# A computational stochastic dynamic model to assess the risk of breakup in a romantic relationship

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# A computational stochastic dynamic model to assess the risk of breakup in a romantic relationship

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**Summary**

We introduce an algorithm to find feedback Nash equilibria of a stochastic differential game. Our computational approach is applied to analyze optimal policies to nurture a romantic relationship in the long term. This is a fundamental problem for the applied sciences, which is naturally formulated in this work as a stochastic differential game with nonlinearities. We use our computational model to analyze the risk of marital breakdown. In particular, we introduce the concept of "love at risk" which allows us to estimate the probability of a couple breaking up in the face of possible unfavorable scenarios.

**KEYWORDS:**
Stochastic Differential Games, Nonlinear problems, Random Differential Equations, Dynamical Analysis, Human Behaviour

## 1 | INTRODUCTION

The purpose of this work is twofold. Firstly, we introduce an algorithm to find feedback Nash equilibria of a stochastic differential game (SDG) and, secondly, we apply our methodology to a problem of significance in the social sciences, related to human behavior.

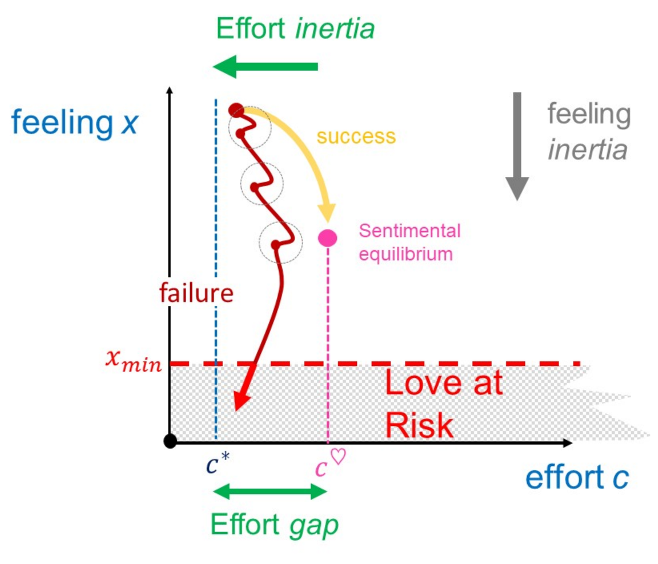
The numerical analysis of SDGs is currently a topic of growing interest (see early publications [1,2] and recent contributions [3,4,5,6,7,8]). Most contributions in the literature focus on the study of the theoretical properties of certain classes of SDGs. Also, various works that formulate economic and social problems as SDGs (see e.g. [9,10,11]) obtain their solutions by using heuristic approximations to avoid solving the stochastic Hamilton-Jacobi-Bellman (HJB) equations of the problem. The most extended approach to dealing with problems with more than two players is the approximation by a linear quadratic problem or the use of open-loop solutions (see, for example, [5]). Also, in recent years, there has been an increase in computational methods to address differential game problems. For example, [12,13] propose an algorithm based on deep learning and fictitious play to find feedback Nash equilibria for a class of finite-horizon SDGs. Additionally, [14] addresses a similar problem from the perspective of policy iteration and the Chebyshev spectral collocation method. Furthermore, in the literature on multi-agent reinforcement learning there are model-free and discrete-time counterparts to solve similar problems (see e.g. [15]).

In this paper, our goal is to solve a model-based infinite-horizon autonomous SDG, which is a common problem in economics and management (see e.g. [16]). Our computational approach involves solving a stationary HJB system. This seems like a useful contribution to the field since most of the available algorithms are designed to solve non-stationary HJB equations. Regarding the specific SDG application considered in this paper, [17] proposes a numerical procedure to solve the HJB problem of the original deterministic and one-dimensional version [18]. Here, our computational analysis deals with a two-dimensional stochastic framework, which is required to address the two-person decision-making problem considered in this paper.

Our numerical approach extends the idea in[19], where an algorithm –called RaBVItG (Radial Basis Value Iteration Game)– is introduced to solve the HJB system to find feedback Nash equilibria of deterministic differential games. The core of the algorithm consists of two main loops: value iteration, as in[20,21,22], plus game iteration, as introduced in[19]. More precisely, RaBVItG uses game iteration to find the Nash equilibrium corresponding to a fixed value of the game and then uses value iteration to find a fixed point solution for the coupled system of value functions (one per player). The feedback Nash equilibrium of the deterministic differential game is found as the convergent solution of both iterations. We introduce below a stochastic version of the RaBVItG algorithm to find feedback Nash equilibria of an SDG. Notice that game iteration is similar to the fictitious play idea mentioned above. Fictitious play fixes, at an arbitrary stage of the game, the previous strategies of a player's opponents to find the Nash equilibrium. The game iteration method uses Krasnoselski iteration[23] to find each player's stage strategy by combining the previous strategies of all players with the updated ones through the HJB equations. Both philosophies rely on the idea of decoupling the problem into $N$ individual problems to be solved iteratively, instead of dealing with the complete set of coupled HJB equations.

The second objective of this work seeks to estimate the risk of rupture in a dyadic romantic relationship that is intended to last. This is a problem of enormous interest in the social sciences, due to the relevance of long-term romantic relationships, marriage in particular, in most societies[24]. Furthermore, there is an epidemic of failed marriages in the West (see e.g.[25]) which is not well understood in the field of marital psychology (see[26]). To formulate our problem we model a long-term romantic relationship as an optimal control problem, as originally proposed in[18] and[27], and then extended in[28]. The quality of the relationship is monitored by a state variable $x(t)$ (called *feeling*) whose evolution is controlled by the effort exerted by both partners to keep the relationship alive and well. The couple's problem consists of finding the optimal effort control paths to stay together forever. In particular, for the relationship to be viable the feeling must stay above a certain critical value $x_{min} > 0$. Once $x(t)$ drops below the level $x_{min}$, the relationship enters a risk zone and is in danger of breaking down. It was found in[18] that two effects contribute to hindering the viability of the relationship. First, the feeling is subject to decay as time goes by and, second, there is a tendency to reduce effort below the level required for the relationship to last, thus moving the relationship away from the unique equilibrium path of feeling-effort for the relationship. These two inertial forces can make the feeling approach the risk zone where the breakup is likely. Figure 1 below illustrates the idea of this hindering mechanism to put love at risk (see[18]). Regarding the problem of love at risk, the point of interest in this paper is to estimate the critical value $x_{min}$ under a more realistic

**Figure 1** Basic mechanism operating to put love at risk (adapted from[18].)




version of the original model, where the effort variable is common for both partners and the evolution of the feeling is governed by a deterministic equation. First, we assume here, as in the differential game formulation[28], that each partner could make effort differently, so there are two different effort variables $c_1(t)$ and $c_2(t)$ controlling the feeling dynamics. Furthermore, we extend both formulations of the couple's relationship by considering that the feeling $x(t)$ is a random variable whose evolution is governed by a stochastic differential equation. We thus introduce a new model formulation of the couple's problem as an SDG. This stochastic generalization allows us to deal with the idea of the probability of breaking up at a particular moment in the relationship, which can be obtained from the probability distribution of $x(t)$ once the threshold value $x_{min}$ is estimated. Using the well-known idea of "value at risk" in finance[29] we provide an estimate of $x_{min}$ that will be called *Love at Risk* (LaR) below.

The paper is organized as follows. In Section 2, we present the mathematical model of the couple's sentimental dynamics as an SDG. We pay attention here to the main output of the model solution, namely the stochastic feedback Nash equilibrium, and the feedback mappings that are required for its numerical approximation. In Section 3 we present the computational model. Firstly, we present the discretization of the involved equations and the way to implement the RaBVItG algorithm to solve numerically the couple's problem. In section 4 we analyze several numerical experiments for different types of couples and how our stochastic computational scheme renders new information compared with the deterministic versions of the couple's problem. We also show how the threshold value LaR can be determined using our stochastic structure to estimate the probability of dissolution of a given couple. In a final appendix, we provide numerical evidence of the accuracy of the algorithm proposed in the paper.

## 2 | MATHEMATICAL MODEL

Our model is a stochastic two–person generalization of the optimal control model for a long-term romantic relationship introduced in[18]. A deterministic differential game model of the problem was introduced in[28]. Also, a mean-field stochastic version of the original model was considered in[30]. In this paper, the state of the relationship at time $t \geq 0$ is described by $x(t)$ –the *feeling* variable–, which is modeled by a stochastic process $\{x(t)\}_{t\geq 0}$, with $x : [0, \infty) \to X \subseteq \mathbb{R}$, being $X$ the state space. The feeling evolves according to a stochastic differential equation

$$\mathrm{d}x(t) = \left[-rx(t) + a_1 c_1(t) + a_2 c_2(t)\right] \mathrm{d}t + \sigma(x(t))\, \mathrm{d}w, \tag{1}$$

where $r, a_1, a_2 > 0$ and, for $i = 1, 2$, $c_i : [0, \infty) \to \mathbb{R}^+$ is a (piece-wise continuous) function that measures the effort put into the relationship by partner $i$ at time $t$, and $w(t)$ is a Wiener process. Equation (1) is a stochastic version of the differential equation presented in[28], called the "second law of thermodynamics for sentimental relationships"[26]. Here the time evolution of the *feeling* includes a random term, due to the fact that the couple's evaluation of the state of the relationship may be subjected to some observational error or uncertainty at any time $t$. The total well-being $W_i$ of each partner $i$ is defined as the conditional expectation

$$W_i\left(c_i\right) = \mathbb{E}\left(\int_0^\infty e^{-\rho_i t}\left(U_i(x(t)) - D_i\left(c_i(t)\right)\right) \mathrm{d}t | x(0) = y\right), \; i = 1, 2, \tag{2}$$

where $U_i$ and $D_i$ are, respectively, the utility of feeling and disutility of effort, while $\rho_i > 0$ is the individual rate of temporal preference. The functions $U_i$ and $D_i$ are assumed to satisfy the same properties as in[28], namely $U_i'(x) > 0$, $U_i''(x) < 0$, and $U_i'(x) \to 0$ as $x \to +\infty$, and also $D_i''(c_i) > 0$, $D_i'(c_i^*) = 0$ for some $c_i^* \geq 0$, $D_i'(c_i) \to +\infty$ as $c_i \to +\infty$, for $i = 1, 2$. Notice that $c_i^*$ gives the effort level preferred (myopically) by partner $i$. The underlying psychological rationale behind these assumptions is explained in detail in[18].

