{}_{Rz}I_{[a,b]}^\alpha\{m\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{l\}(t)]\\ &\geq\ {}_{Rz}I_{[a,b]}^\alpha\{m\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fl\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gl\}(t)],\end{aligned}\tag{23}$$

and

$$\begin{aligned}&{}_{Rz}I_{[a,b]}^\alpha\{n\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{m\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t){}_{Rz}I_{[a,b]}^\alpha\{l\}(t)]\\ &\geq\ {}_{Rz}I_{[a,b]}^\alpha\{n\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{gm\}(t){}_{Rz}I_{[a,b]}^\alpha\{fl\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{fm\}(t){}_{Rz}I_{[a,b]}^\alpha\{gl\}(t)].\end{aligned}\tag{24}$$

Thus, we can obtain (20). □

Theorem 4. *If $\alpha, \beta \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, and $t \in [a, b]$, then*

$$\begin{aligned}&{}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[2{}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\beta\{fgm\}(t)]\\ &\quad+\ {}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t)]+\ {}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{m\}(t){}_{Rz}I_{[a,b]}^\beta\{n\}(t)\\ &\quad+\ {}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\beta\{m\}(t)]\\ &\geq\ {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{mf\}(t){}_{Rz}I_{[a,b]}^\beta\{ng\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{mg\}(t){}_{Rz}I_{[a,b]}^\beta\{nf\}(t)]\\ &\quad+\ {}_{Rz}I_{[a,b]}^\alpha\{m\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{lf\}(t){}_{Rz}I_{[a,b]}^\beta\{ng\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{lg\}(t){}_{Rz}I_{[a,b]}^\beta\{nf\}(t)]\\ &\quad+\ {}_{Rz}I_{[a,b]}^\alpha\{n\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{lf\}(t){}_{Rz}I_{[a,b]}^\beta\{mg\}(t)+\ {}_{Rz}I_{[a,b]}^\alpha\{lg\}(t){}_{Rz}I_{[a,b]}^\beta\{mf\}(t)].\end{aligned}\tag{25}$$

Proof. For (16), setting $u = m$ and $v = n$, we have

$$\begin{aligned}&{}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t)+\ {}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t)\\ &\geq\ {}_{Rz}I_{[a,b]}^\beta\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fm\}(t)+\ {}_{Rz}I_{[a,b]}^\beta\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gm\}(t).\end{aligned}\tag{26}$$

Multiplying (26) by ${}_{Rz}I_R^\alpha\{l\}(t)$, we receive

$$\begin{aligned}&{}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t)+\ {}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t)]\\ &\geq\ {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\beta\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fm\}(t)+\ {}_{Rz}I_{[a,b]}^\beta\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gm\}(t)].\end{aligned}\tag{27}$$

Replacing u, v by l, n and u, v by l, m in (16), we gain

$$\begin{aligned}&{}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t)+\ {}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{l\}(t)\\ &\geq\ {}_{Rz}I_{[a,b]}^\beta\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fl\}(t)+\ {}_{Rz}I_{[a,b]}^\beta\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gl\}(t),\end{aligned}\tag{28}$$

and

$$
\begin{aligned}
&{}_{Rz}I^{\beta}_{[a,b]}\{m\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fgl\}(t) + {}_{Rz}I^{\beta}_{[a,b]}\{fgm\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{l\}(t) \\
&\geq {}_{Rz}I^{\beta}_{[a,b]}\{gm\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{fl\}(t) + {}_{Rz}I^{\beta}_{[a,b]}\{fm\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{gl\}(t).
\end{aligned} \tag{29}
$$

Then multiplying (28) and (29) by ${}_{Rz}I^{\alpha}_{[a,b]}\{m\}(t)$ and ${}_{Rz}I^{\alpha}_{[a,b]}\{n\}(t)$, we can get (25). □

3. Inequalities involving Riesz-type fractional integral operator for bounded functions

In this section, we investigate some inequalities based on the Riesz-type fractional integral operator for bounded functions.

Theorem 5. *Let $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}^+_0$, $t \in [a,b]$, f be an integrable function on $[a,b]$, u, v be continuous function on $[a,b]$, $\phi_1, \phi_2 \in [a,b]$, and $\phi_1(t) \leq f(t) \leq \phi_2(t)$. Then*

$$
\begin{aligned}
&{}_{Rz}I^{\alpha}_{[a,b]}\{u\phi_2\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{vf\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{uf\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{v\phi_1\}(t) \\
&\geq {}_{Rz}I^{\alpha}_{[a,b]}\{u\phi_2\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{v\phi_1\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{uf\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{vf\}(t).
\end{aligned} \tag{30}
$$

Proof. If $\tau \geq [a,b]$ and $\rho \geq [a,b]$, then we have

$$
[\phi_2(\tau) - f(\tau)][f(\rho) - \phi_1(\rho)] \geq 0. \tag{31}
$$

Multiplying (31) by

$$
\frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{u(\tau)}{|\tau - t|^{1-\alpha}}, \tag{32}
$$

and integrating, we have

$$
\begin{aligned}
&{}_{Rz}I^{\beta}_{[a,b]}\{u\phi_2\}(t) f(\rho) + {}_{Rz}I^{\beta}_{[a,b]}\{uf\}(t)\phi_1(\rho) \\
&\geq {}_{Rz}I^{\beta}_{[a,b]}\{u\phi_2\}(t)\phi_1(\rho) + {}_{Rz}I^{\beta}_{[a,b]}\{uf\}(t) f(\rho).
\end{aligned} \tag{33}
$$

