Modifying Kurchatov's method to find multiple roots of nonlinear equations

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Abstract

In this work, we modify the iterative Kurchatov's method to solve nonlinear equations with multiple roots, that is,for approximating the solutions of multiplicity grater than one. Its main feature is that you do not need to know a priori the multiplicity of the root, which does not appear in the iterative expression. We perform a dynamical analysis to see the behaviour of the proposed method. We also carry out some numerical experiments to confirm the theoretical results and compare the proposed method with other known schemes for multiple roots.

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Modifying Kurchatov's method to find multiple roots of nonlinear equations

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In this work, we modify the iterative Kurchatov's method to solve nonlinear equations with multiple roots, that is,for approximating the solutions of multiplicity grater than one. Its main feature is that you do not need to know a priori the multiplicity of the root, which does not appear in the iterative expression. We perform a dynamical analysis to see the behaviour of the proposed method. We also carry out some numerical experiments to confirm the theoretical results and compare the proposed method with other known schemes for multiple roots.

KEYWORDS
Iterative method; Kurchatov scheme; Multiple roots; Dynamical Analysis

Introduction

The need to solve nonlinear equations, $f(x) = 0$, arises in many engineering or applied mathematical problems. They cannot always be solved exactly, therefore approximations to the solution are sometimes obtained. Iterative methods are often used to obtain these approximations. A well-known one is Newton's method which has the following expression:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \text{ for } k = 0, 1, \ldots$$

To ensure that this method converges to a root of $f(x) = 0$, it is required that the derivative of the function evaluated in the solution is not zero.

Therefore, iterative methods appear that allow to obtain solutions with a multiplicity greater than 1. In manuscripts [1, 8, 7, 6, 9, 13] numerous iterative schemes, without memory, involving or not derivatives, are designed for approximating the multiple roots of a nonlinear equation $f(x) = 0$. In the most of them, the authors assume that the multiplicity is known and it appears in the iterative expression of the method.

It is known that Schröder scheme [12]

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)}, \text{ for } k = 0, 1, \ldots$$

has second-order of convergence for multiple roots of the $f(x) = 0$. This method was designed from Newton's scheme applied to $g(x) = \dfrac{f(x)}{f'(x)}$. Its main feature is that you do not need to know a priori the multiplicity of the root, which does not appear in the iterative expression.

In a similar way, in paper [4], the authors construct an iterative method with memory for approximating the multiple roots, that avoids the need to know a priori the multiplicity. In this manuscript, we apply several techniques to Kurchatov's scheme in order to obtain an iterative method with memory and without derivatives for finding multiple roots . We see that the modification of this method maintains the order and has good dynamical behaviour.

Kurchatov's method is an iterative scheme of second-order convergence obtained from Newton's method by replacing the derivative by the divide difference of Kurchatov $f[2x_k - x_{k-1}, x_{k-1}]$

$$x_{k+1} = x_k - \frac{f(x_k)}{f[2x_k - x_{k-1}, x_{k-1}]} = x_k - \frac{2(x_k - x_{k-1})f(x_k)}{f(2x_k - x_{k-1}) - f(x_{k-1})}, k = 1, 2, \ldots$$

In this paper, Section 2 is devoted to the design and convergence analysis of the proposed iterative method, with memory, to find multiple roots without the knowledge of its multiplicity. A dynamical analysis of the rational function obtained by applying the proposed scheme on low-degree polynomials is presented in Section 3. A method with similar characteristics to the one designed in Section 2, but derivative-free, is shown in Section 4. Finally, in Section 5 we perform several numerical experiments with the Kurchatov method for multiple roots and compare the results obtained by this scheme with other known ones designed for multiple roots.

1 | CONVERGENCE ANALYSIS

Let $f : \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open set $D \subset \mathbb{R}$ that contains a root α of $f(x) = 0$. Let us consider the expression of the divided difference operator

$$f[x + h, x](x + h - x) = f(x + h) - f(x), \tag{1}$$

which we use to prove the order of convergence of methods.

To demonstrate the order of an iterative methods with memory we use the Ortega-Rheinboldt theorem, which can be found in [10]:

Theorem 1 *Let ϕ be an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root α, and let this sequence converges to α. If there exist a nonzero constant η and positive numbers t_i, $i = 0, \ldots, m$ such that the* *inequality*

$$|e_{k+1}| \le \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i},$$

holds, then the R-order of convergence of the iterative method ϕ is at least p, where p is the unique positive root of the equation

$$p^{m+1} - \sum_{i=0}^{m} t_i p^{m-i} = 0.$$

To estimate the roots of $f(x) = 0$, we define the following method, denoted by KM,

$$x_{k+1} = x_k - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]}, \qquad k = 0, 1, 2, \ldots$$

where $g(x) = \dfrac{f(x)}{f'(x)}$.