The couple's problem considered in this paper can now be stated as follows. Given the feeling dynamics (1), and the initial feeling level $x(0) = x_0$, find the effort trajectories $c_1(t), c_2(t)$ such that each individual well-being (2) is maximal. This is an infinite-horizon stochastic differential two-person game. It may be assumed that the relationship will be viable as long as the feeling $x(t)$ remains above a certain value $x_{min} > 0$ (see Figure 1). Let us define the pair $\left(c_1^\heartsuit(t), c_2^\heartsuit(t)\right)$ that solves the couple's problem. We aim to find a Nash equilibrium for this differential game. The differential game is autonomous, so we consider stationary feedback solutions of the problem, that are defined as $c_i = S_i(x)$, being $S_i : X \to \mathbb{R}^+$ the *feedback* map that provides the effort by player $i$ for the feeling level $x$. We look for a couple of optimal strategies $\left(S_1^\heartsuit(\cdot), S_2^\heartsuit(\cdot)\right)$, such that $S_i^\heartsuit : X \to \mathbb{R}^+$ is a stationary feedback Nash equilibrium of the SDG. This equilibrium is attained if $S_1^\heartsuit(x(t))$ solves

$$\max_{c_1(t)} \mathbb{E}\left(\int_0^\infty e^{-\rho_1 t}\left(U_1(x(t)) - D_1\left(c_1(t)\right)\right) \mathrm{d}t | x(0) = y\right) \tag{3}$$

with $\mathrm{d}x(t) = \left[-rx(t) + a_1 c_1(t) + a_2 S_2^{\heartsuit}(x(t))\right] \mathrm{d}t + \sigma(x(t))\,\mathrm{d}w$, and also $S_2^{\heartsuit}(x(t))$ solves

$$\max_{c_2(t)} \mathbb{E}\left( \int_0^{\infty} e^{-\rho_2 t} \left(U_2(x(t)) - D_2\left(c_2(t)\right)\right) \mathrm{d}t | x(0) = y \right) \tag{4}$$

with $\mathrm{d}x(t) = \left[-rx(t) + a_1 S_1^{\heartsuit}(x(t)) + a_2 c_2(t)\right] \mathrm{d}t + \sigma(x(t))\,\mathrm{d}w$, with $x(0) = y$, and $c_i(t) \in \mathbb{R}^+$ for $t \geq 0$.

Assume that there exists a stochastic feedback Nash equilibrium $S^{\heartsuit} = \left(S_1^{\heartsuit}, S_2^{\heartsuit}\right)$ for the couple's problem. Let $v_i^{\heartsuit} : X \to \mathbb{R}$ be the *value function* of partner $i$, i=1,2, defined by

$$v_i^{\heartsuit}(y) = W_i\left(S_i^{\heartsuit}(x(t))\right),$$

where, for $t \geq 0$, $S_i^{\heartsuit}(x(t))$ gives the optimal feedback trajectory of partner $i$ for the initial feeling level $x(0) = y$. The value functions $v_i^{\heartsuit}$, $i = 1, 2$, must satisfy the stochastic Hamilton-Jacobi-Bellman (HJB) equations, which in this case are given by

$$\begin{cases} \rho_1 v_1(y) = \max_{c_1 \in \mathbb{R}^+} \left\{ U_1(y) - D_1\left(c_1\right) + v_1'(y)\left(-rx + a_1 c_1 + a_2 S_2^{\heartsuit}(y)\right) + \frac{1}{2} v_1''(y)\,\sigma^2(y) \right\}, \\ \rho_2 v_2(y) = \max_{c_2 \in \mathbb{R}^+} \left\{ U_2(y) - D_2\left(c_2\right) + v_2'(y)\left(-rx + a_1 S_1^{\heartsuit}(y) + a_2 c_2\right) + \frac{1}{2} v_2''(y)\,\sigma^2(y) \right\}. \end{cases} \tag{5}$$

The solution of (5) gives the (optimal) feedback maps $S_i^{\heartsuit} : X \to \mathbb{R}^+$, $i = 1, 2$,

$$\begin{cases} S_1^{\heartsuit}(y) = arg\max_{c_1 \in \mathbb{R}^+} \left\{ U_1(y) - D_1\left(c_1\right) + v_1'(y)\left(-rx + a_1 c_1 + a_2 S_2^{\heartsuit}(y)\right) + \frac{1}{2} v_1''(y)\,\sigma^2(y) \right\}, \\ S_2^{\heartsuit}(y) = arg\max_{c_2 \in \mathbb{R}^+} \left\{ U_2(y) - D_2\left(c_2\right) + v_2'(y)\left(-rx + a_1 S_1^{\heartsuit}(y) + a_2 c_2\right) + \frac{1}{2} v_2''(y)\,\sigma^2(y) \right\}, \end{cases} \tag{6}$$

constitutes a feedback Nash stochastic equilibrium of the problem. Given $x(0) = y$, inserting $S_i^{\heartsuit}(x(t))$, $i = 1, 2$, into (1), we obtain

$$\mathrm{d}x(t) = \left[-rx(t) + a_1 S_1^{\heartsuit}(x(t)) + a_2 S_2^{\heartsuit}(x(t))\right] \mathrm{d}t + \sigma(x(t))\,\mathrm{d}w,$$

which gives the optimal evolution of the stochastic process $\left\{x^{\heartsuit}(t)\right\}_{t \geq 0}$ which solves the couple's problem with initial state $y \in X$.

## 3 | A COMPUTATIONAL MODEL

General existence or uniqueness results for feedback Nash equilibria for differential games are not available in the literature[31], except for some particular cases, namely the so-called Linear Quadratic models[5]. Thus, a computational approach is required to find a solution. The following method can be considered a generalization of the algorithm in[19] to solve an infinite horizon SDG in feedback Nash equilibrium. While it can be applied to a general class on $N$-player SDG, we present the algorithm adapted to the SDG model for the couple's problem described in the preceding section.

The SDG model is discretized in a Semi-Lagrangian way (see, for instance,[32]). This implies first discretizing in time and space and then using numerical interpolation, by means of radial basis functions (see[19]). The discretization of (2) is performed through the rectangle rule, taking $h > 0$ as a time step. Thus, given a value of the state variable $y \in X$, we consider

$$W_i^h\left(c_i^h\right) = \mathbb{E}\left\{ h \sum_{k=0}^{\infty} e^{-\rho_i t_k} \left(U_i\left(x_k\right) - D_i\left(c_{i,k}\right)\right) | x_0 = y \right\},\ i = 1, 2, \tag{7}$$

where $c_i^h = \left\{c_{i,k}\right\}_{k \geq 0}$ is a sequence of admissible controls for partner $i$, defined by the piece-wise constant function $c_i^h(\tau) = c_{i,k},\ \tau \in [t_k, t_{k+1})$, with $t_k = hk, k \in \mathbb{N} \cup \{0\}$. Furthermore, the sequence $x_k = x(t_k)$ is obtained by time discretization of (1) using the Euler-Maruyama scheme (see, for instance,[33]), that is,

$$x_{k+1} = x_k + hf\left(x_k, c_{1,k}, c_{2,k}\right) + \sigma\left(x_k\right)\xi_k, \tag{8}$$

where $\xi_k$ denotes the increment of a standard Brownian motion $w(t)$ in the interval $[t_k, t_{k+1})$, $x_0 = y$, and $f(x, c_1, c_2) = -rx + a_1 c_1 + a_2 c_2$ in our case. Then, the discrete value function for partner $i = 1, 2$ is given by

$$v_i^h(y) = \max_{c_i^h} W_i^h\left(c_i^h\right).$$

Therefore, we can redefine (8) as a set of two equally probable displacements of the state variable,

$$x_{k+1} = x_k + \delta_d(x_k, c_{1,k}, c_{2,k}),$$

where

$$\delta_d(y, c_1, c_2) = hf(y, c_1, c_2) + (-1)^d \sigma(y)\sqrt{h},\ d = 1, 2.$$

The Dynamic Programming Principle in discrete time implies that the discrete value functions $v_i^h$ satisfy (see[32])

$$\begin{cases} v_1^h(y) = \max_{c_1 \in \mathbb{R}^+} \left\{ h\left(U_1(y) - D_1(c_1)\right) + \frac{(1-\rho_1 h)}{2} \sum_{d=1}^2 v_1^h\left(y + \delta_d\left(y, c_1, S_2^h(y)\right)\right) \right\}, \\ v_2^h(y) = \max_{c_2 \in \mathbb{R}^+} \left\{ h\left(U_2(y) - D_2(c_2)\right) + \frac{(1-\rho_2 h)}{2} \sum_{d=1}^2 v_2^h\left(y + \delta_d\left(y, S_1^h(y), c_2\right)\right) \right\}, \end{cases} \tag{9}$$

where

$$\begin{cases} S_1^h(y) = arg\max_{c_1 \in \mathbb{R}^+} \left\{ h\left(U_1(y) - D_1(c_1)\right) + \frac{(1-\rho_1 h)}{2} \sum_{d=1}^2 v_1^h\left(y + \delta_d\left(y, c_1, S_2^h(y)\right)\right) \right\}, \\ S_2^h(y) = arg\max_{c_2 \in \mathbb{R}^+} \left\{ h\left(U_2(y) - D_2(c_2)\right) + \frac{(1-\rho_2 h)}{2} \sum_{d=1}^2 v_2^h\left(y + \delta_d\left(y, S_1^h(y), c_2\right)\right) \right\}. \end{cases} \tag{10}$$