Multiplying (33) by

$$
\frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \frac{v(\rho)}{|\rho - t|^{1-\alpha}}, \tag{34}
$$

and integrating, we can see that (30) is true. □

Theorem 6. *Let $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}^+_0$, $\tau, \rho \geq 0$, f be an integrable function on $[a,b]$, $u, v : [a,b] \to [a,b]$ be continuous function, $\lambda_1, \lambda_2 > 0$, and $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$. Then*

$$
\begin{aligned}
&\frac{1}{\lambda_1}\,{}_{Rz}I^{\alpha}_{[a,b]}\{v\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{u(\phi_2 - f)^{\lambda_1}\}(t) + \frac{1}{\lambda_2}\,{}_{Rz}I^{\alpha}_{[a,b]}\{u\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{v(f - \phi_1)^{\lambda_2}\}(t) \\
&+ {}_{Rz}I^{\alpha}_{[a,b]}\{u\phi_2\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{v\phi_1\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{uf\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{vf\}(t) \\
&\geq {}_{Rz}I^{\alpha}_{[a,b]}\{u\phi_2\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{vf\}(t) + {}_{Rz}I^{\alpha}_{[a,b]}\{uf\}(t)\,{}_{Rz}I^{\alpha}_{[a,b]}\{v\phi_1\}(t).
\end{aligned} \tag{35}
$$

Proof. Based on the well-known Young's inequality [15]

$$\frac{1}{\lambda_1}x^{\lambda_1} + \frac{1}{\lambda_2}y^{\lambda_2} \geq xy, \quad (x, y \geq 0), \tag{36}$$

Setting $x = \phi_2(\tau) - f(\tau)$ and $y = f(\rho) - \phi_1(\rho)$, we have

$$\begin{aligned} &\frac{1}{\lambda_1}[\phi_2(\tau) - f(\tau)]^{\lambda_1} + \frac{1}{\lambda_2}[f(\rho) - \phi_1(\rho)]^{\lambda_2} \\ &\geq [\phi_2(\tau) - f(\tau)][f(\rho) - \phi_1(\rho)]. \end{aligned} \tag{37}$$

Multiplying (37) by

$$\frac{1}{2\Gamma^2(\alpha)\cos^2(\frac{\pi\alpha}{2})}\frac{u(\tau)v(\rho)}{|\tau - t|^{2(1-\alpha)}}, \tag{38}$$

and integrating, we can acquire (35). □

ACKNOWLEDGMENTS

This work is supported by the Fundamental Research Funds for the Central Universities (No.2022XSCX08).

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] Yang XJ, Gao F, Yang J. General fractional derivatives with applications in viscoelasticity. Academic Press; 2020.

[2] Cerone P. On an identity for the Chebychev functional and some ramifications. J Ineq Pure and Appl Math 2002; 3(1).

[3] Dahmani Z, Mechouar O, Brahami v. Certain inequalities related to the Chebyshev's functional involving a Riemann-Liouville operator. Bull Math Anal Appl 2011; 3(4):38-44.

[4] Chebyshev PL. Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. Proc Math Soc Charkov 1882;2:93-98.

[5] Anastassiou GA. Advances on fractional inequalities. Springer Science and Business Media; 2011.

[6] Belarbi S, Dahmani Z. On some new fractional integral inequalities. J Inequal Pure Appl Math 2009;10(3):1-12.

[7] Sulaiman WT. Some new fractional integral inequalities. J Math Anal 2011;2(2):23-28.

[8] Tatevyan SK, Sorokin NA, Zaljotkin SF. Numerical integration of ordinary differential equations with the use of Chebyshev's series. Numer Meth Program 2002;3:52-81.

[9] Alotaibi AM, El-Moneam MA, Badr BS. The Solutions of Legendre's and Chebyshev's Differential Equations by Using the Differential Transform Method Math Probl Engm 2022;2022.

[10] Smith JE. Generalized Chebychev inequalities: theory and applications in decision analysis. Oper Reso 1995;43(5):807-825.

[11] Zou AM, Kumar KD, Hou ZG. Quaternion-based adaptive output feedback attitude control of spacecraft using Chebyshev neural networks. IEEE T Neural Networ 2010;21(9):1457-1471.

[12] Agarwal P. Some inequalities involving Hadamard-type k-fractional integral operators. Math Method Appl Sci 2017;40(11):3882-3891.

[13] Kuang JC. Applied inequalities. Shangdong Science Technic Press, Jina; 2004.

[14] Hrastinski S. Asynchronous and synchronous e-learning. Educause quarterly 2008;31(4):51-55.

[15] Mitrinovic DS, Vasic PM. Analytic inequalities. Berlin: Springer-verlag; 1970.