Theorem 2 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α which we denote by $D \subset \mathbb{R}$ such that α is a multiple root of $f(x) = 0$ with unknown multiplicity $m \in \mathbb{N} - \{1\}$. Then, taking an initial estimation x_0 close enough to α, the sequence of iterates $\{x_k\}$ generated by method KM converges to α with order 2, and the error equation is:*

$$e_{k+1} = \left(\frac{-1}{m} C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} \left(-5e_k^3 + 2e_k^2 e_{k-1} - e_k e_{k-1}^2 \right) \right) + O_4(e_k, e_{k-1}),$$

being $C_j = \dfrac{m!}{(m+j)!} \dfrac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j = 2, 3, \ldots$ and where O_4 denotes all terms for which the sum of the exponents of e_k and e_{k-1} is at least 4.

Proof We first obtain the Taylor expansion of $f(x_k)$ around α where $e_k = x_k - \alpha$:

$$f(x_k) = \frac{f^{(m)}(\alpha)}{m!} \left(e_k^m + C_1 e_k^{m+1} + C_2 e_k^{m+2} + C_3 e_k^{m+3} \right) + O(e_k^{m+4}).$$

Calculating the derivative of the above expression we obtain

$$f'(x_k) = \frac{f^{(m)}(\alpha)}{m!} \left(m e_k^{m-1} + (m+1) C_1 e_k^m + (m+2) C_2 e_k^{m+1} + (m+3) C_3 e_k^{m+2} \right) + O(e_k^{m+3}).$$

Then, from the above expressions, we calculate $g(x_k)$

$$g(x_k) = \frac{f(x_k)}{f'(x_k)} = \frac{1}{m} \left(e_k - \frac{1}{m} C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} e_k^3 \right) + O(e_k^4).$$

In an equivalent way we obtain the following expressions for $g(x_{k-1})$ and $g(2x_k - x_{k-1})$

$$
\begin{aligned}
g(x_{k_1}) &= \frac{f(x_{k-1})}{f'(x_{k-1})} = \frac{1}{m}\left(e_{k-1} - \frac{1}{m}C_1 e_{k-1}^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} e_{k-1}^3\right) + O(e_{k-1}^4),\\
g(2x_k - x_{k-1}) &= \frac{f(2x_k - x_{k-1})}{f'(2x_k - x_{k-1})}\\
&= \frac{1}{m}\left(2e_k - e_{k-1} - \frac{1}{m}C_1(2e_k - e_{k-1})^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}(2e_k - e_{k-1})^3\right) + O_4(e_k, e_{k-1}),
\end{aligned}
$$

with $e_{k-1} = x_{k-1} - \alpha$.

From the above relations, we obtain

$$
\begin{aligned}
g[2x_k - x_{k-1}, x_{k-1}] &= \frac{g\,(2x_k - x_{k-1}) - g\,(x_{k-1})}{2(x_k - x_{k-1})}\\
&= \frac{\left(2e_k - 2e_{k-1} - \frac{1}{m}C_1\left((2e_k - e_{k-1})^2 - e_{k-1}^2\right) + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left((2e_k - e_{k-1})^3 - e_{k-1}^3\right)\right) + O(e_k^4)}{2m\,(e_k - e_{k-1})}\\
&= \frac{1}{m}\left(1 - \frac{2}{m}C_1 e_k + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(4e_k^2 - 2e_k e_{k-1} + e_{k-1}^2\right)\right) + O_3(e_k, e_{k-1}).
\end{aligned}
$$

Thus, applying the above relationship, the following error equation is obtained:

$$
\begin{aligned}
x_{k+1} - \alpha &= x_k - \alpha - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]}\\
&= e_k - \frac{\left(e_k - \frac{1}{m}C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} e_k^3\right) + O(e_k^4)}{\left(1 - \frac{2}{m}C_1 e_k + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(4e_k^2 - 2e_k e_{k-1} + e_{k-1}^2\right)\right) + O_3(e_k, e_{k-1})}\\
&= \frac{-1}{m}C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(-e_k^3 - e_k\left(4e_k^2 - 2e_k e_{k-1} + e_{k-1}^2\right)\right) + O_4(e_k, e_{k-1})\\
&= \frac{-1}{m}C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(-5e_k^3 + 2e_k^2 e_{k-1} - e_k e_{k-1}^2\right) + O_4(e_k, e_{k-1}).
\end{aligned}
$$

We have some different possibilities for the behaviour of e_{k+1} respect to e_k and e_{k-1}.

By the previous expression, we only are going to take into account if the behaviour is like e_k^2 or $e_k e_{k-1}^2$, because e_k^3 and $e_k^2 e_{k-1}$ converge faster to 0 than e_k^2.

