Calculation and analysis of the strong prism of the octagonal-quadrilateral networks

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October 20, 2022

Abstract

Nowadays, with the development of the times, network structure analysis has become a hot issue in some fields. The eigenvalues of normalized Laplacian are very important for some network structure properties. Let Qn be octagonal-quadrilateral networks composed of n octagons and n squares and let Q2n be the strong prism of Qn. The strong product of a complete graph of order 2 and a complete graph of order G forms a strong prism of the graph G. In this paper, the decomposition theorem of the associated matrix is used to completely investigate the normalized Laplacian spectrum of Qn2. In addition, we establish exact formulas for the degree-Kirchhoff index and the number of spanning trees of Qn2.

Calculation and analysis of the strong prism of the octagonal-quadrilateral networks

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Abstract. Nowadays, with the development of the times, network structure analysis has become a hot issue in some fields. The eigenvalues of normalized Laplacian are very important for some network structure properties. Let Q_n be octagonal-quadrilateral networks composed of n octagons and n squares and let Q_n^2 be the strong prism of Q_n. The strong product of a complete graph of order 2 and a complete graph of order G forms a strong prism of the graph G. In this paper, the decomposition theorem of the associated matrix is used to completely investigate the normalized Laplacian spectrum of Q_2^n. In addition, we establish exact formulas for the degree-Kirchhoff index and the number of spanning trees of Q_2^n.

Keywords: Strong prism; Normalized Laplacian; Spanning trees.

1. Introduction

We exclusively consider finite, simple, and connected graphs in this study. Assume that the graph $G = (V_G, E_G)$ has the vertex set $V_G = \{v_{g1},\ v_{g2}, \cdots,\ v_{gn}\}$ and the edge set $E_G = \{e_{g1},\ e_{g2}, \cdots,\ e_{gm}\}$. More fundamental graph notations, one can be referred to [1]. We write the adjacent matrix of G, whose entry is designated by a_{ij}, as $A(G)$, if the vertices v_i and v_j are adjacent, the (i, j)-entry equals 1; otherwise, it equals zero. Let $L_G = D_G - A_G$ represent the graph G'scombinatorial Laplacian matrix, where D_G is the vertex degrees diagonal matrix of order n. A few years ago, the concept of the normalized Laplacian matrix was introduced by Chung [2] and defined as $\mathcal{L}(G) = I - D(G)^{\frac{1}{2}}\big(D(G)^{-1}A(G)\big)D(G)^{-\frac{1}{2}} = D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}}$, whose (p, q)th-entry of $\mathcal{L}(G)$ can be clearly represented as

$$(\mathcal{L}(G))_{mn} = \begin{cases} 1, & m = n; \\ -\frac{1}{\sqrt{d_m d_n}}, & m \neq n\ ,\ v_m \ \textit{is adjacent to}\ v_n; \\ 0, & \textit{otherwise}. \end{cases} \tag{1.1}$$

The standard separation between any two vertices, like v_i and v_j is what is meant by the conventional distance $d_{ij} = d_G(v_i, v_j)$, which is defined as the graph G's vertex v_i and vertex v_j's smallest length. Wiener [3, 4] studied the boiling points of paraffin wax and put forward the concept of wiener index for the first time in 1947, which was expressed as $W(G)$, which is the sum of distances among all pairs of vertices in G, that is

$$W(G) = \sum_{i<j} d_{ij}.$$

Wiener index is the earliest topological exponent of chemical formula configuration. Due to the success of Wiener's index, other topological indexes have been proposed successively, and new developments have been made on the basis of Wiener's index see [5–9]. Then, in anticipation of a further study of topology, researchers looked upon vertex v_i's degree d_i, thereby introducing the Gutman index [10] of the simple graph G and it expressed as

$$Gut(G) = \sum_{i<j} d_i d_j d_{ij}.$$

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An index of topology based on distance is called the Kirchhoff index. Assume that any two vertices v_i and v_j in any connected graph G have a resistance distance is represented by the symbol r_{ij}. The Kirchhoff index of the graph G is similar to the Wiener index in that its definition is that it is the sum of all vertices' resistance distances, namely

$$Kf(G) = \sum_{i<j} r_{ij}.$$

Chen and Zhang [14] introduced the multiplicative degree-Kirchhoff index in 2007., that is

$$Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}.$$

Kirchhoff index and multiplication-Kirchhoff index have attracted wide attention in the world because of their outstanding contributions in the academic field and their practical applications in physics, chemistry and other fields.

The spanning tree, also known as complexity and typically represented by the symbol $\tau(G)$, is the quantity of subgraphs that each include all of the vertices in the graph G. Also, all subgraphs in a graph G must be trees. This is a significant network stability indicator. You can check and the references [15–18] there for further information on some other topics relating to the number of spanning trees.

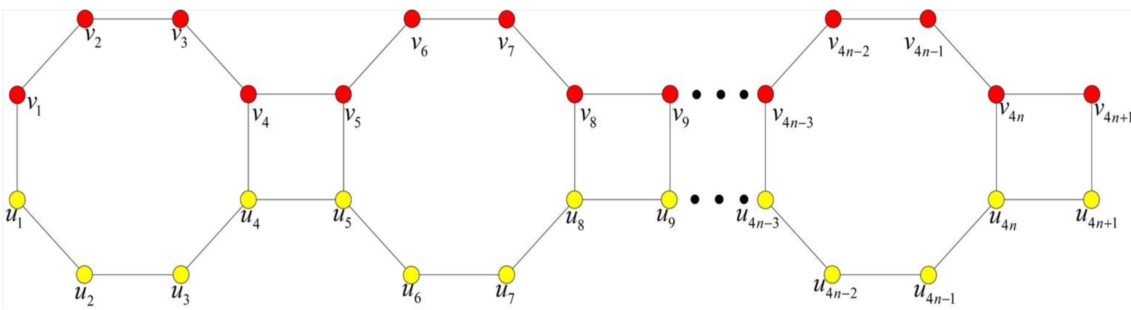

Figure 1: Graph Q_n with labeled vertices.

With the rapid development of science and the effective application of topology in practice, topological theory has gradually emerged in the eyes of the world. Many adjacent octagonal and quadrilateral rings make up octagonal-quadrilateral networks, and each quadrangle only has a maximum of two non-adjacent octagons. The issue of calculating the phenylene Wiener index has been resolved by Gutman [19]. Chen and Zhang [20] came up with an explicit expression for the expected value of the Wiener index of a random phenylene chain. Additionally, Liu and Zheng [21] discovered the degree-Kirchhoff index and the number of spanning trees of the dicyclobutadieno derivative of [n]phenylenes denoted by L_n.