To obtain numerical approximations of $v_i^h$, $i = 1, 2$, in (9), we consider a spatial discretization of the state space. Let $\tilde{X} = \left\{y_j\right\}_{j=1,\dots,Q} \subset X$ be a set of arbitrary $Q$ points. Notice that, in general, the points of the form $y^\sharp := y_j + \delta_d\left(y_j, c_1, c_2\right)$ in (9) do not belong to $\tilde{X}$. To find approximate values $\tilde{v}_i^h\left(y_j\right)$ of $v_i^h\left(y_j\right)$ for $y_j \in \tilde{X}, i = 1, 2$, the values $v_i^h\left(y^\sharp\right)$ are calculated through a collocation mesh-free algorithm using the set of scattered nodes $\tilde{X}$[34]. This idea leads to the following version of (9), for $y_j \in \tilde{X}$,

$$\begin{cases} \tilde{v}_1^h\left(y_j\right) = \max_{c_1 \in \mathbb{R}^+} \left\{ h\left(U_1\left(y_j\right) - D_1(c_1)\right) + \frac{(1-\rho_1 h)}{2} \sum_{d=1}^2 RBF\left[\mathrm{V}_1\right]\left(y_{1,d}^\sharp\right) \right\}, \\ \tilde{v}_2^h\left(y_j\right) = \max_{c_2 \in \mathbb{R}^+} \left\{ h\left(U_2\left(y_j\right) - D_2(c_2)\right) + \frac{(1-\rho_2 h)}{2} \sum_{d=1}^2 RBF\left[\mathrm{V}_2\right]\left(y_{2,d}^\sharp\right) \right\}, \end{cases} \tag{11}$$

together with

$$\begin{cases} \tilde{S}_1^h\left(y_j\right) = arg\max_{c_1 \in \mathbb{R}^+} \left\{ h\left(U_1\left(y_j\right) - D_1(c_1)\right) + \frac{(1-\rho_1 h)}{2} \sum_{d=1}^2 RBF\left[\mathrm{V}_1\right]\left(y_{1,d}^\sharp\right) \right\}, \\ \tilde{S}_2^h\left(y_j\right) = arg\max_{c_2 \in \mathbb{R}^+} \left\{ h\left(U_2\left(y_j\right) - D_2(c_2)\right) + \frac{(1-\rho_2 h)}{2} \sum_{d=1}^2 RBF\left[\mathrm{V}_2\right]\left(y_{2,d}^\sharp\right) \right\}. \end{cases} \tag{12}$$

where for $d = 1, 2$,

$$\begin{cases} y_{1,d}^\sharp = y_j + \delta_d\left(y_j, c_1, \tilde{S}_2^h\left(y_j\right)\right), \\ y_{2,d}^\sharp = y_j + \delta_d\left(y_j, \tilde{S}_1^h\left(y_j\right), c_2\right), \end{cases} \tag{13}$$

and $RBF[\mathrm{V}_i]$ is the approximation of the $i$-th value function obtained using Radial Basis Functions[35]. Specifically, for $y \in X$, $RBF[\mathrm{V}_i](y)$ is obtained by interpolation from the values of the array $\mathrm{V}_i = \left[\tilde{v}_i^h\left(y_1\right), \dots, \tilde{v}_i^h\left(y_Q\right)\right]^T$ as follows,

$$RBF\left[\mathrm{V}_i\right](y) = \sum_{j=1}^{Q} \lambda_{i,j} \Phi\left(\left\|y - y_j\right\|\right),\ i = 1, 2, \tag{14}$$

where $\bar{\lambda}_i = \left[\lambda_{i,1}, \dots, \lambda_{i,Q}\right]^T \in \mathbb{R}^Q$ is an array of weighting coefficients, and $\Phi\left(\left\|y - y_j\right\|\right) = \exp\left(-\frac{\left\|y-y_j\right\|^2}{\nu^2}\right)$, with $\nu > 0$ a shape parameter (see[34] for the details). In addition, for $i = 1, 2$, $\bar{\lambda}_i$ is obtained by solving $\mathrm{A}\bar{\lambda}_i = \mathrm{V}_i$, where A is the matrix with entries $\mathrm{A}_{j,l} = \Phi\left(\left\|y_l - y_j\right\|\right)$, $j, l = 1, \dots, Q$.

## Algorithm pseudocode

The algorithm to find a solution for the discretized problem of the previous section is called RaBVItG, which refers to Radial Basis approximations, Value Iteration and Game Iteration. It essentially consists of two main loops: game iteration to find a Nash Equilibrium for a given value function, and value iteration to improve the approximation of the value function, given a previously obtained equilibrium. Both iterations are sequentially interspersed until convergence is reached. We provide the details below.

Let $\mathrm{V} = \left[\mathrm{V}_1, \mathrm{V}_2\right]$ and $\mathrm{C} = \left[\mathrm{C}_1, \mathrm{C}_2\right]$, denote the arrays to store the updated algorithm information for both partners after each iteration, that is,

$$\mathrm{V}_i = \left[\tilde{v}_i^h\left(y_1\right), \dots, \tilde{v}_i^h\left(y_Q\right)\right]^T,\ \mathrm{C}_i = \left[\tilde{c}_i^h\left(y_1\right), \dots, \tilde{c}_i^h\left(y_Q\right)\right]^T,\ i = 1, 2,$$

are $Q-$dimensional arrays for the value functions and the effort controls of each partner $i = 1, 2$, evaluated at the points $y_j \in \tilde{X}$. Let $T_i = \left[T_{i,1}, ..., T_{i,Q}\right] : \mathbb{R}^Q \to \mathbb{R}^Q$ and $G_i = \left[G_{i,1}, ..., G_{i,Q}\right] : \mathbb{R}^Q \to \mathbb{R}^Q$ be two operators defined component-wise by

$$T_{i,j}\left(\mathrm{V}_i\right) = h\left(U_i\left(y_j\right) - D_i\left(c_i\right)\right) + \frac{\left(1-\rho_i h\right)}{2}\sum_{d=1}^{2} RBF\left[\mathrm{V}_i\right]\left(y^{\sharp}_{i,d}\right),\ j = 1, ..., Q, \tag{15}$$

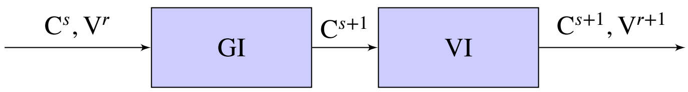
and

$$G_{i,j}\left(\mathrm{V}_i\right) = arg\max_{c_i \in \mathbb{R}^+}\left\{h\left(U_i\left(y_j\right) - D_i\left(c_i\right)\right) + \frac{\left(1-\rho_i h\right)}{2}\sum_{d=1}^{2} RBF\left[\mathrm{V}_i\right]\left(y^{\sharp}_{i,d}\right)\right\},\ j = 1, ..., Q, \tag{16}$$

with $y^{\sharp}_{i,d}$ as defined in (13).

The two main loops of RaBVItG are shown schematically in Figure 2. For $s, r \in \mathbb{N} \cup \{0\}$, let $\mathrm{C}^s$ and $\mathrm{V}^r$ be candidates for optimal controls and values, respectively. The first loop GI obtains a new (optimal) control $\mathrm{C}^{s+1}$, given $\mathrm{V}^r$ and the second loop VI then determines a better value $\mathrm{V}^{r+1}$ using $\mathrm{C}^{s+1}$.

**Figure 2** Scheme of the two main loops of RaBVItG.




We provide the details below.

1. *Game Iteration* (GI). We obtain $\mathrm{C}_i^{s+1}$ as follows:

$$\mathrm{C}_i^{s+1} = \theta \mathrm{C}_i^s + (1-\theta)\,G_i\left(\mathrm{V}_i^r\right),\ i = 1, 2,$$

with $G_i$ as defined in (16), and $\theta \in (0, 1)$ a weighting coefficient, as in the Krasnoselski iteration[23].

The GI loop is thus defined by the scheme

$$\begin{cases} \tilde{c}^{s+1}_{1,j} \equiv \theta \tilde{c}^{s}_{1,j} + (1-\theta)\, arg\max_{c_1 \in \mathbb{R}^+}\left\{h\left(U_1\left(y_j\right) - D_1\left(c_1\right)\right) + \frac{(1-\rho_1 h)}{2}\sum_{d=1}^{2} RBF\left[\mathrm{V}_1\right]\left(y^{\sharp}_{1,d}\right)\right\}, \\ \tilde{c}^{s+1}_{2,j} \equiv \theta \tilde{c}^{s}_{2,j} + (1-\theta)\, arg\max_{c_2 \in \mathbb{R}^+}\left\{h\left(U_2\left(y_j\right) - D_2\left(c_2\right)\right) + \frac{(1-\rho_2 h)}{2}\sum_{d=1}^{2} RBF\left[\mathrm{V}_2\right]\left(y^{\sharp}_{2,d}\right)\right\}, \\ y^{\sharp}_{1,d} = y_j + hf\left(y_j, \left[c_1, \tilde{c}^{s}_{2,j}\right]\right) + (-1)^d \sigma\left(y_j\right)\sqrt{h}, \\ y^{\sharp}_{2,d} = y_j + hf\left(y_j, \left[\tilde{c}^{s}_{1,j}, c_2,\right]\right) + (-1)^d \sigma\left(y_j\right)\sqrt{h},\ d = 1, 2, \end{cases}$$

for $j = 1, ..., Q$, where $\tilde{c}^{s}_{i,j} \equiv \tilde{c}^{h,s}_{i,j}\left(y_j\right)$ denotes the $j$th-component of $\mathrm{C}^s$. This scheme is iterated until a convergence criterion is satisfied, that is, $\left\|\mathrm{C}^{s+1} - \mathrm{C}^s\right\| < \epsilon_1$, for a given $\epsilon_1 > 0$ ($\|\cdot\|$ is the Euclidean norm). A candidate for feedback Nash equilibrium, for the value functions $\mathrm{V}_i^r$, $i = 1, 2$, is obtained as $\mathrm{C}^{s+1} = \left[\mathrm{C}_1^{s+1}, \mathrm{C}_2^{s+1}\right]$.

2. *Value Iteration* (VI). Given $\mathrm{C}^{s+1}$ obtained from the GI loop, the value functions at step $r+1$ are updated as follows:

$$\mathrm{V}_i^{r+1} = T_i\left(\mathrm{V}_i^r\right),\ i = 1, 2,$$

where $T_i = \left[T_{i,j}\right]$, $i = 1, 2$, are defined in (15). The VI loop is defined by the scheme

$$\begin{cases} V^{r+1}_{1,,j} \equiv h\left(U_1\left(y_j\right) - D_1\left(\tilde{c}^{s+1}_{1,j}\right)\right) + \frac{(1-\rho_1 h)}{2}\sum_{d=1}^{2} RBF\left[\mathrm{V}_1^r\right]\left(y^{\sharp}_{1,d}\right), \\ V^{r+1}_{2,j} \equiv h\left(U_2\left(y_j\right) - D_2\left(\tilde{c}^{s+1}_{2,j}\right)\right) + \frac{(1-\rho_2 h)}{2}\sum_{d=1}^{2} RBF\left[\mathrm{V}_2^r\right]\left(y^{\sharp}_{2,d}\right), \\ y^{\sharp}_{1,d} \equiv y^{\sharp}_{2,d} = y_j + hf\left(y_j, \left[\tilde{c}^{s+1}_{1,j}, \tilde{c}^{s+1}_{2,j}\right]\right) + (-1)^d \sigma\left(y_j\right)\sqrt{h},\ d = 1, 2. \end{cases}$$

This scheme is iterated until satisfying the convergence criterion $\left\|\mathrm{V}^{r+1} - \mathrm{V}^r\right\| < \epsilon_2$, with $\epsilon_2 > 0$ given. A new candidate for the value functions is thus obtained, $\mathrm{V}^{r+1} = \left[\mathrm{V}_1^{r+1}, \mathrm{V}_2^{r+1}\right]$.