Then,

$$
e_{k+1} \sim \frac{-1}{m}C_1 e_k^2 - \frac{(m+1)C_1^2 - 2mC_2}{m^2} e_k e_{k-1}^2.
$$

- If $e_{k+1} \sim e_k^2$, then the order of convergence is 2.
- If we assume that $e_{k+1} \sim e_k e_{k-1}^2$. Then, we assume that the method has R-order p, that means,

$$
e_{k+1} \sim e_k^p.
$$

In the same way $e_k \sim e_{k-1}^{p}$. From the above relations, we get that

$$e_{k+1} \sim e_{k-1}^{p^2}.$$

Then, the error equation is

$$e_{k+1} \sim e_k e_{k-1}^2 \sim e_{k-1}^{p+2}.$$

By equating the exponents of e_{k-1} of the above relations, we obtain the following polynomial $p^2 - p - 2 = 0$, whose only positive root is $p = 2$, then, by Theorem 1, the order of convergence of the method is 2.

□

2 | DYNAMICAL ANALYSIS

In this section, we review some of the theoretical concepts to perform the dynamical analysis of an iterative method with memory, since we later perform a dynamical analysis of the proposed method for some family of functions.

The standard form of an iterative method with memory that uses only two previous iterations to calculate the next one is:

$$x_{k+1} = \phi(x_{k-1}, x_k), \ k \geq 1,$$

being x_0 and x_1 the initial estimations. A function defined from $\mathbb{R}^2$ to $\mathbb{R}$ cannot have fixed points. Therefore, an auxiliary vectorial function O is defined by means of $O(x_{k-1}, x_k) = (x_k, x_{k+1}) = (x_k, \phi(x_{k-1}, x_k)),\ k = 1, 2, \ldots$

Thus, the discrete dynamical system $O : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$O(\bar{x}) = O(z, x) = (x, \phi(z, x)),$$

where ϕ is the operator of the iterative scheme with memory.
Then, a point (z, x) is a fixed point of O if $z = x$ and $x = \phi(z, x)$. If a fixed point (z, x) of operator O does not verify that $f(x) = 0$, it is called strange fixed point.

In [11], the stability of a fixed point is defined in the following result:

Theorem 3 *Let O from $\mathbb{R}^2$ to $\mathbb{R}^2$ be a sufficiently differentiable function. Assume that $\bar{x}$ is a fixed point. Let λ_1 and λ_2 be the eigenvalues of the Jacobian matrix of O evaluated at $\bar{x}$. Then,*

- *If all the eigenvalues satisfy $|\lambda_j| < 1$, then $\bar{x}$ is attracting.*
- *If one eigenvalue λ_i satisfy $|\lambda_i| > 1$, then $\bar{x}$ is unstable, that is, repelling or saddle.*
- *If all the eigenvalues satisfy $|\lambda_j| > 1$, then $\bar{x}$ is repelling.*

Moreover, if all the eigenvalues are equal to zero the fixed point is superattracting.

A critical point $\bar{y}$ of operator O satisfies that all the eigenvalues of the Jacobian matrix evaluated at $\bar{y}$ are 0. All superattractor fixed points are critical points.

The basin of attraction of a fixed point x^*, is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \{y \in \mathbb{R}^n : O^r(y) \to x^*,\ r \to \infty\}.$$

We study the stability of the fixed points of the rational operator obtained when the methods is applied on the polynomial $p_m(x) = (x+1)(x-1)^m$, when m is a positive integer greater than 1. Now, we calculate the auxiliar vectorial operator where $z = x_{k-1}$ and $x = x_k$

$$Op(z,x) = \left(x, x - \frac{\left(x^2-1\right)(mz+m+z-1)(2mx-mz+m+2x-z-1)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))} \right).$$

Theorem 4 *The fixed points of the operator $Op(z,x)$ are the roots of the polynomial $p_m(x)$, that is, $(1,1)$ and $(-1,-1)$, both fixed points have superattractor character, and an unestable strange fixed point $\left(\frac{1-m}{1+m}, \frac{1-m}{1+m}\right)$.*

Proof To calculate the fixed points we simultaneously do $z = x$ and $Op(z,x) = (x,x)$. First, we compute $Op(x,x)$

$$Op(x,x) = \left(x, \frac{m(x+1)^2-(x-1)^2}{m(x+1)^2+(x-1)^2} \right).$$

By equating $Op(x,x) = (x,x)$, we obtain that the fixed points satisfy:

$$\begin{aligned}
\frac{m(x+1)^2-(x-1)^2}{m(x+1)^2+(x-1)^2} &= x, \\
m(x+1)^2-(x-1)^2 &= xm(x+1)^2 + x(x-1)^2, \\
m(1-x)(x+1)^2 &= (x+1)(x-1)^2.
\end{aligned}$$

If $x = 1$ or $x = -1$, then it is obvious that the above equation is satisfied.