We denote the strong product of the two automorphic graphs S and K with $V(S) \times V(K)$ by using the notation $S \boxtimes K$. Readers can refer to [22] for more relevant or in-depth definitions and concepts. In recent times, The resistance distance of the strong prism of the graphs P_n and C_n was utilized Pan et al. [23] to calculate the kirchhoff index. Li et al. [24] performed a similar calculation using the strong prism of the star S_n's created graphs' resistance distance-based properties. After much consideration, we achieve the strong prism of Q_n in this study, driven by [22–25]. As plain as the nose on your face, Q_n^2 is the strong prism of Q_n, as seen in Figure 2. It is possible to firmly believe that $|V(Q_n^2)| = 16n + 4$ and $|E(Q_n^2)| = 48n + 6$.

In the paper, we focus on the strong prism of the octagonal-quadrilateral networks. Based on the graph Q_n^2 with $n \geq 1$, we may construct the following. The remainder of the essay is structured as follows:

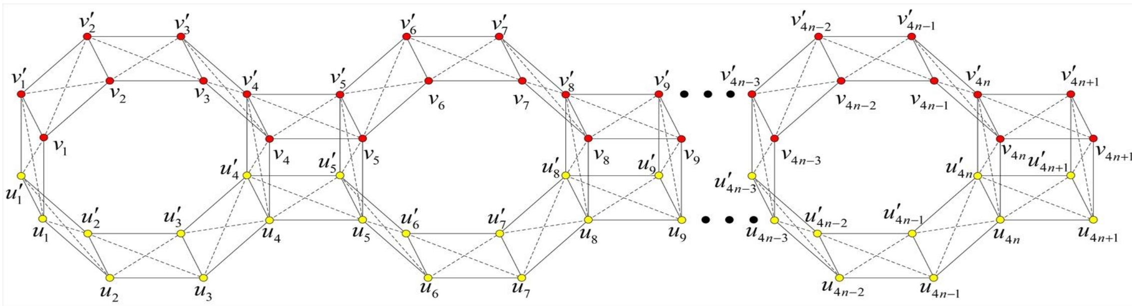

Figure 2: Graph Q_n^2 with labeled vertices.

In Section 2, we summarize the most recent research material pertinent to this work and suggest certain examples, ideas, and lemmas that should be presented in it. In Section 3, the normalized Laplacian spectrum is obtained first, and then the explicit closed formula for the multiplicative degree-Kirchhoff index and the complexity of Q_n^2 is determined. In Section 4, we conclude the thesis.

2. Preliminaries

In this section, let Q_n be the octagonal-quadrilateral networks and the strong prism of the graph Q_n is the graph Q_n^2, where Figures 1 and 2 illustrate, respectively, Q_n and Q_n^2. And $\Phi_A(x) = det(xI_n - A)$ denotes the matrix A's characteristic polynomial.

It is worth noticing that $\pi = (1,1')(2,2')\cdots\big((4n+1),(4n+1)'\big)$ is an automorphism. Let $V_1 = \{u_1, u_2, \cdots, u_{4n+1}, v_1, \cdots, v_{4n+1}\}$, $V_2 = \{u_1', u_2', \cdots, u_{4n+1}', v_1', \cdots, v_{4n+1}'\}$, $|V(L_n^2)| = 16n{+}4$ and $|E(L_n^2)|$ =48n+6. Therefore, we can express the normalized Laplacian matrix in block matrix formthat is

$$\mathcal{L}(L_n^2) = \begin{pmatrix} \mathcal{L}_{V_1V_1} & \mathcal{L}_{V_1V_2} \\ \mathcal{L}_{V_2V_1} & \mathcal{L}_{V_2V_2} \end{pmatrix},$$

in which

$$\mathcal{L}_{V_1V_1} = \mathcal{L}_{V_2V_2},\ \mathcal{L}_{V_1V_2} = \mathcal{L}_{V_2V_1}.$$

Let

$$W = \begin{pmatrix} \frac{1}{\sqrt{2}}I_{8n+2} & \frac{1}{\sqrt{2}}I_{8n+2} \\ \frac{1}{\sqrt{2}}I_{8n+2} & -\frac{1}{\sqrt{2}}I_{8n+2} \end{pmatrix},$$

then

$$W\mathcal{L}(L_n^2)W' = \begin{pmatrix} \mathcal{L}_A & 0 \\ 0 & \mathcal{L}_S \end{pmatrix},$$

where $\mathcal{L}_A(L_n^2) = \mathcal{L}_{V_1V_1} + \mathcal{L}_{V_1V_2}$ and $\mathcal{L}_S(L_n^2) = \mathcal{L}_{V_1V_1} - \mathcal{L}_{V_1V_2}$. Keep in mind that W' is W transposed.

Lemma 2.1. [26] *Suppose $\mathcal{L}_A$ and $\mathcal{L}_S$ are established as previously stated. Then*

$$\Phi_{\mathcal{L}(L_n)}(x) = \Phi_{\mathcal{L}_A}(x) \cdot \Phi_{\mathcal{L}_S}(x).$$

Chen and Zhang [14] determined that the normalized Laplacian spectrum can be used to obtain its corresponding multiplicative degree-Kirchhoff index.

Lemma 2.2. *Let $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$ represent the $\mathcal{L}(G)$'s eigenvalues. Following that, the multiplicative degree-Kirchhoff index could be represented as*

$$Kf^*(G) = 2m\sum_{k=2}^{n}\frac{1}{\sigma_k}.$$

Lemma 2.3. [2] *The complexity of G is also known as the quantity of spanning trees in G. Consequently, G's complexity is*

$$\tau(G) = \frac{1}{2m} \prod_{i=1}^{n} d_i \cdot \prod_{j=2}^{n} \sigma_j.$$

3. Main Results

In this section, we're devoted to talking about how can we use the normalized Laplacian matrix to get an explicit analytical expression for the multiplicative Kirchhoff index. After that, we have the complexity of L_n^2. Afterward, we derive matrices of order $8n+2$ using Eq. (1.1), that are

$$\mathcal{L}_{V_1V_1} = \begin{pmatrix}
1 & -\frac{1}{5} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{5} & 1 & -\frac{1}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{5} & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\
0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & 1 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\
-\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 1 & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 1 & -\frac{1}{5} & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{5} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & 1
\end{pmatrix},$$

and

$$\mathcal{L}_{V_1V_2} = \begin{pmatrix}
-\frac{1}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{5} & -\frac{1}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\
0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\
-\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & -\frac{1}{5}
\end{pmatrix},$$

