A Sufficient Condition for Restoring Sparse Vectors from $\ell_1-\ell_2$-minimization with Cumulative Coherence

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February 2, 2023

Abstract

This paper focuses on the compressed sensing $\ell_1-\ell_2-$minimization model and develops new bounds on cumulative coherence $\mu_1(s)$. We point out that if cumulative coherence $\mu_1(s)$ satisfies (2) or (11), then the sparse signal can stably recover in noise model and exactly recover in free noise by $\ell_1-\ell_2$-minimization model.

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Abstract

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Keywords: $\ell_1 - \ell_2$-minimization, Cumulative Coherence, Sparse Signal, Compressed Sensing, Exactly Recover.

1 Introduction

Compression sensing is mainly used to recover high-dimensional sparse vectors from low dimensional vectors. Its mathematical model can be expressed as ℓ_0-minimization model:

$$\min_{x\in R^n} \|x\|_0 \qquad subject \quad to \qquad y - Ax \in B,$$

where $A \in R^{m\times n}(m \ll n)$, is the measurement matrix, $y \in R^m$ is the measurement vector, $x \in R^n$ is the sparse vector to be recovered, $\|x\|_0$ calculate the number of nonzero components in x, $B = \{0\}$ indicates a noiseless case, and $B = \{\epsilon\}$ indicates a noise case.

The ℓ_0-minimization model is NP-hard, and thus computationally not feasible in high-dimensional sets [2]. Fortunately, however, scholars found that the ℓ_1-minimization model can solve the ℓ_0-minimization model well when measurement matrix meets certain conditions [1, 2, 5].

Although the ℓ_1-minimization model yields considerable results [1, 2, 5], it is not exactly equivalent to the ℓ_0-minimization problem [4]. Hence, the $\ell_1 - \ell_2$-minimization model has been proposed to replace the ℓ_1-minimization model in the case where the ℓ_1-minimization model does not execute well. The $\ell_1 - \ell_2$-minimization model can be expressed as

$$\min_{x\in R^n} \|x\|_1 - \|x\|_2 \qquad subject\ to \qquad \|y - Ax\|_2 \le \epsilon \quad (1)$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$.

In the literature, such as the ℓ_1-minimization model, the $\ell_1 - \ell_2$-minimization model is also solved based on the null space property, coherence, restricted orthogonality constants [3], and restricted isometry property [3].

This paper continues to study the $\ell_1 - \ell_2$-minimization model. The main contribution of our study is that we used the cumulative coherence to solve the $\ell_1 - \ell_2$-minimization model. We point out that if $\mu_1(2s-1)$ combines with $\mu_1(s-1)$ satisfies (2) or cumulative coherence satisfies (11) then the $\ell_1 - \ell_2$-minimization model exactly recovers the s-sparse signal in the noiseless case and stable recovery in the noise case.

We introduce related concepts in Section I. In Section II, we present our main results and we conclude the paper in Section III.

Notations: For $x \in R^n$, $\|x\|_\infty = \max_{i\in[n]} |x_i|$, where $[n] = \{1, 2, 3, \cdots, n\}$. $s \in R$ and $x_{max(s)}$ is defined as the vector x with all but the largest s entries in absolute value set to zero, and $x_{-max(s)} = x - x_{max(s)}$. For $y \in R^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. $T \subset [n]$, x_T is defined as the vector $(x_T)_i = x_i$, if $i \in T$ and $(x_T)_i = 0$ otherwise.

2 Preliminary

Definition 1 (**[2]**) *Let $A \in R^{m\times n}$ be a matrix with ℓ_2-normalized columns $A_1, \ . \ . \ . \ , A_n$ (that is, $\|A_i\|_2 = 1$ for all $i = 1, ..., n$). The cumulative coherence function $\mu_1(s) = \mu_1(A, s)$ of matrix A is defined for $s \in [n-1]$ by*

$$\mu_1(s) = \max_{i\in[n]} max\{\sum_{j\in S} |\langle A_i, A_j\rangle|, S \subset [n], card(S) = s, i \notin S\}$$

When the cumulative coherence of a matrix grows slowly, we can informally say that the dictionary is quasi-incoherent.

The following lemmas are needed in the proof of our main results and we list them below.