Suppose that $x \neq 1$ and $x \neq -1$. Then, the above equation can be rewritten as:

$$\begin{aligned}
-m(x-1)(x+1)^2 &= (x+1)(x-1)^2, \\
-m(x+1) &= x-1, \\
(-m-1)x &= -1+m, \\
x &= \frac{-1+m}{-m-1} = \frac{1-m}{1+m}.
\end{aligned}$$

So, we obtain two fixed point from the roots of the equation, that is, $z = x = 1$ and $z = x = -1$, and one strange fixed point when $z = x = \frac{1-m}{1+m}$.

We are going to see below that the fixed points coming from the roots are superattractors. First, we have to calculate the Jacobian matrix $Op'(z,x)$.

$$Op'(z,x) = \begin{pmatrix} 0 & 1 \\ dOp_z(z,x) & dOp_x(z,x) \end{pmatrix},$$

where

$$dOp_z(z,x) = -\frac{8m(m+1)\left(x^2-1\right)(x-z)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2},$$

$$dOp_x(z,x) = -\frac{4m^3(z+1)\left(x^2(5z+1)+x\left(-4z^2+2z-2\right)+z^3-z^2-2\right)+8m^2\left(x^2\left(5z^2-3\right)-4xz^3+z^4-z^2+2\right)}{(mx+m+x-1)^2(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}$$
$$-\frac{4m(z-1)\left(x^2(5z-1)-2x\left(2z^2+z+1\right)+z^3+z^2+2\right)}{(mx+m+x-1)^2(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}.$$

The eigenvalues of $Op'(x,x)$ are 0 and $-\dfrac{8m\left(x^2-1\right)}{\left(m(x+1)^2+(x-1)^2\right)^2}$.
Then, both eigenvalues are 0 when $x^2-1=0$, that is, $x=1$ or $x=-1$, so we find that the fixed points coming from the roots are superattractor fixed points.

In the case $x=\dfrac{1-m}{1+m}$, we obtain that the second eigenvalue is 2, so is a point with an unstable character (repulsor or saddle).

□

Theorem 5 *The operator $Op(z,x)$ does not have free critical points, that is, the operator has only two critical points that are the superattractor fixed points.*

Proof First, we calculate the determinant of $Op'(z,x)$, because when the determinant is 0, it means that at least one of the eigenvalues is 0.

$$\det(Op'(z,x)) = \frac{8m(m+1)\left(x^2-1\right)(x-z)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}.$$

By equating that expresion to 0, we obtain 3 types of possible critical points:

- The points (z,x) where $x=-1$. The eigenvalues of $Op'(z,-1)$ are 0 and $-\dfrac{m(1+m)(1+z)^2}{-3+2z+z^2+m(1+z)^2}$.
  Then, the second eigenvalue is 0 if $z=-1$. Then, there is only one critical point with this structure which is the fixed point $(-1,-1)$.
- The points (z,x) where $x=1$. The eigenvalues of $Op'(z,1)$ are 0 and $-\dfrac{(1+m)(-1+z)^2}{m((z-1)^2+m(z^2-2z-3))}$.
  Then, the second eigenvalue is 0 if $z=1$. Then, there is only one critical point with this structure which is the fixed point $(1,1)$.
- The points (z,x) where $z=x$. The eigenvalues of $Op'(z,z)$ are 0 and $-\dfrac{8m(-1+z^2)}{((-1+z)^2+m(1+z)^2)^2}$.
  The second eigenvalue is 0 if $z=\pm 1$. So, the critical points that verify this structure are the non strange fixed points, that is, $(1,1)$ and $(-1,-1)$.

Then, the operator does not have free critical points.

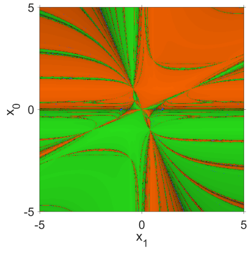
□

Below we show some real dynamical planes to see the behaviour of the method and the basins of attraction for the function p_m by varying the value of m.

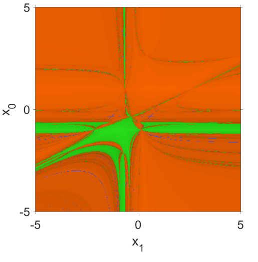
These planes have been generated by making a mesh of 400 points by 400 points, where each point of the mesh is considered as the initial iterations of the iterative method, on the abscissa axis we have the iteration x_1 and on the ordinate axis the iteration x_0.

If the distance between iterations of the method to one of the roots of the function is less than 10^{-3}, then we say that the initial point converges to that root. Moreover, this convergence must happen before 100 iterations.