Once convergence is reached, the algorithm renders the outputs $\mathrm{C}^{\heartsuit} = \left[\mathrm{C}_1^{\heartsuit}, \mathrm{C}_2^{\heartsuit}\right]$, $\mathrm{V}^{\heartsuit} = \left[\mathrm{V}_1^{\heartsuit}, \mathrm{V}_2^{\heartsuit}\right]$, which constitute the computational solutions for the value functions and control policies of the couple's problem. Once $\left(\mathrm{C}^{\heartsuit}, \mathrm{V}^{\heartsuit}\right)$ are obtained, we can recover the corresponding approximated feedback maps defined in (12).

For the purpose of our model analysis below, we take $f\left(y, \left[c_1, c_2\right]\right) = -ry + a_1 c_1 + a_2 c_2$, and $\sigma\left(y_j\right) \equiv \sigma$ constant.

## 4 | NUMERICAL ANALYSIS

We present here the numerical analysis of the couple's problem defined in section 3 for the functional structure and parameter values given in Table 1. Notice that these model inputs satisfy all assumptions specified in section 2. Furthermore, it is a convenient choice for the sake of comparison with previous works, i.e.[17] and[28], where the same set of inputs are considered. The algorithm code has been written and run in MATLAB[36]. The set of parameter values used in the computational experiments below are $h = 1/12$, $X = [0, 5]$, $\tilde{X} = \left\{ y_j := j \frac{\text{length (X)}}{Q-1} \right\}_{j=0,1,\ldots,Q-1} \subset X$, $Q = 15$, $\theta = 0.05$, $\epsilon_1 = 0.001$, $\epsilon_2 = 0.0001$, and $\nu = 0.5$. Our routine is initialized with the admissible array of controls $\mathrm{C}_i^0 = [c_i^*, \ldots, c_i^*]^T$ and of values $\mathrm{V}_i^0 = [hU_i(y_1), \ldots, hU_i(y_Q)]^T$, for $i = 1, 2$, where $c_i^*$ and $U_i$ are the model inputs introduced in section 2.

**Table 1** Model inputs: functions and parameters.

<table>
<tr><th></th><th>$r$</th><th>$a_1$</th><th>$a_2$</th><th>$\sigma$</th><th>$D_i$</th><th>$c_i^*$</th><th>$U_i$</th><th>$\rho_i$</th></tr>
<tr><td rowspan="4">Homogamous</td><td rowspan="8">$-2$</td><td rowspan="4">1</td><td rowspan="4">1</td><td>1.75</td><td rowspan="8">$\frac{(c_i - c_i^*)^2}{2}$</td><td rowspan="8">0.2</td><td rowspan="8">$5 \ln(x+1)$</td><td rowspan="8">0.1</td></tr>
<tr><td>1.25</td></tr>
<tr><td>0.5</td></tr>
<tr><td>0</td></tr>
<tr><td rowspan="4">Heteterogamous</td><td rowspan="4">1.75</td><td rowspan="4">1</td><td>1.75</td></tr>
<tr><td>1.25</td></tr>
<tr><td>0.5</td></tr>
<tr><td>0</td></tr>
</table>

### 4.1 | Preliminary analysis: the effect of uncertainty

In Figures 3 and 4, we show the effort feedback policies and the value (well-being) functions for each partner, for two types of couples, homogamous and heterogamous, respectively. They differ here only in the effort efficiency of each partner, which is represented by $a_1$ and $a_2$. Homogamous couples are formed by partners with $a_1 = a_2$, otherwise, they are heterogamous. Different implications of this asymmetry are discussed in detail in[28]. The effort and value curves in Figures 3 and 4 correspond to different levels of stochasticity, i.e. $\sigma = 0.5, 1.25, 1.75$. The curves corresponding to the deterministic case ($\sigma = 0$) are also provided, so our results can be compared with those in[28], where the non-stochastic case is analyzed. It allows us to study the impact of stochasticity on effort policies and well-being compared with the benchmark case of deterministic feeling dynamics.

It follows from the analysis that, as the uncertainty, $\sigma$, about the actual state of the relationship increases, both partners' effort curves monotonically shift upwards and their welfare curves (value functions) shift downwards. As a consequence, in the face of uncertainty, couples must make more effort and expect less reward in terms of well-being. This qualitative effect holds in general for both homogamous and heterogamous couples, as Figures 3 and 4 show.

### 4.2 | Love at Risk

To assess the probability of the breakup of a romantic relationship we now pay attention to the model parameter $x_{min}$ below which the feeling variable must remain to guarantee a sufficiently rewarding relationship –see section 1. This is the threshold feeling level for the relationship to start facing a risk of breakup (see Figure 1). A suitable version of that threshold value can be thought of as a “value at risk", which is defined in finance as a probabilistic measure of incurring a given loss[29]. In a similar

**Figure 3** Computational feedback analysis of a homogamous couple ($a_1 = a_2 = 1$) at different $\sigma$ values.

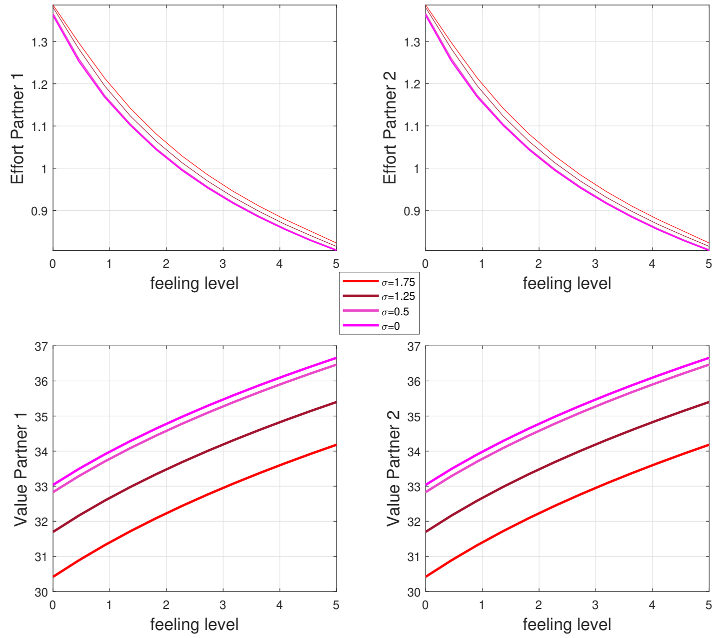
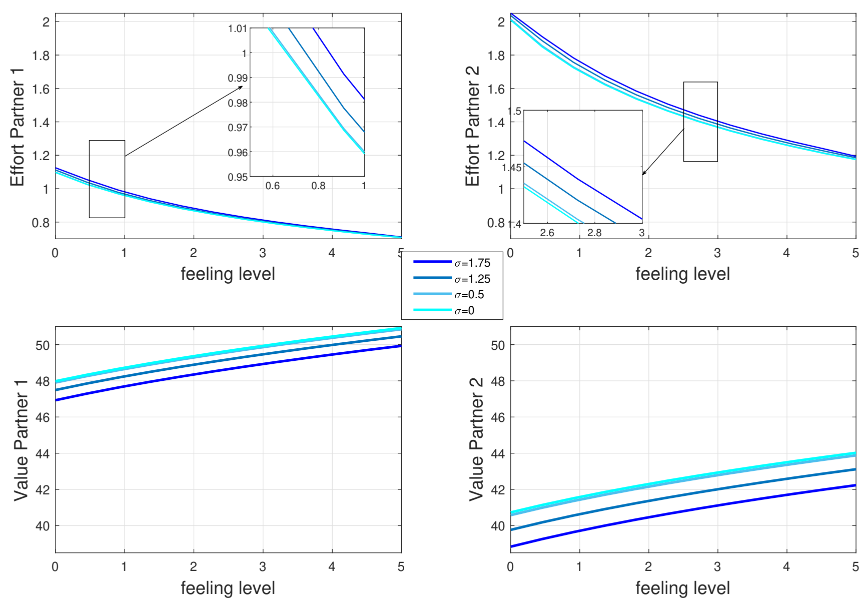



**Figure 4** Computational feedback analysis of a heterogamous couple ($a_1 = 1, a_2 = 1.75$) at different $\sigma$ values.




fashion, for a given probability $\alpha \in (0, 1)$, we define *Love at Risk* (LaR) (at time $k > 0$) as the feeling value such that