Lemma 1 (**[2]**) *Let $A \in R^{m\times n}$ be a matrix with ℓ_2-normalized columns and $s \in [n]$. For all s-sparse vectors $x \in R^n$,*

$$(1 - \mu_1(s-1))\|x\|_2^2 \le \|Ax\|_2^2 \le (1 + \mu_1(s-1))\|x\|_2^2.$$

Lemma 2 (**[5]**) *Suppose that x is s-sparse and y is t-sparse; then,*

$$|\langle Ax, Ay\rangle - \langle x, y\rangle| \le \mu_1(s + t - 1)\|x\|_2\|y\|_2.$$

Moreover, if $supp(x) \cap supp(y) = \emptyset$, then

$$|\langle Ax, Ay\rangle| \le \mu_1(s + t - 1)\|x\|_2\|y\|_2.$$

Lemma 3 (**[1]**) *Let q and r be positive integers satisfying $r \le q \le 3r$. Descending chain of real numbers*

$$a_1 \ge a_2 \ge \cdots a_r \ge b_1 \ge \cdots \ge b_q \ge c_1 \cdots \ge c_r \ge 0$$

satisfies

$$\sqrt{\sum_{i=1}^q b_i^2 + \sum_{i=1}^r c_i^2} \le \frac{\sum_{i=1}^r a_i + \sum_{i=1}^q b_i}{\sqrt{q+r}}.$$

3 Main result

In this section, we present the main results. Theorem 1 shows that when $\mu_1(3s-1)$ combines with $\mu_1(2s-1)$ satisfy some conditions, the $\ell_1 - \ell_2-$minimization model can stably recover an unknown signal.

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Theorem 1 *Suppose that*

$$\mu_1(3s-1)+(\frac{\sqrt{s}+1}{\sqrt{2s}-1})^2\mu_1(2s-1)<1-(\frac{\sqrt{s}+1}{\sqrt{2s}-1})^2, \quad (2)$$

then the solution $\overline{x}$ of $\ell_1-\ell_2$-minimization model and the original signal x satisfies

$$\|\overline{x}-x\|_2\leq\frac{(4+2\sqrt{2})s}{a_1}\epsilon+\frac{a_3}{a_2}\|x_{S_0^c}\|_1.$$

where $a_1=(2s-\sqrt{2s})\sqrt{1-\mu_1(3s-1)}-(\sqrt{2}s+\sqrt{2s})\sqrt{1+\mu_1(2s-1)}$, $a_2=(2s-\sqrt{2s})\sqrt{1-\mu_1(3s-1)}-(\sqrt{2}s+\sqrt{2s})\sqrt{1+\mu_1(2s-1)}$, $a_3=2\sqrt{2s}\sqrt{1-\mu_1(3s-1)}+2\sqrt{2s}\sqrt{1+\mu_1(2s-1)}$

Proof: Set S_0 as the indices of the s largest entries of x, S_1 are the indices of the t largest entries of $h_{S_0^c}$, S_2 are the indices of the next t largest entries of $h_{S_0^c}$, and so on.

Firstly, we know that the kth largest value of $h_{S_0^c}$ obeys

$$|h_{S_0^c}|_k\leq\|h_{S_0^c}\|_1/k.$$

Set $S_{01}=S_0\cup S_1$; therefore,

$$\|h_{S_{01}^c}\|_2^2\leq\|h_{S_0^c}\|_1^2\sum_{k\geq t+1}1/k^2\leq\|h_{S_0^c}\|_1^2/t. \quad (3)$$

On the other hand, set $x=\overline{x}-h$, then we have

$$\|x\|_1-\|x\|_2\geq\|x+h\|_1-\|x+h\|_2.$$

Thus,

$$\|h\|_2+\|x\|_1\geq\|x+h\|_2-\|x\|_2+\|x\|_1\geq\|x+h\|_1.$$

Additionally,

$$\begin{aligned}\|x+h\|_1&=\|(x+h)_{S_0}\|_1+\|(x+h)_{S_0^c}\|_1\\&\geq\|x_{S_0}\|_1-\|h_{S_0}\|_1+\|h_{S_0^c}\|_1-\|x_{S_0^c}\|_1.\end{aligned}$$

Combining the above two inequalities yield

$$\|h_{S_0^c}\|_1\leq\|h_{S_0}\|_1+2\|x_{S_0^c}\|_1+\|h\|_2. \quad (4)$$

The inequalities (4) and (3) give

$$\begin{aligned}\|h_{S_{01}^c}\|_2&\leq(\|h_{S_0}\|_1+2\|x_{S_0^c}\|_1+\|h\|_2)/\sqrt{t}\\&\leq\sqrt{s/t}\|h_{s_0}\|_2+(2\|x_{S_0^c}\|_1+\|h\|_2)/\sqrt{t},\end{aligned}$$

and thus

$$\begin{aligned}&\|h\|_2\leq\|h_{S_{01}}\|_2+\|h_{S_{01}^c}\|_2\\&\leq(1+\sqrt{s/t})\|h_{S_{01}}\|_2+2\|x_{S_0^c}\|_1/\sqrt{t}+\|h\|_2/\sqrt{t}.\end{aligned}$$