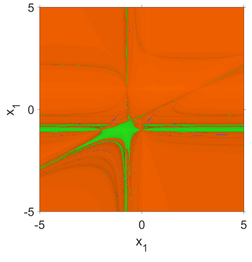
We paint the initial point in different colours according to its convergence. We paint in orange the initial points that converge to the fixed point 1 and in green the initial points that converge to the fixed point -1. We would also paint in black those initial points that do not converge to any of the roots, but in this case, that does not happen for this mesh.



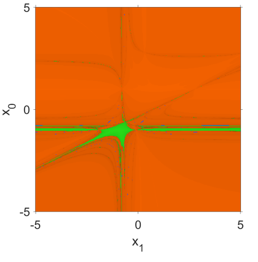
(a) Dynamical plane for $m = 1$



(b) Dynamical plane for $m = 2$



(c) Dynamical plane for $m = 3$



(d) Dynamical plane for $m = 4$

FIGURE 1 Real dynamical planes

As we can see in Figures 1a,1b,1c,1d, if we increase the value of m, the zone of convergence to the root 1 increases, which is the root of multiplicity m. As can be seen in all the dynamical planes, all the initial points coming from the mesh converge to one of the roots. With this study we show what happens with a family of polynomials with one simple root and one multiple root. Now we perform a dynamical analysis to see what happens when we have two multiple roots.

The polynomials are $f_{m,n}(x) = (x+1)^n(x-1)^m$ where $m > 1$ and $n > 1$.
Now, we calculate the auxiliar vectorial operator

$$Of(z,x) = \left(x, \frac{m^2(x+1)(z+1)(2x-z+1) + 2mn\left(2xz - z^2 - 1\right) - n^2(x-1)(z-1)(2x-z-1)}{(m(x+1)+n(x-1))(m(z+1)(2x-z+1)+n(z-1)(2x-z-1))} \right).$$

Theorem 6 *The fixed points of the operator $Of(z,x)$ are the roots of the polynomial $f_{m,n}(x)$, that is, $(1,1)$ and $(-1,-1)$, both fixed points have superattractor character, and an unestable strange fixed point that is $\left(\frac{n-m}{n+m}, \frac{n-m}{n+m}\right)$.*

Proof To calculate the fixed points we simultaneously do $z = x$ and $Of(z,x) = (x,x)$. First, we compute $Of(x,x)$

$$Of(x,x) = \left(x, \frac{m(x+1)^2 - n(x-1)^2}{m(x+1)^2 + n(x-1)^2}\right).$$

By equating $Of(x,x) = (x,x)$, we obtain that the fixed points are those that are satisfied:

$$\begin{aligned} \frac{m(x+1)^2 - n(x-1)^2}{m(x+1)^2 + n(x-1)^2} &= x, \\ m(x+1)^2 - n(x-1)^2 &= xm(x+1)^2 + xn(x-1)^2, \\ m(1-x)(x+1)^2 &= n(x+1)(x-1)^2. \end{aligned}$$

If $x = 1$ or $x = -1$, then it is obvious that the above equation is satisfied. Suppose that $x \neq 1$ and $x \neq -1$. Then, the above equation can be rewritten as:

$$\begin{aligned} -m(x-1)(x+1)^2 &= n(x-1)^2, \\ -m(x+1) &= n(x-1), \\ (-m-n)x &= -n+m, \\ x = \frac{-n+m}{-m-n} &= \frac{n-m}{n+m}. \end{aligned}$$

So, we obtain two fixed point from the roots of the equation, that is, $z = x = 1$ and $z = x = -1$, and one strange fixed point when $z = x = \frac{n-m}{n+m}$.

We see below that the fixed points coming from the roots are superattractors. First, we calculate the eigenvalues of the Jacobian matrix $Of'(x,x)$, that are 0 and $-\frac{8mn\left(z^2-1\right)}{\left(m(z+1)^2+n(z-1)^2\right)^2}$.

Then, both eigenvalues are 0 when $x^2 - 1 = 0$, that is, $x = 1$ or $x = -1$, so we find that the fixed points coming from the roots are superattractor fixed points.

In the case that $x = \frac{n-m}{n+m}$, we obtain that the second eigenvalue is 2, so is a point with an unstable character (repulsor or saddle).