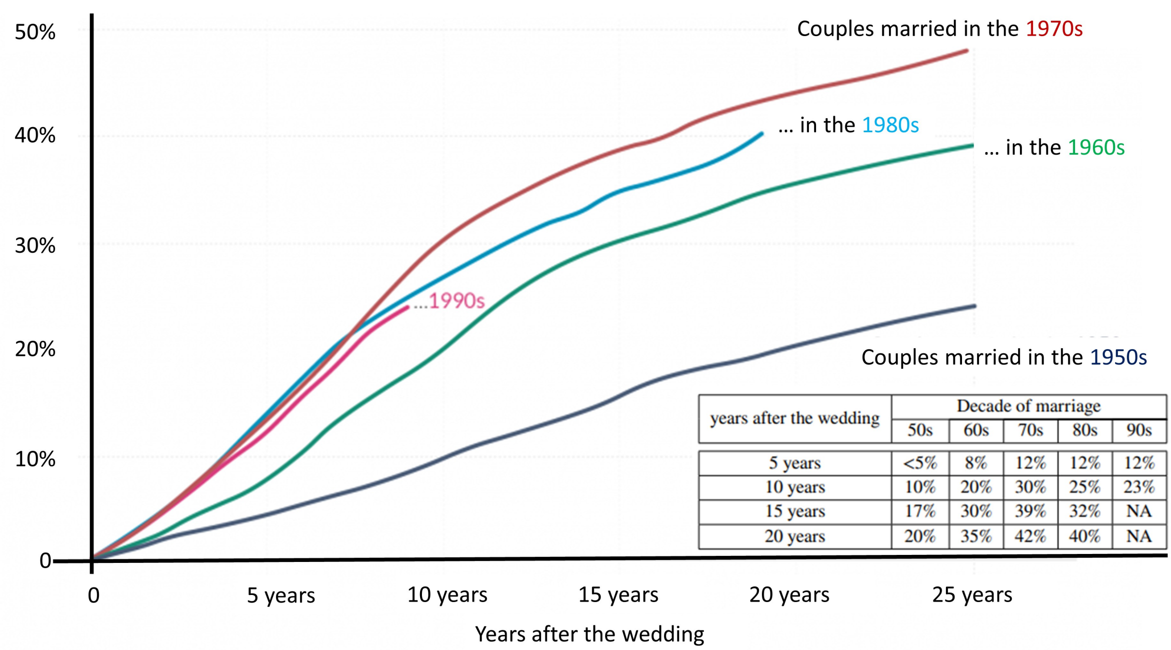
$$\mathbb{P}\left(x_k^{\heartsuit} \leq \mathrm{LaR}\right) = \alpha$$

where $x_k^{\heartsuit}$ is the (optimal) solution of the computational couple's problem defined in section 3, and $\mathbb{P}$ is its probability function, so that LaR is the $\alpha$-percentile of the distribution of $x_k^{\heartsuit}$.

In order to illustrate our methodology, we consider realistic estimates of the probability of divorce in the US. They are shown in Figure 5, where different values of $\alpha = \alpha(k)$ are given, for different cohorts of marriages, $k$ months after the wedding, for $k = 60, 120, 180, 240$.

We consider a heterogamous marriage, as specified in Table 1, which may be facing a certain uncertainty $\sigma$ in their feeling dynamics (1). To estimate the value LaR for such kind of marriage, five years after the wedding, we proceed as follows. We compute a large sample of realizations of the optimal solution $x^{\heartsuit}(k)$ for the computational stochastic model in section 3, and for $\sigma = 0, 0.5, 1.25, 1.75$. Given that the time variable $k$ in our computational model corresponds to months, we generate an estimate of the probability densities of the random variable $x^*(60)$ for the different values of $\sigma$. They are displayed in Figure 6. According to Figure 5, $\alpha(k = 60) \approx 0.10$, on average, over the marriage cohorts. The LaR level at five years can thus be estimated as the first decile of the feeling distribution corresponding to each $\sigma$ value in Figure 6.

**Figure 5** Share of marriages ending in divorce in the US: percentage of straight couples who divorced after a given number of years of marriage (Source: Our World in Data and[37]).



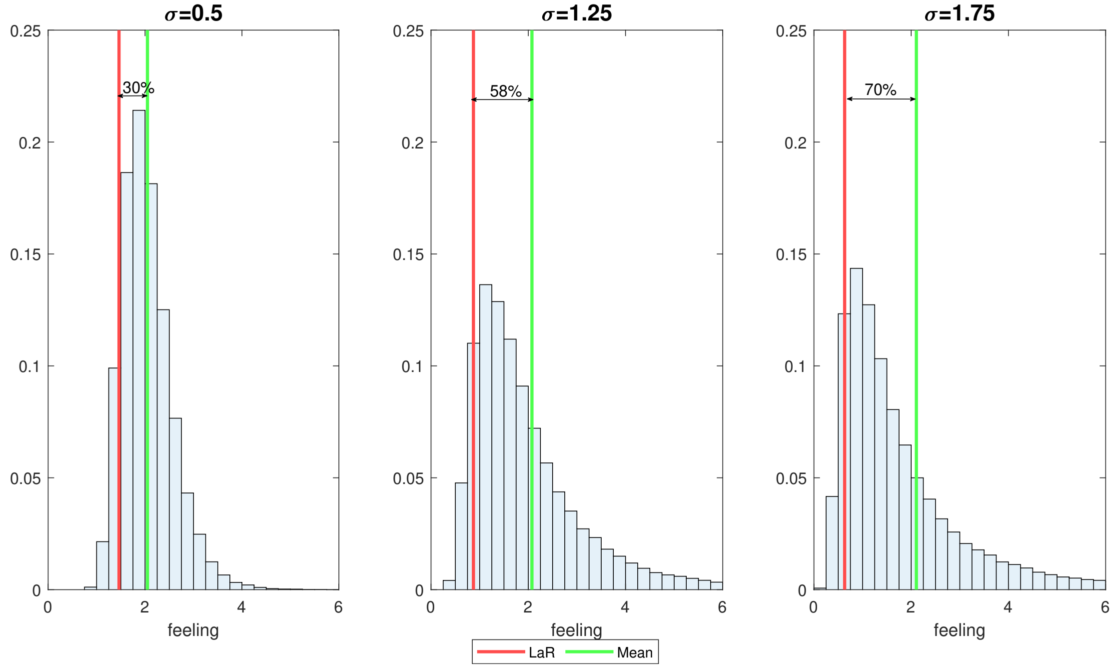

| years after the wedding | Decade of marriage | | | | |
|---|---|---|---|---|---|
| | 50s | 60s | 70s | 80s | 90s |
| 5 years | <5% | 8% | 12% | 12% | 12% |
| 10 years | 10% | 20% | 30% | 25% | 23% |
| 15 years | 17% | 30% | 39% | 32% | NA |
| 20 years | 20% | 35% | 42% | 40% | NA |

In general, the LaR level fluctuates with the type of couple, the time after the wedding, and the noise term in the feeling dynamics. For the heterogamous couple under consideration, it is apparent from Figure 6 that the LaR level after five years decreases as $\sigma$ increases, both in absolute value and relative to the mean of the feeling values.

Notice that both the probability estimates of rupture $\alpha = \alpha(k)$ in the US, given in Figure 5, and the distribution of the (controlled) feeling variable $x^{\heartsuit} = x_k^{\heartsuit}$ of the couple's problem vary with $k$. As a consequence, the LaR level also varies with $k$, LaR=LaR($k$), and it can be estimated in a dynamic fashion using our computational model. To obtain the sequence LaR($k$) for the different levels of uncertainty $\sigma$, we proceed as follows. Given $\sigma$, we generate the distribution of the feeling levels $x_k^{\heartsuit}$ for each $k$ from a sample of 10 000 realizations of the following stochastic numerical scheme obtained in section 3:

$$\text{(SM1)} \begin{cases} c_{i,k} = \tilde{S}_i^h\left(x_k\right),\ i = 1, 2, \\ x_{k+1} = x_k + hf(x_k, c_{1,k}, c_{2,k}) + \sqrt{h}\sigma\xi_k, \\ x_0 \in X. \end{cases}$$

**Figure 6** Love at Risk (LaR) for a heterogamous couple with $a_1 = 1$,$a_2 = 1.75$ at $k = 60$ for different $\sigma$ values. Empirical densities are obtained from a sample of 10 000 feeling trajectories. The gap –in relative terms– between the LaR and the mean feeling level of the process is also provided, for each value of $\sigma$.




The scheme above defines a stabilization mechanism for the relationship since the control policies, computed by the stochastic feedback Nash maps, allow partners to react optimally to perturbations of the feeling at any time. We note that the feedback control maps $\tilde{S}_i^h$ in (SM1) are defined for any $x \in X$ using $RBF$ approximation, in the same fashion as in Section 3 –see (14). That is, $\tilde{S}_i^h(x)$ is computed for any $x \in X$ from the array of controls $C_i^{\heartsuit}$ as $RBF\left[C_i^{\heartsuit}\right](x) \equiv \sum_{j=1}^{Q} \mu_{i,j}\Phi\left(\left\|x - y_j\right\|\right)$, being $\bar{\mu}_i = \left[\mu_{i,1}, ..., \mu_{i,Q}\right]^T$, for $i = 1, 2$, a vector of weighting coefficients obtained from $A\bar{\mu}_i = C_i^{\heartsuit}$, where A is the matrix with entries $A_{j,l} = \Phi\left(\left\|y_l - y_j\right\|\right)$, $j, l = 1, ..., Q$.

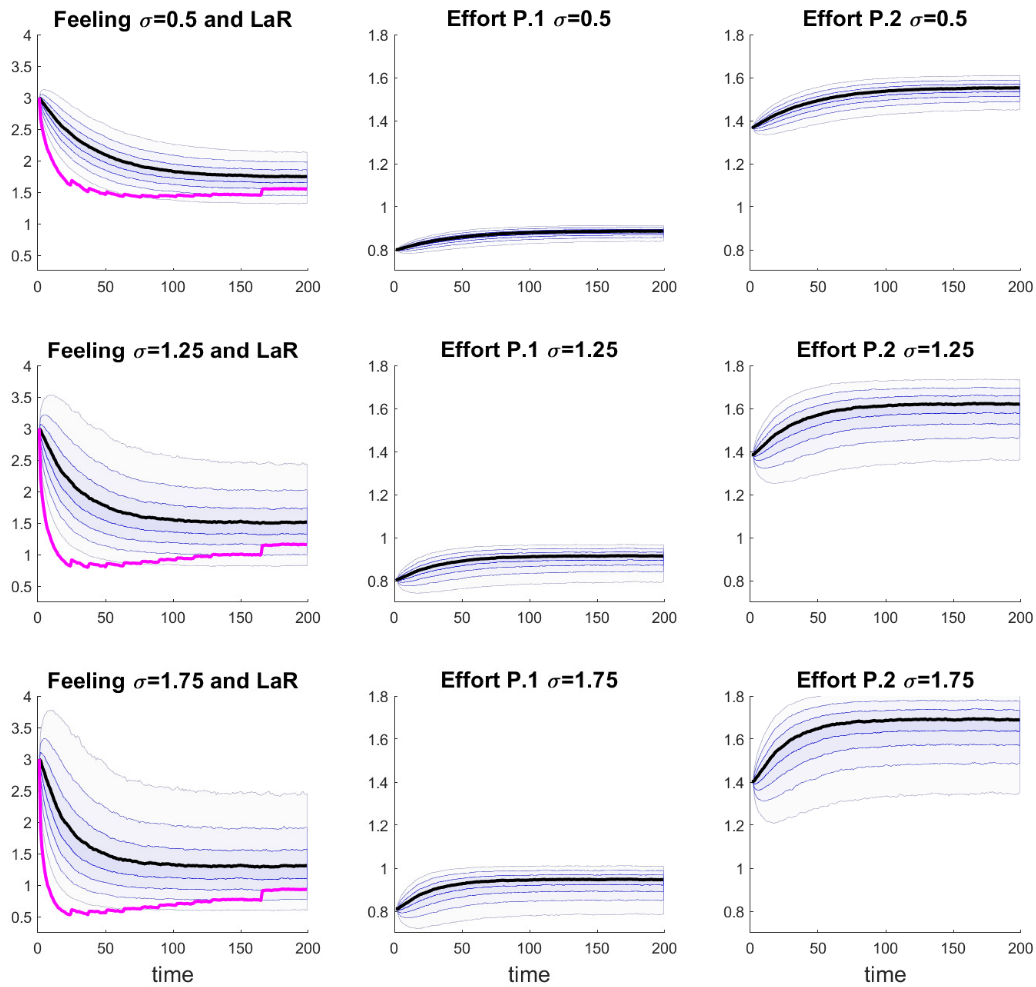
Once the distribution $x_k^{\heartsuit}$ is simulated, we estimate LaR($k$) from the condition $\mathbb{P}(x_k^{\heartsuit} \le \text{LaR}(k)) = \alpha(k)$, where the probability values $\alpha(k)$ are obtained from the data source of Figure 5.