Owing to $\mathcal{L}_A = \mathcal{L}_{V_1V_1}(L_n^2) + \mathcal{L}_{V_1V_2}(L_n^2)$ and $\mathcal{L}_S = \mathcal{L}_{V_1V_1}(L_n^2) - \mathcal{L}_{V_1V_2}(L_n^2)$, we have $8n+2$ order matrices. It's possible to be informed that

$$\mathcal{L}_A \quad = \quad 2\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{3}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\ -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & \frac{3}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{pmatrix},$$

and

$$\mathcal{L}_S = diag(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{8}{7}, \frac{8}{7}, \cdots \frac{6}{5}, \frac{8}{7}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{8}{7}, \frac{8}{7}, \cdots \frac{6}{5}, \frac{8}{7}, \frac{6}{5}).$$

Therefore, by understanding the Lemma 2.1, we discover that the $\mathcal{L}_A$ and $\mathcal{L}_S$ eigenvalues make up the L_n^2 normalized Laplacian spectrum. Determining that $\frac{6}{5}$ and $\frac{8}{7}$, the $\mathcal{L}_S$'s eigenvalues, have multiplicities of $(4n+4)$ and $(4n-2)$, respectively. Then, in order to determine the graph invariants of Q_n^2, we must obtain the eigenvalues of $\mathcal{L}_A$. Let

$$A \quad = \quad \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & \frac{3}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{pmatrix}_{(4n+1)\times(4n+1)},$$

and

$$C = diag(-\frac{1}{5}, 0, 0, -\frac{1}{7}, -\frac{1}{7}, \cdots, 0, -\frac{1}{7}, -\frac{1}{5}),$$

which are $4n+1$ order matrices.

The following block matrix would thus serve as a representation of $\frac{1}{2}\mathcal{L}_A$.

$$\frac{1}{2}\mathcal{L}_A = \begin{pmatrix} A & C \\ C & A \end{pmatrix}.$$

Suppose that

$$K = \begin{pmatrix} \frac{1}{\sqrt{2}} I_{4n+1} & \frac{1}{\sqrt{2}} I_{4n+1} \\ \frac{1}{\sqrt{2}} I_{4n+1} & -\frac{1}{\sqrt{2}} I_{4n+1} \end{pmatrix}$$

is the block matrix. Hence, one has

$$K(\frac{1}{2}\mathcal{L}_A)K' = \begin{pmatrix} A+C & 0 \\ 0 & A-C \end{pmatrix}.$$

Let $P = A + C$ and $Q = A - C$. Then

$$P = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{35}} & \frac{1}{5} \end{pmatrix}_{(4n+1)\times(4n+1)},$$

and

$$Q = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{4}{7} & \frac{1}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{4}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{5} \end{pmatrix}_{(4n+1)\times(4n+1)},$$

Lemma 2.1 makes it simple to prove that the eigenvalues of $\frac{1}{2}\mathcal{L}_A$ are equal to the eigenvalues of S and K. Assuming that σ_i and ς_j $(i, j = 1, 2, \cdots, 2n+1)$ are, respectively, S and K's eigenvalues, with $\sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \cdots \leq \sigma_{4n+1}$, $\varsigma_1 \leq \varsigma_2 \leq \varsigma_3 \leq \cdots \leq \varsigma_{4n+1}$. We verify $\sigma_1 \geq 0$ and $\varsigma_1 \geq 0$ and it is clear that the normalized Laplacian spectrum of L_n^2 is $\{2\sigma_1,\ 2\sigma_2,\ \cdots,\ 2\sigma_{4n+1},\ 2\varsigma_1,\ 2\varsigma_2,\ \cdots,\ 2\varsigma_{4n+1}\}$. Keep in mind that the following, $|E(L_n^2)| = 48n + 6$, is a direct consequence of Lemma 2.2.

Lemma 3.1. Consider Q_n^2 to be the strong prism of the octagonal-quadrilateral networks. Then

$$Kf^*(L_n^2) = 2(48n+6)\left((4n+4)\frac{5}{6} + (4n-2)\frac{7}{8} + \frac{1}{2}\sum_{i=2}^{4n+1}\frac{1}{\sigma_i} + \frac{1}{2}\sum_{j=1}^{4n+1}\frac{1}{\varsigma_j}\right), \tag{3.2}$$

as desired.

The computations of $\sum_{i=2}^{4n+1}\frac{1}{\sigma_i}$ will be our focus in the sentences that follow.

Lemma 3.2. Suppose $\sigma_i (i = 1, 2, \cdots, 4n+1)$ is defined as stated above. one has

$$\sum_{i=2}^{4n+1}\frac{1}{\sigma_i} = \frac{406n^3 + 743n^2 + 299n + 105}{16n+2}.$$

Proof. Suppose that $\Phi(S) = x^{4n+1} + a_1 x^{4n} + \cdots + a_{4n}x^2 + a_{4n+1}x = x(x^{4n+1} + a_1 x^{4n} + \cdots + a_{4n}x + a_{4n+1})$. Then $\sigma_2, \sigma_3, \cdots, \sigma_{4n+1}$ satisfy the equation below:

$$x^{4n+1} + a_1 x^{4n} + \cdots + a_{4n}x + a_{4n+1} = 0,$$

and we discover that the following equation's roots are $\frac{1}{\sigma_2}, \frac{1}{\sigma_3}, \cdots, \frac{1}{\sigma_{4n+1}}$.

$$a_{4n}x^{4n} + a_{4n-1}x^{4n-1} + \cdots + a_1 x + 1 = 0.$$

Thus according Vieta's Theorem, one has

$$\sum_{i=2}^{4n+1} \frac{1}{\sigma_i} = \frac{(-1)^{4n-1} a_{4n-1}}{(-1)^{4n} a_{4n}}. \tag{3.3}$$