□

Theorem 7 *The operator $Of(z,x)$ does not have free critical points, that is, the operator has only two critical points that are the superattractor fixed points.*

Proof First, we analyze the determinant of $Of'(z,x)$, because when the determinant is 0, it means that at least one of the eigenvalues is 0,

$$\det\left(Of'(z,x)\right) \frac{8mn\left(x^2-1\right)(m+n)(x-z)}{(m(x+1)+n(x-1))(m(z+1)(2x-z+1)+n(z-1)(2x-z-1))^2}.$$

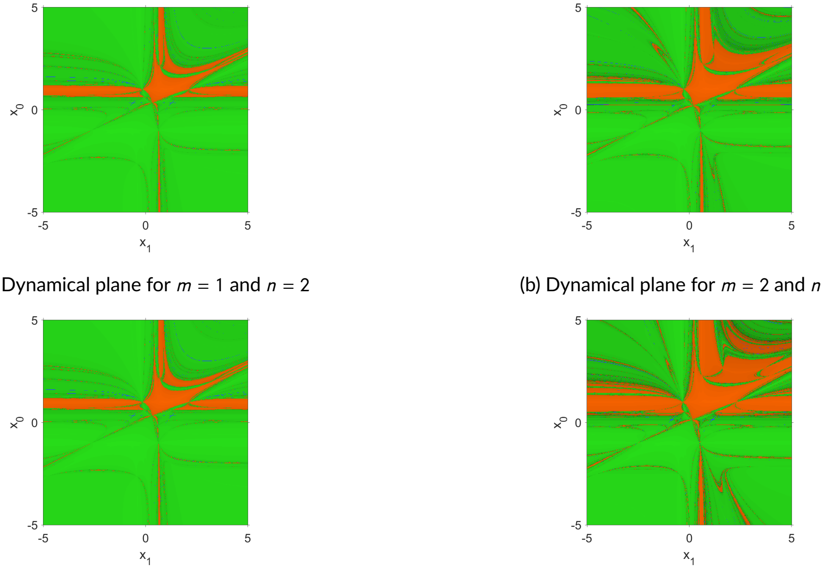
By equating that expresion to 0, we obtain 3 types of possible critical points:

- The points (z, x) where $x = -1$. The eigenvalues of $Of'(z, -1)$ are 0 and $-\dfrac{m(z+1)^2(m+n)}{n\left(m(z+1)^2+n\left(z^2+2z-3\right)\right)}$.
  Then, the second eigenvalue is 0 if $z = -1$. Then, there is only one critical point with this structure which is the fixed point $(-1, -1)$.
- The points (z, x) where $x = 1$. The eigenvalues of $Of'(z, 1)$ are 0 and $-\dfrac{n(z-1)^2(m+n)}{m\left(m\left(z^2-2z-3\right)+n(z-1)^2\right)}$.
  Then, the second eigenvalue is 0 if $z = 1$. Then, there is only one critical point with this structure which is the fixed point $(1, 1)$.
- The points (z, x) where $z = x$. The eigenvalues of $Of'(z, z)$ are 0 and $-\dfrac{8mn\left(z^2-1\right)}{\left(m(z+1)^2+n(z-1)^2\right)^2}$.
  The second eigenvalue is 0 if $z = \pm 1$. So, the critical points that verify this structure are the non strange fixed points, that is, $(1, 1)$ and $(-1, -1)$.
  Then, the operator does not have free critical points.

□

Below we show some real dynamical planes to see the behaviour of the method and the basins of attraction for the function $f_{m,n}$ varying the value of m and the value of n.

These planes have been generated in the same way as the previous dynamical planes, making a mesh of 400 points by 400 points, where each point of the mesh is considered as the initial iteration of the iterative method, in the abscissa axis we have the iteration x_1 and in the ordinate axis the iteration x_0. The convergence criteria are the same as in the previous dynamical planes. Remember that we paint in orange the initial points that converge to the fixed point 1 and in green the initial points that converge to the fixed point -1.



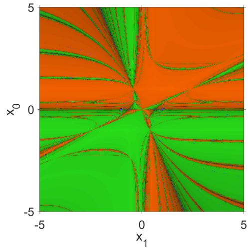
(a) Dynamical plane for $m = 1$ and $n = 2$

(b) Dynamical plane for $m = 2$ and $n = 3$

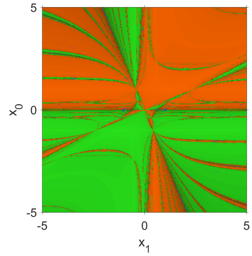
(c) Dynamical plane for $m = 2$ and $n = 4$

(d) Dynamical plane for $m = 3$ and $n = 4$

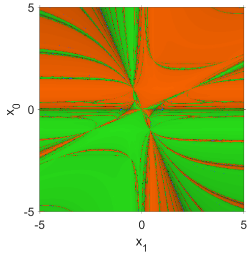
FIGURE 2 Real dynamical planes



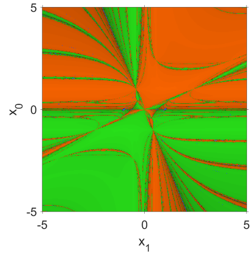
(a) Dynamical plane for $m = 1$ and $n = 1$



(b) Dynamical plane for $m = 2$ and $n = 2$



(c) Dynamical plane for $m = 3$ and $n = 3$



(d) Dynamical plane for $m = 4$ and $n = 4$

FIGURE 3 Real dynamical planes

As we can see in Figures 2 and 3, if we the value of n is greater than the value of m, the zone of convergence to the root -1 is greater than the zone of convergence to the root 1. If both values are equal, then the convergence zones do not change if we increase the multiplicity value.