This yields

$$\|h\|_2\leq\frac{\sqrt{t}+\sqrt{s}}{\sqrt{t}-1}\|h_{S_{01}}\|_2+\frac{2}{\sqrt{t}-1}\|x_{S_0^c}\|_1. \quad (5)$$

From Lemma 1, we have

$$\begin{aligned}&\|Ah\|_2=\|A_{S_{01}}h_{S_{01}}+\sum_{i\geq2}A_{S_i}h_{S_i}\|_2\\&\geq\|A_{S_{01}}h_{S_{01}}\|_2-\|\sum_{i\geq2}A_{S_i}h_{S_i}\|\\&\geq\|A_{S_{01}}h_{S_{01}}\|_2-\sum_{i\geq2}\|A_{S_i}h_{S_i}\|_2\\&\geq\sqrt{1-\mu_1(s+t-1)}\|h_{S_{01}}\|_2-\sqrt{1+\mu_1(t-1)}\sum_{i\geq2}\|h_{S_i}\|_2\end{aligned} \quad (6)$$

Now, we note that for $i\geq1$ and $k\in S_{i+1}$,

$$|h_k|\leq\|h_{S_i}\|_1/t,$$

then

$$\|h_{S_{i+1}}\|_2^2\leq\|h_{S_i}\|_1^2/t.$$

This and inequality (4) give

$$\begin{aligned}&\sum_{i\geq2}\|h_{S_i}\|_2\leq\sum_{i\geq1}\|h_{S_i}\|_1/\sqrt{t}=\|h_{S_0^c}\|_1/\\&\sqrt{t}\leq\sqrt{s/t}\|h_{S_0}\|_2+2\|x_{S_0^c}\|_1/\sqrt{t}+\|h\|_2/\sqrt{t}.\end{aligned} \quad (7)$$

Combining (6) and (7), we obtain:

$$\begin{aligned}&\|Ah\|_2\geq(\sqrt{1-\mu_1(s+t-1)}-\sqrt{s/t}\sqrt{1+\mu_1(t-1)})\|h_{S_{01}}\|_2\\&-\frac{2\sqrt{1+\mu_1(t-1)}}{\sqrt{t}}\|x_{S_0^c}\|_1-\frac{\sqrt{1+\mu_1(t-1)}}{\sqrt{t}}\|h\|_2.\end{aligned}$$

If

$$\sqrt{1-\mu_1(s+t-1)}-\sqrt{s/t}\sqrt{1+\mu_1(t-1)}>0, \quad (8)$$

then, by combining $\|Ah\|_2\leq2\epsilon$ and the above two inequalities, we have

$$\|h_{S_{01}}\|_2\leq\frac{1}{a_4}(2\epsilon+\frac{2\sqrt{1+\mu_1(t-1)}}{\sqrt{t}}\|x_{S_0^c}\|_1+\frac{\sqrt{1+\mu_1(t-1)}}{\sqrt{t}}\|h\|_2).$$

where $a_4=\sqrt{1-\mu_1(s+t-1)}-\sqrt{s/t}\sqrt{1+\mu_1(t-1)}$.
If

$$(t-\sqrt{t})\sqrt{1-\mu_1(s+t-1)}-(\sqrt{st}+\sqrt{t})\sqrt{1+\mu_1(t-1)}>0, \quad (9)$$

it follows from the above two inequalities and inequality (5) that

$$\begin{aligned}\|h\|_2\leq&\frac{2t+2\sqrt{st}}{(t-\sqrt{t})\sqrt{1-\mu_1(s+t-1)}-(\sqrt{st}+\sqrt{t})\sqrt{1+\mu_1(t-1)}}\epsilon\\&+\frac{2\sqrt{t}\sqrt{1-\mu_1(s+t-1)}+2\sqrt{t}\sqrt{1+\mu_1(t-1)}}{(t-\sqrt{t})\sqrt{1-\mu_1(s+t-1)}-(\sqrt{st}+\sqrt{t})\sqrt{1+\mu_1(t-1)}}\|x_{S_0^c}\|_1.\end{aligned} \quad (10)$$

It is easy to know that inequality (9) is sufficient for inequality (8), and inequality (9) is equivalent to

$$\mu_1(s+t-1)+(\frac{\sqrt{s}+1}{\sqrt{t}-1})^2\mu_1(t-1)<1-(\frac{\sqrt{s}+1}{\sqrt{t}-1})^2.$$

Condition (2) ensures that the inequality above holds when $t=2s$. Hence, from inequality (10), we can get the conclusion of the theorem. □

The following theorem provides another condition that can ensure stable recovery of unknown signals by $\ell_1-\ell_2$-minimization model. This condition is accompanied through additional parameters. We can choose appropriate parameters to meet our requirements. The following lemma is needed in the proof of the following theorem.