In Figure 7 we show the simulation of the model above for the heterogamous couple under study and for initial feeling $x_0 = 3$. Different percentile trajectories (from 10 to 80) of the feeling variable $x_k^{\heartsuit}$ for the different levels of $\sigma$, as well as the corresponding effort trajectories of each partner, are displayed in the figure. The curve in pink corresponds to the dynamic LaR levels estimated by the computational model. As in the static exercise above ($k = 60$), it can be seen that, for every $k > 0$, the LaR curves are convex, monotonically decreasing as $\sigma$ increases, and they eventually approach a stationary value. Our numerical experiment allows us to obtain synthetic long-term feeling trajectories for different couples in a variety of uncertainty scenarios. It is worth mentioning that they seem to be a good approximation to the feeling trajectories of real couples. In fact, longitudinal studies using survey data –see e.g. p.149 in[38] and Figure 4 in[39]– find that the typical trajectories of marital happiness, a proxy of our feeling variable, show a steep decline at first and then a tendency to stabilize, like the feeling curves in Figure 7. This is a piece of evidence that our model can reproduce stylized facts that other researchers find empirically.

## 4.3 | Estimating the probability of breakup in the face of a shock

Regarding the odds of survival of a relationship whose evolution is described by our control model, we may also analyze how the couple reacts optimally in the face of a shock. This is a relevant question since relationships are subjected to external shocks over the life course (see e.g.[40]). Notice that the feedback control mechanism provided by our analysis in section 3 is particularly

**Figure 7** Simulations of the stochastic process $x^{\heartsuit}(k)$ of the feeling for different values of $\sigma$, together with the corresponding optimal effort trajectories of both partners, for a heterogamous couple with $a_1 = 1$ and $a_2 = 1.75$. The sequence LaR$(k)$ is plotted in pink, for the different $\sigma$ values. Values of $\alpha(k)$ are approximated using data from Table (2). The trajectories corresponding to the 10th to 80th percentiles of the process are also shown in the graphs (the trajectory in black corresponds to the mean value).




useful here, since it allows partners to adjust their effort levels after a perturbation of the feeling to drive it back to a successful path.

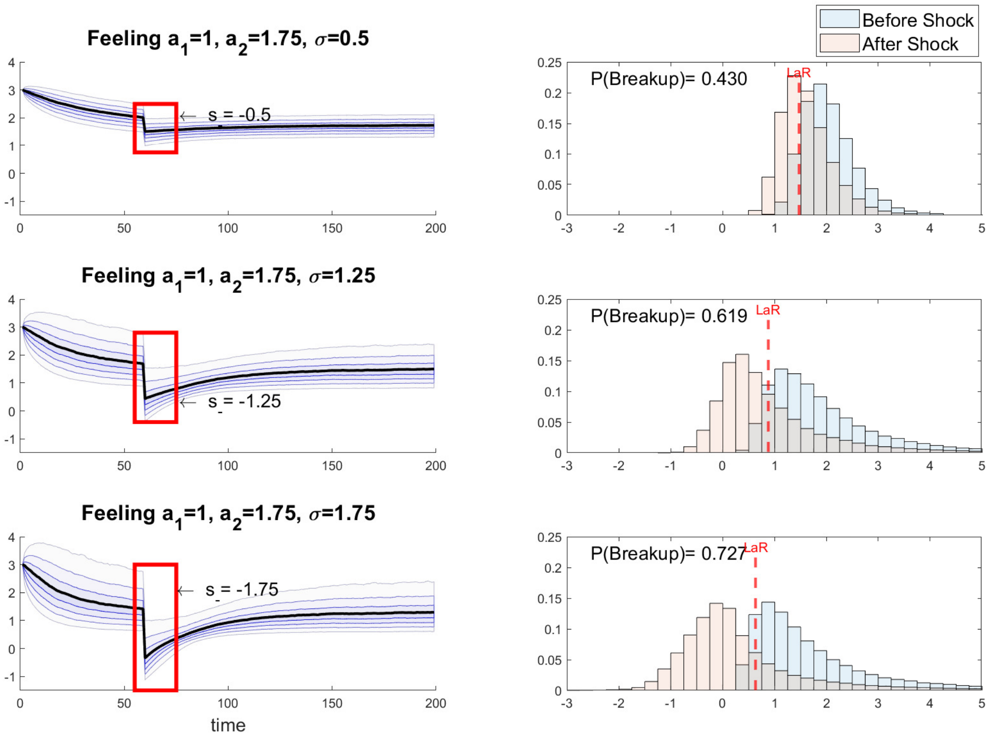
We address the shock problem by estimating the change in the probability of breakup after a shock of the feeling has occurred at a given time $k > 0$. In general, the feeling may be affected by a certain sequence of different shocks $s_- = \{s_k\}_{k\geq 0}$. Then the stabilization mechanism provided by the feedback analysis reads as follows

$$\text{(SM2)} \begin{cases} c_{i,k} = \tilde{S}_i^h\left(x_k\right),\ i = 1, 2, \\ x_{k+1} = x_k + hf(x_k, c_{1,k}, c_{2,k}) + \sqrt{h}\sigma\xi_k + s_k, \\ x_0 \in X. \end{cases}$$

Again, the feedback maps $\tilde{S}_i^h$ can be computed here for any $x \in X$ using $RBF$ approximation, as described above for the scheme (SM1).

Even though the stabilization mechanism (SM2) is working, the perturbed feeling trajectory may enter the zone of risk of breakup at a certain moment $k$ (that is, below the level LaR$(k)$ with some probability), and then remain within the risk zone for some time. This is a critical period that can be painful, or even impossible to get through, so it can eventually cause the relationship to break up. Thus the probability that a perturbed trajectory controlled by (SM2) spends a certain period below the curve LaR($k$) –see Figure 7– serves as a measure of the risk to the survival of the relationship. This probability can be estimated from an ensemble of realizations of the process steered by (SM2).

**Figure 8** Left: Feedback response to a one-period negative shock of size $s_{-}$, proportional to $\sigma$, five years after the wedding ($k = 60$) for a heterogamous couple with $a_1 = 1, a_2 = 1.75$, and for $\sigma = 0.5, 1.25, 1.75$. Feeling trajectories are obtained using the numerical scheme (SM2). Right: Empirical distribution for the feeling variable obtained from a sample of 10 000 trajectories before the shock ($k = 60$) and over one year after the shock ($60 < k \leq 72$) for the different values of $\sigma$. The values LaR correspond to the unperturbed process, as shown in Figure 6.




To illustrate the method described above, consider the case that $s_{-}$ consists of a large one-period shock (of size $\sigma$) taking place five years after the wedding ($k = 60$). In Figure 8 (left) we show the percentile trajectories of the stochastic process steered by the stabilization mechanism (SM2) for a particular heterogamous couple and for different values of $\sigma$. They coincide with the corresponding trajectories of Figure 7 before the shock at $k = 60$. Computing a large ensemble of trajectories, we produce an estimate of the distribution of the feeling values for the perturbed process over a whole year ($60 < k \leq 72$) after the shock. In Figure 8 (right) we show the empirical distributions of the feeling variable before the shock and over one year after the shock. Using the LaR level at $k = 60$ as the benchmark, the probability of breakup over a year after the shock can be estimated from the empirical distribution after the shock (in pink) for different values of $\sigma$. As shown in Figure 8 (right), given that the one-period shock is of size $\sigma$, the probability of breakup over one year after the event increases significantly as $\sigma$ increases.

We also analyze how the probability of breakup after a shock varies with respect to the size of the shock and the uncertainty of the feeling dynamics. For the same type of heterogamous couple considered above, Table 2 shows the probabilities of breakup for different values of $\sigma$ and different sizes of a one-period shock occurring five years after the wedding. Our estimates show that, for any level $\sigma$, the probability of breakup increases as the size of the shock increases. Also, a higher level of uncertainty entails a lower LaR level and, in addition, it makes it more likely that the level of feeling remains in the secure zone (i.e. over LaR).