As can be seen in all the dynamical planes, all the initial points coming from the mesh converge to one of the roots. With this study we show that the method is stable for that family of polynomials that have two multiple roots.

3 | WITHOUT DERIVATIVES

To estimate the roots of $f(x) = 0$ with the KM method we calculate the derivative of $f(x)$. In the following iterative method, which we denote by KMD, we modify the KM method, so that we do not use derivatives in the iterative expression:

$$x_{k+1} = x_k - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]},$$

where $g(x) = \dfrac{f(x)}{f[x + f(x), x]}$.

Theorem 8 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α which we denote by $D \subset \mathbb{R}$ such that α is a multiple root of $f(x) = 0$ with unknown multiplicity $m \in \mathbb{N} - \{1\}$. Then, taking an initial estimation x_0 close enough to α, the sequence of iterates $\{x_k\}$ generated by the KMD method converges to α with order 2.*

Proof We first obtain the Taylor expansion of $f(x_k)$ around α where $e_k = x_k - \alpha$:

$$f(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left(e_k^m + C_1 e_k^{m+1}\right) + O(e_k^{m+2}).$$

being $C_j = \dfrac{m!}{(m+j)!}\dfrac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j = 2, 3, \ldots$

In the same way,

$$f(x_k + f(x_k)) = \frac{f^{(m)}(\alpha)}{m!}\left((e_k + f(x_k))^m + C_1\,(e_k + f(x_k))^{m+1}\right) + O(e_k^{m+2}).$$

Then,

$$f(x_k + f(x_k)) - f(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left((e_k + f(x_k))^m - e_k^m + C_1\left((e_k + f(x_k))^{m+1} - e_k^{m+1}\right)\right) + O(e_k^{m+2}).$$

Using Newton's binomial and the Taylor expansion of $f(x_k)$ around α we obtain

$$\frac{f(x_k + f(x_k)) - f(x_k)}{x_k + f(x_k) - x_k} = \frac{f^{(m)}(\alpha)}{m!}\left(me_k^{m-1} + (m+1)C_1 e_k^m\right) + O(e_k^{m+1}).$$

We then calculate $g(x_k)$ from the above expressions:

$$\begin{aligned} g(x_k) &= \frac{f(x_k)}{f[x_k + f(x_k), x_k]} = \frac{e_k^m + C_1 e_k^{m+1} + O(e_k^{m+2})}{me_k^{m-1} + (m+1)C_1 e_k^m + O(e_k^{m+1})} \\ &= \frac{1}{m}\left(e_k - \frac{1}{m}C_1 e_k^2\right) + O(e_k^3). \end{aligned}$$

In an equivalent way we obtain the following expressions for $g(x_{k-1})$ and $g(2x_k - x_{k-1})$

$$g(x_{k-1}) = \frac{1}{m}\left(e_{k-1} - \frac{1}{m}C_1 e_{k-1}^2\right) + O(e_{k-1}^3),$$

$$g(2x_k - x_{k-1}) = \frac{1}{m}\left(2e_k - e_{k-1} - \frac{1}{m}C_1(2e_k - e_{k-1})^2\right) + O_3(e_k, e_{k-1}),$$

with $e_{k-1} = x_{k-1} - \alpha$.

Then, appyling the above relations, we obtain

$$\begin{aligned} g[2x_k - x_{k-1}, x_{k-1}] &= \frac{g(2x_k - x_{k-1}) - g(x_{k-1})}{2(x_k - x_{k-1})} \\ &= \frac{\left(2e_k - 2e_{k-1} - \frac{1}{m}C_1((2e_k - e_{k-1})^2 - e_{k-1}^2)\right) + O_3(e_k, e_{k-1})}{2m(e_k - e_{k-1})} \\ &= \frac{1}{m}\left(1 - \frac{2}{m}C_1 e_k\right) + O_2(e_k, e_{k-1}). \end{aligned}$$

Thus, the following error equation is obtained

$$\begin{aligned}
x_{k+1} - \alpha = x_k - \alpha - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]} \\
&= e_k - \frac{\left(e_k - \frac{1}{m} C_1 e_k^2\right) + O(e_k^3)}{\left(1 - \frac{2}{m} C_1 e_k\right) + O_2(e_k, e_{k-1})} \\
&= e_k - \frac{2}{m} C_1 e_k^2 + e_k O_2(e_k, e_{k-1}) - e_k + \frac{1}{m} C_1 e_k^2 + O(e_k^3) \\
&= -\frac{1}{m} C_1 e_k^2 + e_k O_2(e_k, e_{k-1}) + O(e_k^3).
\end{aligned}$$

We have some different possibilities for the behaviour of e_{k+1} respect to e_k and e_{k-1}.