Lemma 4 *If positive integer a, $b\geq2$, s satisfies*

$$\mu_1(s+a-1)+\frac{\sqrt{s}+1}{\sqrt{b}-1}\mu_1(s+a+b-1)\leq1,$$

then

$$b-\sqrt{b}-(b-\sqrt{b})\mu_1(s+a-1)-(\sqrt{sb}+\sqrt{b})\mu_1(s+a+b-1)\geq0,$$

and

$$1-\mu_1(s+a-1)-\sqrt{\frac{s}{b}}\mu_1(s+a+b-1)\geq0.$$

This lemma is simple, therefore we can ignore its proof.

Theorem 2 *Suppose that*

$$\mu_1(s+a-1) + \frac{\sqrt{s}+1}{\sqrt{b}-1}\mu_1(s+a+b-1) \le 1 \tag{11}$$

holds for some positive integers a and b, satisfying $2a \le b \le 4a$. Then, the solution $\overline{x}$ of $\ell_1 - \ell_2$-minimization model and the original signal x satisfy

$$\|\overline{x} - x\|_2 \le \frac{2(b+\sqrt{sb})\sqrt{1+\mu_1(s+a-1)}}{c_1}\epsilon + \frac{c_2}{c_1}\|x_{-max(s)}\|_1.$$

where $c_1 = b - \sqrt{b} - (b-\sqrt{b})\mu_1(s+a-1) - (\sqrt{sb}+\sqrt{b})\mu_1(s+a+b-1)$, $c_2 = 2\sqrt{b} - 2\sqrt{b}\mu_1(s+a-1) + 2\sqrt{b}\mu_1(s+a+b-1)$.

Proof: Without a loss of generality, we assume the first s coordinates of x are the largest in magnitude. Making a rearrangement if necessary, we may also assume that

$$|h(s+1)| \ge |h(s+2)| \ge \cdots.$$

Set $T_0 = \{1, 2, \cdots, s\}$, $T_* = \{s+1, s+2, \cdots, s+a\}$ and $T_i = \{s+a+(i-1)b+1, \cdots, s+a+ib\}$, $i = 1, 2\cdots$, with the last subset of size less than or equal to b. Let $h_0 = h_{T_0}$, $h_* = h_{T_*}$ and $h_i = h_{T_i}$ for $i \ge 1$.

First, we divide each vector h_i into two pieces. Set $T_{i1} = \{s+a+(i-1)b+1, \cdots, s+ib\}$ and $T_{i2} = T_i - T_{i1} = \{s+1+ib, \cdots, s+a+ib\}$. We note that $|T_{i1}| = b-a$ and $|T_{i2}| = a$ for all $i \ge 1$. Let $h_{i1} = h_{T_{i1}}$ and $h_{i2} = h_{T_{i2}}$. Note that $a \le b-a \le 3a$. Applying Lemma 3 to the vectors h_*, h_{11}, h_{12} and $h_{(i-1)2}, h_{i1}, h_{i2}$ for $i = 2, 3, \cdots$, we obtain,

$$\|h_1\|_2 \le \frac{\|h_*\|_1 + \|h_{11}\|_1}{\sqrt{b}}, \|h_2\|_2 \le \frac{\|h_{12}\|_1 + \|h_{21}\|_1}{\sqrt{b}}, \cdots,$$
$$\|h_i\|_2 \le \frac{\|h_{(i-1)2}\|_1 + \|h_{i1}\|_1}{\sqrt{b}}, \cdots.$$