**Table 2** Probability of breakup of a heterogamous relationship with $a_1 = 1, a_2 = 1.75$ for different uncertainty levels $\sigma$ and a -period shock $s_-$ of different sizes, five years after the wedding ($k = 60$). The simulation is obtained using the scheme (SM2).

| $\sigma$ | LaR | $s_-$ | $\mathbb{P}\,(breakup)$ |
|---|---|---|---|
| 0.5 | 1.51 | −0.1 | 0.1488 |
| | | −0.5 | 0.4304 |
| | | −1.25 | 0.8636 |
| | | −1.75 | 0.9593 |
| 1.25 | 1.02 | −0.1 | 0.1334 |
| | | −0.5 | 0.3080 |
| | | −1.25 | 0.6195 |
| | | −1.75 | 0.7505 |
| 1.75 | 0.77 | −0.1 | 0.1339 |
| | | −0.5 | 0.3135 |
| | | −1.25 | 0.6114 |
| | | −1.75 | 0.7266 |

## 5 | CONCLUSIONS

In this article, we have introduced an algorithm to find feedback Nash equilibria for a class of stochastic differential games. The algorithm extends the idea of a previous scheme for deterministic games (called RaBVItG) to a stochastic environment. It builds on a combination of two iterations: a first one to find the Nash equilibrium by fixing the value of the game, and a second iteration to find the value of the game given a Nash equilibrium. The algorithm can be applied to a general class of $N$-player infinite horizon stochastic games. We have also addressed a substantial issue in the applied sciences, namely the design of a happy long-term romantic relationship. We formulate this problem as a two-person optimal control problem to govern the feeling of the relationship in a stochastic environment. The algorithm allows us to find approximate solutions to a computational version of the control problem for different stochastic dynamics. In particular, we have focused on estimating the risk of the breakup of a long-term relationship at a certain time after the initial commitment. Using divorce data in the US, the proposed algorithm gives an estimate of the feeling level at different times below which the relationship can probably break up -called Love at Risk here. Also, the computational model allows us to estimate the probability of breaking up in the face of an external shock. The numerical analysis can be applied to different types of couples and different levels of stochasticity in the feeling dynamics. While actual breakup data is used to synthesize a stochastic process that allows us to estimate the LaR parameter, no other data is used in the article. Once the model has been calibrated using longitudinal data on marital quality, the methodology presented in the article can be used to provide accurate estimates of Love at Risk for romantic relationships. This seems an intriguing topic for future work.

# APPENDIX

We present here numerical evidence of the convergence and accuracy of the stochastic RaBVItG algorithm. Below we first consider two benchmark problems with known analytical solutions. These are the problems discussed in[14] but adapted to an infinite-horizon context. We then perform a further test that serves to evaluate the accuracy of the algorithm in solving our problem, i.e. we show that the numerical value functions rendered by the feedback scheme of the algorithm satisfy the discretized versions of the HJB equations. We thank an anonymous referee for suggesting this numerical test.

All experiments were performed on a computer with an Intel(R) Core(TM) i7-4600U CPU and 4 Gb of RAM. Additionally, we used the following set of parameters: $\theta = 0.05$, $\epsilon_1 = 1 \times 10^{-4}, \epsilon_2 = h^2$.

## Test 1: Stochastic Linear Quadratic

Consider a two-player SDG such that the $i$-th player seeks to determine $u_i$ that minimizes the following quadratic cost functional

$$J_i(y, u_i) = \mathbb{E}\left\{ \int_0^\infty \left[q_i x(t)^2 + r_i u_i(t)^2\right] \mathrm{d}t | x(0) = y \right\},\ i = 1, 2,$$

subject to the stochastic linear differential equation

$$\mathrm{d}x(t) = \left[ax(t) + b_1 u_1(t) + b_2 u_2(t)\right] \mathrm{d}t + \sigma x(t) \mathrm{d}w(t),$$

where $x(0) = y$ and $y \in \mathbb{R}$, and $q_i$, $r_i$, $b_i$, for $i = 1, 2$, are parameters.

According to[41], the analytic feedback Nash equilibrium for this game is defined by the linear expressions

$$u_1^*(t) = k_1 x(t),\ u_2^*(t) = k_2 x(t),$$

where $k_1, k_2$ are solutions to the following scalar Riccati algebraic equations:

$$2p_1(a + b_2 k_2) + q_1 + \sigma^2 p_1 + k_1 b_1 p_1 = 0,\ k_1 = -\frac{1}{r_1} b_1 p_1,$$
$$2p_2(a + b_1 k_1) + q_2 + \sigma^2 p_2 + k_2 b_2 p_2 = 0,\ k_2 = -\frac{1}{r_2} b_2 p_{2.}$$

In Table 3, we present the accuracy results of our algorithm for this particular problem. We consider here $X = [0, 1]$ and $Q = 5$. Also, $\nu = 0.5$. The set of model parameter values is as follows: $a = 2, b_1 = b_2 = 1, q_1 = q_2 = 4, r_1 = r_2 = 1$. We show the error values for different time steps, $h =\in \{0.2, 0.1, 0.05, 0.001\}$, and different stochasticity levels, $\sigma \in \{0.1, 0.3, 0.7\}$. The error values are given by

$$L^\infty - \text{error} := \max_{y_j \in \tilde{X}} \left[|u_1^*(y_j) - \tilde{c}_1^h(y_j)|, |u_2^*(y_j) - \tilde{c}_2^h(y_j)|\right], \tag{17}$$

where $\mathrm{C}_i^\heartsuit = \left[\tilde{c}_i^h\left(y_1\right), ..., \tilde{c}_i^h\left(y_Q\right)\right],\ i = 1, 2$, are the arrays of optimal controls computed by the algorithm. The error values decrease as the time step gets smaller, and it is not significantly affected by the value of $\sigma$. We also report the average CPU time obtained for each experiment. The good error values reported in the Table serve to illustrate the relative precision of the numerical solutions. Additionally, the computational times are all below 1 second.

As expected, the algorithm does not always converge. For some experiments (not reported here), with $h > 0.1$ or $\sigma > 0.9$, convergence is not attained in general. The values $\sigma$ and $h$ must be studied a priori to achieve the convergence of the algorithm.

## Test 2: Vidale-Wolfe advertising Model

In the literature on differential games for marketing problems (see[42]), the Vidale-Wolfe model is used to find feedback advertising policies of minimal cost. In the infinite-horizon stochastic version, the following cost functions are considered:

$$J_1(x_0, u_1) = \mathbb{E}\left\{ \int_0^\infty e^{-\rho_1 t} \left[m_1 x(t) + c_1 u_1(t)^2\right] \mathrm{d}t | x(0) = x_0 \right\},$$

**Table 3** Results obtained with RaBVItG for a stochastic linear quadratic problem. In each cell, we report two values for the corresponding experiment: (top) $L^\infty$–error (17) and (bottom) number of iterations until convergence.

| | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | $h = 0.01$ |
|---|---|---|---|---|
| $\sigma = 0.1$ | 0.81<br>9 | 0.19<br>29 | 0.09<br>42 | 0.02<br>210 |
| $\sigma = 0.3$ | 0.82<br>10 | 0.19<br>27 | 0.09<br>42 | 0.02<br>188 |
| $\sigma = 0.7$ | 0.79<br>15 | 0.20<br>27 | 0.10<br>48 | 0.02<br>195 |
| Avg. CPU time (s) | 0.44 | 0.52 | 0.69 | 0.75 |

and

$$J_2(y_0, u_2) = \mathbb{E}\left\{ \int_0^\infty e^{-\rho_2 t} \left[ m_2 y(t) + c_2 u_2(t)^2 \right] \mathrm{d}t | y(0) = y_0 \right\},$$

subject to the stochastic dynamical system –here $x(t)$ and $y(t)$ are the market shares of each player at time $t \geq 0$–

$$\mathrm{d}x = \left[ r_1 u_1 \sqrt{1-x} - r_2 u_2 \sqrt{x} - \delta(x-y) \right] \mathrm{d}t + \sigma \sqrt{xy} \mathrm{d}w,$$
$$\mathrm{d}y = \left[ r_2 u_2 \sqrt{1-y} - r_1 u_1 \sqrt{y} - \delta(y-x) \right] \mathrm{d}t - \sigma \sqrt{xy} \mathrm{d}w,$$

where $x(0) = x_0$, $y(0) = 1 - x_0$ with $0 \leq x_0 \leq 1$ and $x(t) + y(t) = 1$ for all $t \geq 0$, and $\rho_i, m_i, c_i, r_i$, $i = 1, 2$, and $\delta > 0$ are parameters.

According to [10], the analytic solution is given by

$$u_1^*(x) = \beta_1 r_1 \frac{\sqrt{1-x}}{2c_1}, \quad u_2^*(y) = \beta_2 r_2 \frac{\sqrt{1-y}}{2c_2},$$

where, in our case, $\beta_i = \frac{\sqrt{(\rho_i + 2\delta)^2 + \frac{12 r_i^2}{4c_i} m_i} - (\rho_i + 2\delta)}{\frac{3r_i^2}{2c_i}}$. For our numerical test, we use the following parameter values (similar to those chosen in the literature): $\rho_1 = \rho_2 = 0.1, m_1 = m_2 = 1, c_1 = c_2 = 0.1, r_1 = r_2 = 1$ and $\delta = 0.5$. We test the performance of the algorithm for three levels of stochasticity $\sigma \in \{0.5, 1, 1.5\}$. Additionally, $x, y \in [0, 1]$, with $Q = 5$. Also, we use $\nu = 0.1$ Notice that the explicit solution is independent of $\sigma$.

In Table 4, we report the results of the experiments in the same format as Table 3.

Test 2 again illustrates the accuracy of the numerical solutions obtained by the algorithm. Error estimates are already small for relatively high values of the time step $h$ and they decrease as $h$ approaches zero. Also, the accuracy does not depend on the level of stochasticity. We note that for some experiments (not reported here), with $\sigma > 2$ or $h > 0.2$, the algorithm does not converge.

**Table 4** Results obtained with RaBVItG for a version of the Vidale-Wolfe advertising model. In each cell, we report two values for the corresponding experiment: (top) the value of the $L^\infty$–error (17) and (bottom) the number of iterations until convergence.

| | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | $h = 0.01$ |
|---|---|---|---|---|
| $\sigma = 0.5$ | 0.0087<br>6 | 0.0109<br>12 | 0.0062<br>25 | 0.0043<br>121 |
| $\sigma = 1$ | 0.015<br>6 | 0.0085<br>11 | 0.0032<br>24 | 0.0047<br>121 |
| $\sigma = 1.5$ | 0.013<br>11 | 0.0052<br>12 | 0.0035<br>24 | 0.0046<br>120 |
| Avg CPU time (s) | 6.4 | 9.3 | 19.2 | 64.7 |

## Test 3: Double-checking the value functions.

When analytical solutions are not available, an alternative way to test the accuracy of the algorithm consists of double-checking the value functions of the problem. That is, checking whether the approximate values of the problem given by (7) using the feedback scheme (SM1) –see section 4.2– satisfy the discretized versions (11) of the HJB equations.