We consider S_i and set $s_i := det\ S_i$ for $1 \le i \le 4n$. We should discover the s_i equation, which may be used to calculate $(-1)^{4n}a_{4n}$ and $(-1)^{4n-1}a_{4n-1}$. Then, one has

$$w_1 = \frac{1}{5},\ w_2 = \frac{1}{25},\ w_3 = \frac{1}{125},\ w_4 = \frac{1}{875},\ w_5 = \frac{1}{6125},\ w_6 = \frac{1}{30625},\ w_7 = \frac{1}{153125},\ w_8 = \frac{1}{1071875}$$

and

$$\begin{cases} w_{4i} = \frac{2}{7} w_{4i-1} - \frac{1}{35} w_{4i-2},\ 1 \le i \le n; \\ w_{4i+1} = \frac{2}{7} w_{4i} - \frac{1}{49} w_{4i-1},\ 0 \le i \le n-1; \\ w_{4i+2} = \frac{2}{5} w_{4i+1} - \frac{1}{35} w_{4i},\ 0 \le i \le n-1; \\ w_{4i+3} = \frac{2}{5} w_{4i+2} - \frac{1}{25} w_{4i+1},\ 0 \le i \le n-1; \end{cases}$$

These generic formulas can be obtained by doing the direct computation shown below. One follows that

$$\begin{cases} w_{4i} = \frac{7}{5} \cdot (\frac{1}{1225})^i,\ 1 \le i \le n; \\ w_{4i+1} = \frac{1}{5} \cdot (\frac{1}{1225})^i,\ 0 \le i \le n-1; \\ w_{4i+2} = \frac{1}{25} \cdot (\frac{1}{1225})^i,\ 0 \le i \le n-1; \\ w_{4i+3} = \frac{1}{125} \cdot (\frac{1}{1225})^i,\ 0 \le i \le n-1. \end{cases} \tag{3.4}$$

On the other hand, take into account the i-th order primary submatrix U_i of P, which is made up of the last i rows and columns, $i = 1, 2, \cdots, 4n$. Let $u_i = det\ U_i$. Then

$$u_1 = \frac{1}{5},\ u_2 = \frac{1}{35},\ u_3 = \frac{1}{175},\ u_4 = \frac{1}{875},\ u_5 = \frac{1}{6125},\ u_6 = \frac{1}{42875},\ u_7 = \frac{1}{214375},\ u_8 = \frac{1}{1071875}$$

and

$$\begin{cases} u_{4i} = \frac{2}{5} u_{4i-1} - \frac{1}{25} u_{4i-2},\ 1 \le i \le n; \\ u_{4i+1} = \frac{2}{7} u_{4i} - \frac{1}{35} u_{4i-1},\ 0 \le i \le n-1; \\ u_{4i+2} = \frac{2}{7} u_{4i+1} - \frac{1}{49} u_{4i},\ 0 \le i \le n-1; \\ u_{4i+3} = \frac{2}{5} u_{4i+2} - \frac{1}{35} u_{4i+1},\ 0 \le i \le n-1. \end{cases}$$

These generic formulas could well be calculated explicitly and appear as follows. Then we have

$$\begin{cases} u_{4i} = \frac{7}{5} \cdot (\frac{1}{1225})^i,\ 1 \le i \le n; \\ u_{4i+1} = \frac{1}{5} \cdot (\frac{1}{1225})^i,\ 0 \le i \le n-1; \\ u_{4i+2} = \frac{1}{35} \cdot (\frac{1}{1225})^i,\ 0 \le i \le n-1; \\ u_{4i+3} = \frac{1}{175} \cdot (\frac{1}{1225})^i,\ 0 \le i \le n-1. \end{cases} \tag{3.5}$$

We will move on to two assertions in terms of $(-1)^{4n}a_{4n}$ and $(-1)^{4n-1}a_{4n-1}$ expressions. □

Fact 3.3

$$(-1)^{4n} a_{4n} = \frac{112n + 14}{25} \big(\frac{1}{1225}\big)^n.$$

Proof. Since S is a left-right symmetric matrix and the number $(-1)^{4n}a_{4n}$ has $(4n)$-row and $(4n)$-column and reflects the total number of S's primary minors, we may obtain

$$(-1)^{4n}a_{4n} = \sum_{i=1}^{4n+1} det\ \mathcal{L}_A[i] = \sum_{i=1}^{4n+1} det\begin{pmatrix} S_{i-1} & 0 \\ 0 & S_{4n+1-i} \end{pmatrix} = \sum_{i=1}^{4n+1} s_{i-1}\cdot s_{4n+1-i}, \qquad (3.6)$$

where

$$S_{4n+1-i} = \begin{pmatrix} l_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & l_{4n,4n} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & l_{4n+1,4n+1} \end{pmatrix}.$$

By Eqs.(3.4) and (3.5), we have

$$\begin{aligned}(-1)^{4n}a_{4n} &= 2w_{4n} + \sum_{k=1}^{n} w_{4(k-1)+3}\cdot u_{4(n-k)+1} + \sum_{k=1}^{n-1} w_{4k}\cdot u_{4(n-k)+1} \\ &\quad + \sum_{k=0}^{n-1} w_{4k+1}\cdot u_{3(n-k-l)+3} + \sum_{k=0}^{n-1} w_{4k+2}\cdot u_{4(n-k-l)+2} \\ &= 2\cdot\frac{7}{5}\cdot\Big(\frac{1}{1225}\Big)^{n} + \frac{n}{625}\cdot\Big(\frac{1}{1225}\Big)^{1-n} + \frac{49(n-1)}{25}\cdot\Big(\frac{1}{1225}\Big)^{n} \\ &\quad + \frac{n}{875}\cdot\Big(\frac{1}{1225}\Big)^{1-n} + \frac{n}{875}\cdot\Big(\frac{1}{1225}\Big)^{1-n} \\ &= \frac{112n+14}{25}\Big(\frac{1}{1225}\Big)^{n}.\end{aligned}$$

as desired. $\square$

Fact 3.4

$$(-1)^{4n-1}a_{4n-1} = \frac{406n^3+743n^2+299n+105}{4375}\Big(\frac{1}{1225}\Big)^{n}.$$

Proof. Observing that $(-1)^{4n-1}a_{4n-1}$ is the product of all main minors of P with $4n-1$ rows and columns, we have

$$(-1)^{4n-1}a_{4n-1} = \sum_{1\le i<j}^{4n-1}\begin{vmatrix} W_{i-1} & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & U \end{vmatrix},\ 1\le i<j\le 4n+1,$$

where

$$Z = \begin{pmatrix} l_{i+1,i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_{j-1,j-1} \end{pmatrix},$$

and

$$U = \begin{pmatrix} l_{j+1,j+1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & l_{4n,4n} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & l_{4n+1,4n+1} \end{pmatrix}.$$

Note that

$$(-1)^{4n-1}a_{4n-1} = \sum_{1\le i<j}^{4n+1} det\ S_{i-1}\cdot det\ Z\cdot det\ S_{4n+1-j} = \sum_{1\le i<j}^{4n+1} det\ Z\cdot s_{i-1}\cdot s_{4n+1-j}. \tag{3.7}$$

By Eq. (3.6), we are aware that the values of i and j affect how $det\ Z$ turns out. The following cases can then be categorized.