By the expression, we only are going to take into account if the behaviour is like e_k^2 or $e_k e_{k-1}^2$, because e_k^3 and $e_k^2 e_{k-1}$ converge faster to 0 than e_k^2.

Then

- If $e_{k+1} \sim e_k^2$, then the order of convergence is 2.
- If we assume that $e_{k+1} \sim e_k e_{k-1}^2$. Then, we assume that the method has R-order p, that means,

$$e_{k+1} \sim D_{k,p} e_k^p.$$

At the same time, $e_k \sim e_{k-1}^p$, then we obtain that

$$e_{k+1} \sim e_{k-1}^{p^2}.$$

From the error equation and the last relation, we have

$$e_{k+1} \sim e_k e_{k-1}^2 \sim e_{k-1}^{p+2}.$$

By equating the exponents of e_{k-1} of the last two equation, we obtain the following polynomial $p^2 - p - 2 = 0$, whose only positive root is $p = 2$, then the order of convergence of the method is 2.

□

4 | NUMERICAL EXPERIMENTS

We use Matlab R2020b with arithmetic precision of 500 digits for the computational calculations. As a stopping criterion we use that the absolute value of the function at the last iteration is less than a tolerance of 10^{-25}. We also use a maximum of 100 iterations as a stopping criterion. We compare the proposed methods with the method coming from [4], which we denote by gTM.

The numerical results we are going to compare the methods in the different examples are:

- the approximation obtained,
- the norm of the equation evaluated in that approximation,
- the norm of the distance between the last two approximations,
- the number of iterations necessary to satisfy the required tolerance,

- the computational time and the approximate computational convergence order (ACOC), defined by Cordero and Torregrosa in [5], which has the following expression

$$p \approx ACOC = \frac{\ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}.$$

We are going to solve two nonlinear equations:

- The equation $f_1(x) = (x^3 - 1)^4 = 0$, which has three roots with multiplicity four.
- In [14], they considered the isothermal CSTR problem, with the following equation for the transfer function of the reactor: $KC2.98(x + 2.25)/((x + 1.45)(x + 2.85)^2(x + 4.35)) = -1$, where KC is the gain of the proportional controller. If we choose KC = 0, the nonlinear equation to solve is the following one:

$$f_2(x) = x^4 + 11.50x^3 + 47.49x^2 + 86.0325x + 51.23266875 = 0.$$

There is one multiple root with multiplicity 2.

TABLE 1 Results for equation $f_1(x) = 0$.

	x_0	x_{-1}	x_{-2}	$\|x_{k+1} - x_k\|$	$\|g(x_{k+1})\|$	Iter	ACOC
KM	0.5	0.1		1.5776e-13	0	8	1.9994
KMD	0.5	0.1		6.1173e-14	0	6	1.8434
gTM	0.5	0.1	-0.1	1.7764e-15	0	42	1.5850

As we can see in Table 1, all the methods obtain good results for the chosen initial points. The approximate computational convergence order coincides with the theoretical one. What is interesting from the table is that, for the initial points chosen, we see that the KMD method performs less iterations to verify the stopping criterion than KM, but both perform far less iterations than the gTM method.

TABLE 2 Results for equation $f_2(x) = 0$.

	x_0	x_{-1}	x_{-2}	$\|x_{k+1} - x_k\|$	$\|g(x_{k+1})\|$	Iter	ACOC
KM	-3	-3.25		1.9884e-09	1.6566e-30	4	2.2725
KMD	-3	-3.25		2.4269e-08	2.0293e-29	4	2.0649
gTM	-3	-3.25	-3.5	2.5116e-11	1.0354e-29	5	1.7914

As we can see in Table 2, all the methods obtain good results for the chosen initial points. The approximate computational convergence order coincides with the theoretical one and the number of iterations to verify the stopping criterion is almost the same for all methods.

5 | CONCLUSIONS

In this work, we have modified Kurchatov's method to make it applicable to obtaining multiple roots while maintaining the quadratic order of convergence of Kurchatov's method.

We have modified the method so that it does not use the multiplicity of the solution in its expression, so that it is not necessary to know this value before applying the iterative method.

We have performed the dynamical analysis of the iterative method for two family of functions, one of the polynomials with one simple root and one multiple root, and another with two multiple roots, showing that the method is stable in both cases.

We also modify the method we propose to obtain the KMD method, which is a method with free memory of derivatives, with the same characteristics as the KM method, that is, it can be applied to obtain solutions with multiplicity greater than one, and does not involve the value of this multiplicity in its iterative expression.

Conflict of interest

The authors declare no potential conflict of interests.

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