Additionally, inequality (4) holds. Cauchy–Buniakowsky–Schwarz inequality and (4) yield

$$\begin{aligned}\sum_{i\ge1}\|h_i\|_2 &\le \frac{\|h_*\|_1 + \sum_{i\ge1}\|h_i\|_1}{\sqrt{b}} = \frac{\|h-h_0\|_1}{\sqrt{b}} \\ &\le \frac{\|h_0\|_1 + 2\|x_{-max(s)}\|_1 + \|h\|_2}{\sqrt{b}} \\ &\le \sqrt{\frac{s}{b}}\|h_0\|_2 + \frac{2\|x_{-max(s)}\|_1 + \|h\|_2}{\sqrt{b}} \\ &\le \sqrt{\frac{s}{b}}\|h_0 + h_*\|_2 + \frac{2\|x_{-max(s)}\|_1 + \|h\|_2}{\sqrt{b}}.\end{aligned} \tag{12}$$

From Lemmas 1, 2, and (12), we have

$$\begin{aligned}&|\langle Ah, A(h_0+h_*)\rangle| \\ &= |\langle A(h_0+h_*), A(h_0+h_*)\rangle + \sum_{i\ge1}\langle Ah_i, A(h_0+h_*)\rangle| \\ &\ge (1-\mu_1(s+a-1))\|h_0+h_*\|_2^2 \\ &- \sum_{i\ge1}\mu_1(s+a+b-1)\|h_0+h_*\|_2\|h_i\|_2 \\ &\ge \|h_0+h_*\|_2((1-\mu_1(s+a-1)) - \sqrt{\frac{s}{b}}\mu_1(s+a+b \\ &-1))\|h_0+h_*\|_2 - \mu_1(s+a+b-1)\frac{2\|x_{-max(s)}\|_1 + \|h\|_2}{\sqrt{b}}).\end{aligned} \tag{13}$$

Additionally, $\|Ah\|_2 = \|A(\overline{x}-x)\|_2 \le \|A\overline{x} - y\|_2 + \|Ax - y\|_2 \le 2\epsilon$. Combining Cauchy–Buniakowsky–Schwarz inequality and Lemma 1 yields

$$\begin{aligned}|\langle Ah, A(h_0+h_*)\rangle| &\le \|Ah\|_2\|A(h_0+h_*)\|_2 \\ &\le 2\epsilon\sqrt{1+\mu_1(s+a-1)}\|h_0+h_*\|_2.\end{aligned} \tag{14}$$

Combining (13), (14), (11) and Lemma 4 give

$$\|h_0+h_*\|_2 \le \frac{2\epsilon\sqrt{1+\mu_1(s+a-1)}}{1-\mu_1(s+a-1)-\sqrt{\frac{s}{b}}\mu_1(s+a+b-1)} + \frac{\mu_1(s+a+b-1)\frac{2\|x_{-max(s)}\|_1+\|h\|_2}{\sqrt{b}}}{1-\mu_1(s+a-1)-\sqrt{\frac{s}{b}}\mu_1(s+a+b-1)}.$$

Therefore, the above inequality and (12) produce

$$\begin{aligned}\|h\|_2 &\le \|h_0+h_*\|_2 + \sum_{i\ge1}\|h_i\|_2 \\ &\le (1+\sqrt{\frac{s}{b}})\|h_0+h_*\|_2 + \frac{2\|x_{-max(s)}\|_1 + \|h\|_2}{\sqrt{b}} \\ &\le ((1+\sqrt{\frac{s}{b}})\frac{\mu_1(s+a+b-1)}{c_4} + 1)\frac{2\|x_{-max(s)}\|_1 + \|h\|_2}{\sqrt{b}} + \\ &(1+\sqrt{\frac{s}{b}})\frac{2\epsilon\sqrt{1+\mu_1(s+a-1)}}{c_4}.\end{aligned}$$

where $c_4 = 1 - \mu_1(s+a-1) - \sqrt{\frac{s}{b}}\mu_1(s+a+b-1)$. Simplifying the above inequality, from (11) and Lemma 4, We can get the conclusion of the theorem. □

Theorem 2 and 1 naturally leads to the following conclusion.

Theorem 3 *Assume that the cumulative coherence of the measurement matrix satisfies condition* (2) *or* (11)*, and $\epsilon = 0$ in $\ell_1 - \ell_2$-minimization model, then any $s-$sparse vector can be accurately recovered through $\ell_1 - \ell_2$-minimization model.*

4 Conclusion

From this paper, we find that based on some condition of cumulative coherence, the $\ell_1 - \ell_2-$minimization model can exactly recover s-sparse signals in noiseless cases and stably recover s-sparse signals in the noise cases.

DECLARATIONS

The authors have no relevant financial or non-financial interests to disclose. All authors contributed to the study conception and proof. The authors declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted. This study focuses on theoretical analysis and does not involve ethical issues. All data generated or analysed during this study are included in this published article. The corresponding authors of this paper is Meijiao Zhang.

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