Case 1. $i = 4s$ and $j = 4t$ for $1 \le s < t \le n$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{vmatrix}_{(4t-4s-1)} \\ &= \frac{4}{175}(t-s)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 2. $i = 4s$ and $j = 4t+1$ for $1 \le s \le t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s)} \\ &= \big[4(t-s)+1\big]\Big(\frac{1}{1225}\Big)^{t-s}, \end{aligned}$$

Case 3. $i = 4s$ and $j = 4t+2$ for $1 \le s \le t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(4t-4s+1)} \\ &= \frac{1}{7}(4t-4s+2)\Big(\frac{1}{1225}\Big)^{t-s}, \end{aligned}$$

Case 4. $i = 4s$ and $j = 4t+3$ for $1 \le s \le t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{vmatrix}_{(4t-4s+2)} \\ &= \frac{1}{35}(4t-4s+3)\Big(\frac{1}{1225}\Big)^{t-s}, \end{aligned}$$

Case 5. $i = 4s$ and $j = 4n+1$ for $1 \le s \le n$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s)} \\ &= (4n-4s+1)\Big(\frac{1}{1225}\Big)^{n-s}, \end{aligned}$$

Case 6. $i = 4s+1$ and $j = 4t$ for $0 \le s < t \le n$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{vmatrix}_{(4t-4s-2)} \\ &= \frac{1}{25}(4t-4s-1)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 7. $i = 4s+1$ and $j = 4t+1$ for $0 \le s < t \le n-1$, and

$$\begin{aligned} det\ Z_9 &= \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s-1)} \\ &= \frac{4}{175}(t-s)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 8. $i = 4s+1$ and $j = 4t+2$ for $0 \le s \le t \le n-1$, and

$$
\begin{aligned}
det\ Z_9 &= \begin{vmatrix}
\frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\frac{2}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5}
\end{vmatrix}_{(4t-4s)} \\
&= (4t-4s+1)\Big(\frac{1}{1225}\Big)^{t-s}.
\end{aligned}
$$

Case 9. $i = 4s+1$ and $j = 4t+3$ for $0 \le s \le t \le n-1$, and

$$
\begin{aligned}
det\ Z &= \begin{vmatrix}
\frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5}
\end{vmatrix}_{(4t-4s+1)} \\
&= \frac{1}{5}(4t-4s+2)\Big(\frac{1}{1225}\Big)^{t-s}.
\end{aligned}
$$

Case 10. $i = 4s+1$ and $j = 4n+1$ for $0 \le s \le n-1$, and

$$
\begin{aligned}
det\ Z &= \begin{vmatrix}
\frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7}
\end{vmatrix}_{(4t-4s-1)} \\
&= \frac{4}{25}(n-s)\Big(\frac{1}{1225}\Big)^{n-s-1}.
\end{aligned}
$$

Case 11. $i = 4s+2$ and $j = 4t$ for $0 \le s < t \le n$, and

$$
\begin{aligned}
det\ Z_9 &= \begin{vmatrix}
\frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5}
\end{vmatrix}_{(4s-4t-3)} \\
&= \frac{1}{5}(4t-4s+2)\Big(\frac{1}{1225}\Big)^{t-s-1}.
\end{aligned}
$$

Case 12. $i = 4s+2$ and $j = 4t+1$ for $0 \le s < t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s-2)} \\ &= \frac{1}{35}(4t-4s-1)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 13. $i = 4s+2$ and $j = 4t+2$ for $0 \le s < t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(4t-4s-1)} \\ &= \frac{4}{245}(t-s)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 14. $i = 4s+2$ and $j = 4t+3$ for $0 \le s < t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{vmatrix}_{(4t-4s)} \\ &= (4t-4s+1)\Big(\frac{1}{1225}\Big)^{t-s}. \end{aligned}$$

Case 15. $i = 4s+2$ and $j = 4n+1$ for $0 \le s \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s-2)} \\ &= \frac{1}{35}(4n-4s-1)\Big(\frac{1}{1225}\Big)^{n-s-1}. \end{aligned}$$

Case 16. $i = 4s+3$ and $j = 4t$ for $0 \le s < t \le n$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{vmatrix}_{(4t-4s-4)} \\ &= (4t-4s-3)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 17. $i = 4s+3$ and $j = 4t+1$ for $0 \le s < t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s-3)} \\ &= \frac{1}{7}(4t-4s-2)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 18. $i = 4s+3$ and $j = 4t+2$ for $0 \le s < t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s-2)} \\ &= \frac{1}{49}(4t-4s-1)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 19. $i = 4s+3$ and $j = 4t+1$ for $0 \le s < t \le n-1$, and

$$\begin{aligned} det\ Z &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(4t-4s-3)} \\ &= \frac{1}{245}(4t-4s)\Big(\frac{1}{1225}\Big)^{t-s-1}. \end{aligned}$$

Case 20. $i = 4s+3$ and $j = 4n+1$ for $0 \leq s \leq n-1$, and

$$\begin{aligned}
det\ Z &= \begin{vmatrix}
\frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7}
\end{vmatrix}_{(4n-4s-3)} \\
&= \frac{1}{7}(4n-4s-2)\Big(\frac{1}{1225}\Big)^{n-s-1}.
\end{aligned}$$