We thus compute the following sample approximations of (7):

$$W_i^h\left(c_i^h\right) \approx h\sum_{k=0}^{T} e^{-\rho_i t_k}\left[\frac{1}{S}\sum_{j=1}^{S}\left(U_i\left(x_{k,j}\right)-D_i\left(c_{i,k,j}\right)\right)\right],\ i=1,2, \tag{18}$$

where, for each $j=1,...,S$, $x_{k,j}$ and $c_{i,k,j}$ obey (SM1), that is,

$$\begin{cases} x_{k+1,j} = x_{k,j} + h\left(-rx_{k,j} + a_1 c_{1,k,j} + a_2 c_{2,k,j}\right) + \sigma\left(x_{k,j}\right)\sqrt{h}\xi_{k,j}, \\ c_{i,k,j} = \tilde{S}_i^h\left(x_{k,j}\right),\ i=1,2, \\ x_{0,j} = y, \end{cases}$$

with $\xi_{k,j} \to N(0,h)$ and $y \in X$. Notice that the expectation in (7) is replaced by a (large) sample average of $S$ realizations of the scheme (SM1). We thus expect that, given $y \in X$, the value of the problem $W_i^h\left(c_i^h\right)$ obtained as described above gives a good approximation of the value $\tilde{v}_i^h(y)$ obtained by RaBVItG from (11), provided that $T$ is sufficiently large.

In Table 5, we show the results of the numerical test for a particular choice of $y$ and different values of $\sigma$. We report the values $W_i^h$ for $i=1,2$, obtained for a set of increasing time horizons $T$ ($h$ fixed) and, in brackets, the percentage error of each value with respect to the target values $\tilde{v}_i^h(y)$ -shown in the last column of the table. As the numbers in the table show, the discrepancy between both values becomes very small as the time horizon gets large. Again, the experiment shows that errors do not vary significantly as the stochasticity of the dynamics increases. Test 3 thus provides further numerical evidence for the accuracy of the algorithm presented in this paper.

**Table 5** Results of the numerical double-check of the value functions. Comparison of $W_i^h\left(c_i^h\right)$ (for a increasing sequence of $T$) with $\tilde{v}_i^h(y)$, $i=1,2$, for $y=3$ and for different levels of stochasticity $\sigma$. We report -in brackets- the percentage error of approximation for each trial. We take $S=1000$, $r=2$, $a_1=1$, $a_2=1.75$, $\rho_1=\rho_2=0.10$, $h=1/12$, $\nu=0.5$.

| $i$ | $y$ | $\sigma$ | $T=2000$ | $T=5000$ | $T=10000$ | $T=20000$ | $\tilde{v}_i^h(y)$ |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 0.5 | 43.08 (13.7%) | 49.39 (1.1%) | 49.71 (0.5%) | 49.73 (0.44%) | 49.95 |
| 2 | | | 37.40 (12.8%) | 42.78 (0.25%) | 43.01 (0.27%) | 43.00 (0.25%) | 42.89 |
| 1 | | 1.25 | 42.58 (13.9%) | 48.86 (1.2%) | 49.15 (0.62%) | 49.20 (0.52%) | 49.46 |
| 2 | | | 36.56 (12.9%) | 41.92 (0.19%) | 42.07 (0.16%) | 42.07 (0.16%) | 42 |
| 1 | | 1.75 | 41.89 (14.3%) | 48.10 (1.6%) | 48.37 (1.1%) | 48.44 (0.9%) | 48.92 |
| 2 | | | 35.58 (13.5%) | 40.84 (0.60%) | 41.05 (0.14%) | 41.13 (0.04%) | 41.11 |

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## References


1. Basar T, Olsder GJ. *Dynamic Noncooperative Game Theory, 2nd Edition*. 23. Society for Industrial and Applied Mathematics . 1998.
2. Friedman A. Stochastic differential games. *Journal of Differential Equations* 1972; 11(1): 79–108.
3. Bensoussan A, Siu CC, Yam SCP, Yang H. A class of non-zero-sum stochastic differential investment and reinsurance games. *Automatica* 2014; 50(8): 2025–2037.
4. Deng C, Zeng X, Zhu H. Non-zero-sum stochastic differential reinsurance and investment games with default risk. *European Journal of Operational Research* 2018; 264(3): 1144–1158.
5. Engwerda J. *LQ Dynamic Optimization and Differential Games*. John Wiley & Sons . 2005.
6. Josa-Fombellida R, Rincón-Zapatero JP. New approach to stochastic optimal control. *Journal of Optimization Theory and Applications* 2007; 135(1): 163–177.
7. Mannucci P. Nonzero-sum stochastic differential games with discontinuous feedback. *SIAM Journal on Control and Optimization* 2004; 43(4): 1222–1233.
8. Marín-Solano J, Shevkoplyas EV. Non-constant discounting and differential games with random time horizon. *Automatica* 2011; 47(12): 2626–2638.
9. Huang J, Leng M, Liang L. Recent developments in dynamic advertising research. *European Journal of Operational Research* 2012; 220(3): 591–609.
10. Prasad A, Sethi SP. Competitive advertising under uncertainty: A stochastic differential game approach. *Journal of Optimization Theory and Applications* 2004; 123(1): 163–185.
11. Sethi SP. Deterministic and stochastic optimization of a dynamic advertising model. *Optimal Control Applications and Methods* 1983; 4(2): 179–184.
12. Han J, Hu R. Deep fictitious play for finding Markovian Nash equilibrium in multi-agent games. In: Proceedings of Machine Learning Research, vol. 107. ; 2020: 221–245.
13. Han J, Hu R, Long J. Convergence of deep fictitious play for stochastic differential games. *Frontiers of Mathematical Finance* 2022; 1(2): 287–319.
14. Nikooeinejad Z, Heydari M. Nash equilibrium approximation of some class of stochastic differential games: A combined Chebyshev spectral collocation method with policy iteration. *Journal of Computational and Applied Mathematics* 2019; 362: 41–54.
15. Zhang K, Yang Z, Başar T. Multi-agent reinforcement learning: A selective overview of theories and algorithms. *Handbook of reinforcement learning and control* 2021: 321–384.
16. Dockner EJ, Jorgensen S, Van Long N, Sorger G. *Differential Games in Economics and Management Science*. Cambridge University Press . 2000.
17. Goudon T, Lafitte P. The lovebirds problem: why solve Hamilton-Jacobi-Bellman equations matters in love affairs. *Acta Applicandae Mathematicae* 2015; 136(1): 147–165.
18. Rey JM. A mathematical model of sentimental dynamics accounting for marital dissolution. *PLoS One* 2010; 5(3): e9881.
19. Herrera J, Ivorra B, Ramos ÁM. An Algorithm for Solving a Class of Multiplayer Feedback-Nash Differential Games. *Mathematical Problems in Engineering* 2019; 2019.
20. Bokanowski O, Falcone M, Ferretti R, Grüne L, Kalise D, Zidani H. Value iteration convergence of"-monotone schemes for stationary Hamilton-Jacobi equations. *Discrete and Continuous Dynamical Systems-Series A* 2015; 35(9): 4041–4070.

21. Powell WB. *Approximate Dynamic Programming: Solving the curses of dimensionality*. 703. John Wiley & Sons . 2007.

22. Tassa Y, Erez T. Least squares solutions of the HJB equation with neural network value-function approximators. *IEEE Transactions on Neural Networks* 2007; 18(4): 1031–1041.

23. Berinde V, Takens F. *Iterative Approximation of Fixed Points*. 1912. Springer . 2007.

24. Coontz S. *Marriage, a history*. New York, Viking . 2005.

25. Kazdin AE. *Encyclopedia of Psychology*. American Psychological Association and Oxford University Press . 2000.

26. Gottman JM, Murray JD, Swanson CC, Tyson R, Swanson KR. *The Mathematics of Marriage: Dynamic Nonlinear Models*. MIT Press . 2005.

27. Rey JM. Sentimental equilibria with optimal control. *Mathematical and Computer Modelling* 2013; 57(7-8): 1965–1969.

28. Herrera J, Rey JM. Controlling forever love. *PLoS One* 2021; 16(12): e0260529.

29. Duffie D, Pan J. An overview of value at risk. *Journal of Derivatives* 1997; 4(3): 7–49.

30. Bauso D, Mansour DB, Djehiche B, Tembine H, Tempone R. Mean-field games for marriage. *PLoS One* 2014; 9(5): e94933.

31. Bressan A, Shen W. Small BV solutions of hyperbolic noncooperative differential games. *SIAM Journal on Control and Optimization* 2004; 43(1): 194–215.

32. Falcone M. Numerical methods for differential games based on partial differential equations. *International Game Theory Review* 2006; 8(02): 231–272.

33. Higham DJ. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review* 2001; 43(3): 525–546.

34. Fasshauer GE, Zhang JG. On choosing "optimal" shape parameters for RBF approximation. *Numerical Algorithms* 2007; 45(1-4): 345–368.

35. Fasshauer GE. *Meshfree approximation methods with MATLAB*. 6. World Scientific . 2007.

36. MATLAB . *Version 7.10.0 (R2010a)*. Natick, Massachusetts: The MathWorks Inc . 2010.

37. Stevenson B, Wolfers J. Marriage and divorce: Changes and their driving forces. *Journal of Economic Perspectives* 2007; 21(2): 27–52.

38. Amato PR, James SL. Changes in Spousal Relationships over the Marital Life Course. *Alwin, Duane F.; Felmlee, Diane H.; and Kreager, Derek A. Editors (2018) Social Networks and the Life Course. Integrating the Development of Human Lives and Social Relational Networks* 2018: 139-158.

39. Van Laningham J, Johnson DR, Amato P. Marital Happiness, Marital Duration, and the U-Shaped Curve: Evidence from a Five Wave Panel Study. *Social Forces* 2001; 79(4): 1313-1341.

40. Kulu H. Marriage Duration and Divorce: The Seven-Year Itch or a Lifelong Itch?. *Demography* 2014; 51(3): 881-893.

41. Zhou H, Zhu H, Zhang C. Linear quadratic Nash differential games of stochastic singular systems. *Journal of Systems Science and Information* 2014; 2(6): 553–560.

42. Haurie A, Krawczyk JB, Zaccour G. *Games and Dynamic Games*. 1. World Scientific Publishing Company . 2012.


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