Therefore, we can obtain

$$(-1)^{4n-1}a_{4n-1} = \sum_{1 \leq i < j \leq 4n+2} det\ Z \cdot w_{i-1} \cdot u_{4n+1-j} = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4, \tag{3.8}$$

where

$$\begin{aligned}
\zeta_1 &= \sum_{1 \leq s < t \leq n} P[4s, 4t] + \sum_{1 \leq s \leq t \leq n-1} P[4s, 4t+1] + \sum_{1 \leq s \leq t \leq n-1} P[4s, 4t+2] \\
&\quad + \sum_{1 \leq s \leq t \leq n-1} P[4s, 4t+3] + \sum_{1 \leq s \leq n} P[4s, 4n+1] \\
&= \frac{2n(n^2-1)}{328125}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{n(4n^2-9n+5)}{3750}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{n(2n^2-3n+1)}{375}\Big(\frac{1}{1225}\Big)^{n-1} \\
&\quad + \frac{n(4n^2-3n-1)}{750}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{2n^2-n}{125}\Big(\frac{1}{1225}\Big)^{n-1} \\
&= \frac{12n^3+n^2-8n}{625}\Big(\frac{1}{1225}\Big)^{n-1},
\end{aligned}$$

$$\begin{aligned}
\zeta_2 &= \sum_{0 \leq s < t \leq n} P[4s+1, 4t] + \sum_{0 \leq s < t \leq n-1} P[4s+1, 4t+1] + \sum_{0 \leq s \leq t \leq n-1} P[4s+1, 4t+2] \\
&\quad + \sum_{0 \leq s \leq t \leq n-1} P[4s+1, 4t+3] + \sum_{1 \leq s \leq n-1} P[4s+1, 4n+1] \\
&= \frac{7n(4n^2+9n-1)}{3750}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{14n(n^2-1)}{1875}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{n(4n^2+3n-1)}{750}\Big(\frac{1}{1225}\Big)^{n-1} \\
&\quad + \frac{n(2n^2+3n+1)}{375}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{14n(n+1)}{125}\Big(\frac{1}{1225}\Big)^{n-1} \\
&= \frac{16n^3+88n^2+65n}{625}\Big(\frac{1}{1225}\Big)^{n-1},
\end{aligned}$$

$$\begin{aligned}
\zeta_3 &= \sum_{0 \leq s < t \leq n} P[4s+2, 4t] + \sum_{0 \leq s < t \leq n-1} P[4s+2, 4t+1] + \sum_{0 \leq s < t \leq n-1} P[4s+2, 4t+2] \\
&\quad + \sum_{0 \leq s < t \leq n-1} P[4s+2, 4t+3] + \sum_{0 \leq s \leq n-1} P[4s+2, 4n+1] \\
&= \frac{n(2n^2+9n+13)}{375}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{4n^3-3n^2-7n+6}{750}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{2n(n^2-1)}{525}\Big(\frac{1}{1225}\Big)^{n-1} \\
&\quad + \frac{4n^3+3n^2-n-6}{525}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{2n^2+2n}{175}\Big(\frac{1}{1225}\Big)^{n-1} \\
&= \frac{26n^3+20n^2+23n+2}{875}\Big(\frac{1}{1225}\Big)^{n-1},
\end{aligned}$$

and

$$\begin{aligned}
\zeta_4 &= \sum_{0\le s<t\le n} P[4s+3,4t] + \sum_{0\le s<t\le n-1} P[4s+3,4t+1] + \sum_{0\le s<t\le n-1} P[4s+3,4t+2] \\
&\quad + \sum_{0\le s<t\le n-1} P[4s+3,4t+3] + \sum_{0\le s\le n-1} P[4s+3,4n+1] \\
&= \frac{n(4n^2+3n-19)}{750}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{2n^3-3n^2-5n+6}{375}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{4n^3+3n^2-7n+6}{1050}\Big(\frac{1}{1225}\Big)^{n-1} \\
&\quad + \frac{2n^3-2n}{375}\Big(\frac{1}{1225}\Big)^{n-1} + \frac{2n^2}{175}\Big(\frac{1}{1225}\Big)^{n-1} \\
&= \frac{16n^3+4n^2-43n+19}{875}\Big(\frac{1}{1225}\Big)^{n-1},
\end{aligned}$$

Hence, substituting ζ_1, ζ_2, ζ_3 and ζ_4 into Eq. (3.7), we can obtain

$$(-1)^{4n-1}a_{4n-1} = \zeta_1+\zeta_2+\zeta_3+\zeta_4 = \frac{406n^3+743n^2+299n+105}{4375}\Big(\frac{1}{1225}\Big)^{n-1}.$$

Fact 3.4 has been proved. □

Lemma 3.5. Let $0<\varsigma_1<\varsigma_2\le\varsigma_3\le\cdots\le\varsigma_{4n+1}$ be the eigenvalues of Q. Then

$$\sum_{i=2}^{4n+1}\frac{1}{\varsigma_i} = \frac{(-1)^{4n-1}b_{4n-1}}{(-1)^{4n}b_{4n}} = \frac{\theta_1+\theta_2}{128\left[217+50\sqrt{14}(15+4\sqrt{14})^{n-1}+(217-50\sqrt{14})(15-4\sqrt{14})^{n-1}\right]}.$$

where $\theta_1 = 5488000+1615425\sqrt{14}+24500n(392+193\sqrt{14})(15+4\sqrt{14})^{n-1}$ and $\theta_2 = 5488000-1615425\sqrt{14}+24500n(392-193\sqrt{14})(15-4\sqrt{14})^{n-1}$

***Proof*.** Suppose that $\Phi(Q) = x^{4n}+b_1x^{4n-1}+\cdots+b_{4n-1}x+b_{4n}$. So $\frac{1}{\varsigma_1},\frac{1}{\varsigma_2},\cdots,\frac{1}{\varsigma_{4n}}$ satisfy the equation below:

$$b_{4n}x^{4n}+b_{4n-1}x^{4n-1}+\cdots+b_1x+1=0.$$

By Vieta's Theorem, one holds that

$$\sum_{j=1}^{4n+1}\frac{1}{\varsigma_j} = \frac{(-1)^{4n}b_{4n}}{det\ K}. \tag{3.9}$$

Let $r_p = det\ R_p$ and let R_p be the p-th order principal minors of matrix T for convenience's sake. We will discover the r_p equation for $1\le p\le 4n$, which can be used to calculate $(-1)^{4n}b_{4n}$ and $det\ K$. Next, we achieve

$$r_1=\frac{3}{5},\ r_2=\frac{1}{5},\ r_3=\frac{7}{125},\ r_4=\frac{23}{875},\ r_5=\frac{17}{1225},\ r_6=\frac{147}{30625},\ r_7=\frac{209}{153125},\ r_8=\frac{689}{1071875}$$

and

$$\begin{cases}
r_{4i} = \frac{4}{7}r_{4i-1}-\frac{1}{35}r_{4i-2},\ 1\le i\le n; \\
r_{4i+1} = \frac{4}{7}r_{4i}-\frac{1}{49}r_{4i-1},\ 0\le i\le n-1; \\
r_{4i+2} = \frac{2}{5}r_{4i+1}-\frac{1}{35}r_{4i},\ 0\le i\le n-1; \\
r_{4i+3} = \frac{2}{5}r_{4i+2}-\frac{1}{25}r_{4i+1},\ 0\le i\le n-1.
\end{cases}$$

These general formulas can be obtained directly by computation and are shown below. One follows that

$$\begin{cases} r_{4i} = \frac{7+\sqrt{14}}{10}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{7-\sqrt{14}}{10}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 1\le i\le n; \\ r_{4i+1} = \frac{42+10\sqrt{14}}{140}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{42-10\sqrt{14}}{140}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 0\le i\le n-1; \\ r_{4i+2} = \frac{35+9\sqrt{14}}{350}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{35-9\sqrt{14}}{350}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 0\le i\le n-1; \\ r_{4i+3} = \frac{49+13\sqrt{14}}{1750}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{49-13\sqrt{14}}{1750}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 0\le i\le n-1. \end{cases}$$

We go on in what follows by taking into account the next Facts. □

Fact 3.6

$$det\ K = \frac{217+58\sqrt{14}}{30625}(\frac{15+4\sqrt{14}}{1225})^{n-1} + \frac{217-58\sqrt{14}}{30625}(\frac{15-4\sqrt{14}}{1225})^{n-1}.$$

Proof. By expanding $det\ Q$ with regards to the last row, we have

$$det\ K = \frac{3}{5}r_{4n} - \frac{1}{35}r_{4n-1} = \frac{217+58\sqrt{14}}{30625}(\frac{15+4\sqrt{14}}{1225})^{n-1} + \frac{217-58\sqrt{14}}{30625}(\frac{15-4\sqrt{14}}{1225})^{n-1}.$$

The result as desired. On the other hand, we consider the $i-th$ order principal sub-matrix of Q, denoted by S_i, generated by the last i rows and columns, $1\le i\le 4n$. Let $s_i = detS_i$. Then $s_1 = \frac{3}{5}, s_2 = \frac{11}{35}, s_3 = \frac{19}{175}, s_4 = \frac{27}{875}, s_5 = \frac{89}{6125}, s_6 = \frac{329}{42875}, s_7 = \frac{569}{214375}, s_8 = \frac{809}{1071875}$ and

$$\begin{cases} s_{4i} = \frac{2}{5}s_{4i-1} - \frac{1}{25}s_{4i-2},\ 1\le i\le n; \\ s_{4i+1} = \frac{4}{7}s_{4i} - \frac{1}{35}s_{4i-1},\ 0\le i\le n-1; \\ s_{4i+2} = \frac{4}{7}s_{4i+1} - \frac{1}{49}s_{4i},\ 0\le i\le n-1; \\ s_{4i+3} = \frac{2}{5}s_{4i+2} - \frac{1}{35}s_{4i+1},\ 0\le i\le n-1. \end{cases}$$

In a manner similar to that explained before, we have

$$\begin{cases} s_{4i} = \frac{14+3\sqrt{14}}{20}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{14-3\sqrt{14}}{20}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 1\le i\le n; \\ s_{4i+1} = \frac{42+11\sqrt{14}}{140}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{42-11\sqrt{14}}{140}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 0\le i\le n-1; \\ s_{4i+2} = \frac{88+25\sqrt{14}}{560}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{88-25\sqrt{14}}{560}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 0\le i\le n-1; \\ s_{4i+3} = \frac{266+71\sqrt{14}}{4900}\cdot(\frac{15+4\sqrt{14}}{1225})^i + \frac{266-71\sqrt{14}}{4900}\cdot(\frac{15-4\sqrt{14}}{1225})^i,\ 0\le i\le n-1. \end{cases}$$

□

Fact 3.7

$$\begin{aligned}(-1)^{4n}b_{4n} \quad = \quad & \frac{31360+9231\sqrt{14}+140n(392+193\sqrt{14})}{22400}(\frac{15+4\sqrt{14}}{1225})^{n-1} \\ & +\frac{31360-9231\sqrt{14}+140n(392-193\sqrt{14})}{22400}(\frac{15-4\sqrt{14}}{1225})^{n-1}.\end{aligned}$$

Proof. Since the number $(-1)^{4n}b_{4n}$, which has a $(4n)$-row and $(4n)$-column, represents the total of all the major minors of T. For simplification, let's assume that in the following, diagonal elements of K are represented by g_{ii}. K is left-right symmetric, so we may obtain

$$(-1)^{4n}b_{4n} \quad = \quad \sum_{i=1}^{4n+1} det\ K[i] = \sum_{i=1}^{4n+1} det\begin{pmatrix} R_{i-1} & 0 \\ 0 & R_{4n+1-i}\end{pmatrix} = \sum_{i=1}^{4n+1} r_{i-1}\cdot r_{4n+1-i}, \qquad (3.10)$$

where

$$R_{4n+1-i} \quad = \quad \begin{pmatrix} g_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & g_{4n,4n} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & g_{4n+1,4n+1} \end{pmatrix}.$$

In line with Eq. (3.9), we have

$$\begin{aligned}(-1)^{4n}b_{4n} =& 2r_{4n} + \sum_{k=1}^{n} det\ K[4k] + \sum_{k=0}^{n-1} det\ Q[4k+1] + \sum_{k=0}^{n-1} det\ K[4k+2] + \sum_{k=0}^{n-1} det\ K[4k+3] \\ =& r_{4n} + s_{4n} + \sum_{k=1}^{n} r_{4(k-1)+3} \cdot s_{4(n-k)+1} + \sum_{k=0}^{n-1} r_{4k} \cdot s_{4(n-k)} \\ &+ \sum_{k=0}^{n-1} r_{4k+1} \cdot s_{4(n-k-1)+3} + \sum_{k=0}^{n-1} r_{4k+2} \cdot s_{4(n-k-1)+2}. \end{aligned} \tag{3.11}$$

The following forms can be generated by using above equations.

$$\begin{aligned}\sum_{k=1}^{n} r_{4(k-1)+3} \cdot s_{4(n-k)+1} =& n\left[\frac{28+7\sqrt{14}}{40}(\frac{15+4\sqrt{14}}{1225})^{n-1} + \frac{28-7\sqrt{14}}{40}(\frac{15-4\sqrt{14}}{1225})^{n-1}\right] \\ &+ \frac{\sqrt{14}}{200}(\frac{15+4\sqrt{14}}{1225})^{n} - \frac{\sqrt{14}}{200}(\frac{15-4\sqrt{14}}{1225})^{n}. \end{aligned} \tag{3.12}$$

$$\begin{aligned}\sum_{k=0}^{n-1} r_{4k} \cdot s_{4(n-k)} =& n\left[\frac{28+7\sqrt{14}}{40}(\frac{15+4\sqrt{14}}{1225})^{n} + \frac{28-7\sqrt{14}}{40}(\frac{15-4\sqrt{14}}{1225})^{n}\right] \\ &+ \frac{11\sqrt{14}}{100}(\frac{15+4\sqrt{14}}{1225})^{n-1} - \frac{11\sqrt{14}}{100}(\frac{15-4\sqrt{14}}{1225})^{n-1}, \end{aligned} \tag{3.13}$$

$$\begin{aligned}\sum_{k=0}^{n-1} r_{4k+1} \cdot r_{4(n-k-1)+3} =& n\left[\frac{42+13\sqrt{14}}{40}(\frac{15+4\sqrt{14}}{1225})^{n-1} + \frac{42-13\sqrt{14}}{40}(\frac{15-4\sqrt{14}}{1225})^{n-1}\right] \\ &+ \frac{11\sqrt{14}}{280}(\frac{15+4\sqrt{14}}{1225})^{n} - \frac{11\sqrt{14}}{280}(\frac{15-4\sqrt{14}}{1225})^{n}, \end{aligned} \tag{3.14}$$

$$\begin{aligned}\sum_{k=0}^{n-1} r_{4k+2} \cdot s_{4(n-k-1)+2} =& n\left[\frac{98+85\sqrt{14}}{160}(\frac{15+4\sqrt{14}}{1225})^{n-1} + \frac{98-85\sqrt{14}}{160}(\frac{15-4\sqrt{14}}{1225})^{n-1}\right] \\ &+ \frac{\sqrt{14}}{128}(\frac{15+4\sqrt{14}}{1225})^{n} - \frac{\sqrt{14}}{128}(\frac{15-4\sqrt{14}}{1225})^{n}, \end{aligned} \tag{3.15}$$

and

$$r_{4n} + s_{4n} = \frac{28+5\sqrt{14}}{20}(\frac{15+4\sqrt{14}}{1225})^{n} + \frac{28-5\sqrt{14}}{20}(\frac{15-4\sqrt{14}}{1225})^{n}. \tag{3.16}$$

Equations (3.12) to (3.16) can be substituted into Fact 2 to produce the desired outcome (3.11). Lemma 3.3 holds right away in light of Facts 1 and 2 and (3.11), respectively. $\square$

The theorem that follows is derived from Lemmas 3.1 through 3.3.

Theorem 3.8. Assume that Q_n^2 serves as the strong prism for the networks of octagons-quadrilateral. One has

$$\begin{aligned} Kf^*(Q_n^2) &= 2486n^3 + 5114n^2 + 2028n + 649 \\ &+(48n+6)\left[\frac{(-1)^{4n}b_{4n}}{detK}\right], \end{aligned}$$

where

$$\begin{aligned} (-1)^{4n}b_{4n} &= \frac{31360 + 9231\sqrt{14} + 140n(392 + 193\sqrt{14})}{22400}\left(\frac{15+4\sqrt{14}}{1225}\right)^{n-1} \\ &+\frac{31360 - 9231\sqrt{14} + 140n(392 - 193\sqrt{14})}{22400}\left(\frac{15-4\sqrt{14}}{1225}\right)^{n-1}. \end{aligned}$$

and

$$det\ K = \frac{217+58\sqrt{14}}{30625}\left(\frac{15+4\sqrt{14}}{1225}\right)^{n-1} + \frac{217-58\sqrt{14}}{30625}\left(\frac{15-4\sqrt{14}}{1225}\right)^{n-1}.$$

According to the aforementioned theorem, Q_n^2's multiplicative degree-Kirchhoff index.

Theorem concerning the expatiatory formula of the spanning trees of Q_n^2 is what we will demonstrate next.

Theorem 3.9. Consider Q_n^2 to be the strong prism for the networks of octagonal-quadrilateral. One has

$$\tau(Q_n^2) = \frac{3^{4n+4}\cdot 2^{16n-2}}{8575}\Big((217+58\sqrt{14})(15+4\sqrt{14})^{n-1} + (217-58\sqrt{14})(15-4\sqrt{14})^{n-1}\Big).$$

Proof. According to Lemma 2.2, we have

$$\prod_{i=2}^{4n+1}\sigma_i = (-1)^{4n}a^{4n}.$$

By Fact 3.3, we have

$$\prod_{i=2}^{4n+1}\sigma_i = \frac{112n+14}{25}\Big(\frac{1}{1225}\Big)^n.$$

At the same time, we are easy to get $\prod\limits_{v\in V_{Q_n^2}} d(Q_n^2) = 5^{8n+8}\cdot 7^{8n-4}$ and according to the $|E(Q_n^2)| = 48n+4$ mentioned in the previous text. Let't combine that with Lemma 2.3, we have

$$\begin{aligned} \tau(Q_n^2) &= \frac{1}{2|E(L_n^2)|}\Big((\frac{6}{5})^{4n+4}\cdot(\frac{8}{7})^{4n-2}\cdot\prod_{i=2}^{4n+1}2\sigma_i\cdot\prod_{j=1}^{4n+1}2\varsigma_j\cdot\prod_{v\in V_{Q_n^2}}d(Q_n^2)\Big) \\ &= \frac{2^{16n-2}\cdot 3^{4n+4}}{8575}\Big((217+58\sqrt{14})(15+4\sqrt{14})^{n-1} + (217-58\sqrt{14})(15-4\sqrt{14})^{n-1}\Big). \end{aligned}$$

At this point, the proof of the theorem has already completed. □

4. Conclusion

In this paper, employing the principles and characteristics of the theory of the decomposition of the spectrum of the normalized Laplacian matrix, we have calculated the spanning tree and multiplicative degree-Kirchhoff index of the graphs produced by the strong product of Q_n^2.

Funding

This work was supported in part by Anhui Provincial Natural Science Foundation under Grant 2008085J01 and Natural Science Fund of Education Department of Anhui Province under Grant KJ2020-A0478.

Funding

No potential conflict of interest was reported by the author(s).

Data Availability Statement

Included in the article are the graphs and other data that were utilized to support this investigation. Additionally, upon reasonable request, any pertinent data are accessible from the respective author.

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