A nonlocal diffusion SIR epidemic model with nonlocal incidence rate and free boundaries

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Abstract

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Keywords: nonlocal diffusion; nonlocal incidence rate; free boundaries; spreading; vanishing

AMS subject classifications (2000): 35K57, 35R35, 92D30

1 Introduction

Epidemic models are interesting and significant topics in mathematical biology. Kermack and McKendrick gave the classical theoretical papers on epidemic models in 1927 ([5]) and 1932 ([6]), which have a substantial influence in the development of mathematical models and are still relevant in a surprising number of epidemic situations.

In the classical SIR model, the population is divided into three distinct classes: the susceptible but uninfected class S, who can catch the infectious disease, the infective class I, who have the infectious disease and can transmit it, and the removed class R, namely, those who have had the infectious disease, either are recovered, immune or isolated until recovered. Since the population can migrate, the diffusion terms should be taken into consideration and thus the diffusion SIR system appeared, and one of them is as follows

$$\begin{cases} S_t - d_1\Delta S = -\beta SI - \mu_1 S + b, & t > 0,\ x \in \Omega, \\ I_t - d_2\Delta I = \beta SI - \mu_2 I - \alpha I, & t > 0,\ x \in \Omega, \\ R_t - d_3\Delta R = \alpha I - \mu_3 R, & t > 0,\ x \in \Omega, \\ \partial_\eta S = \partial_\eta I = \partial_\eta R, & t > 0, x \in \partial\Omega, \\ S(0,x) = S_0(x), I(0,x) = I_0(x), R(0,x) = R(x), & x \in \Omega, \end{cases} \tag{1.1}$$

with homogeneous Neumann boundary condition. We know the solution of system (1.1) is always positive for any time $t > 0$ no matter what the nonnegative nontrivial initial date is. It means that the disease spreads to the whole area immediately even when the infectious is confined to a small part of the area in the beginning. It doesn't match the observed fact that disease always spreads gradually. So Kim et al.([10]) considered a SIR epidemic model in a radially symmetric domain

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with a free boundary, which describes the spreading frontier of the disease

$$\begin{cases} S_t - d_1\Delta S = -\beta SI - \mu_1 S + b, & t > 0,\ r > 0, \\ I_t - d_2\Delta I = \beta SI - \mu_2 I - \alpha I, & t > 0,\ 0 < r < h(t), \\ R_t - d_3\Delta R = \alpha I - \mu_3 R, & t > 0,\ 0 < r < h(t), \\ S_r(t,0) = R_r(t,0) = I_r(t,0) = 0, & t > 0, \\ I(t,r) = R(t,r) = 0, & t > 0,\ r \geq h(t), \\ h'(t) = -\mu I_r(t,h(t)),\ h(0) = h_0 > 0, & t > 0, \\ S(r,0) = S_0(r),\ I(r,0) = I_0(r),\ R(r,0) = R_0(r), & r \geq 0. \end{cases} \tag{1.2}$$

The existence and uniqueness of the global solution were given by the contraction mapping theorem. They showed that the disease will not spread to the whole area if the basic reproduction number $R_0 := \frac{b\beta}{\mu_1(\mu_2+\alpha)} < 1$ or the initial infected radius h_0 and μ are sufficiently small even that $R_0 > 1$. Moreover, the disease will spread to the whole area if $R_0 > 1$ and the initial infected radius h_0 is suitably large.

In fact, because of the propagation of viruses in the air, the susceptible at the point x and time t should be influenced (infected) by the infectives around them. Hence the mechanism of infection is governed by a nonlocal law

$$(K * I)(t,x) = \int_{\mathbb{R}} K(x,y)I(t,y)dy,$$

where the kernel function $K(x,y) \geq 0$ denotes the probability density that weights the contributions of infectious at location y to the infection of susceptible individuals at location x.

In [7], Cao et al. concerned with a diffusion SIS model with nonlocal incidence rate and double free boundaries which somehow can be viewed a nonlocal version of (1.2). They define $R_0^F(t)$ as a critical function and got the full information about the sufficient conditions that ensure the disease spreading or vanishing, which exhibits a detailed description of the communicable mechanism of the disease. Their results imply that the nonlocal interaction may enhance the spread of the disease.

In [9], Huang and Wang changed the equation of S in (1.2) and studied the following SIR epidemic model

$$\begin{cases} S_t - S_{xx} = a(I+R) - bSI, & t > 0, x > 0, \\ I_t - I_{xx} = bSI - (a+c)I, & t > 0, 0 < x < h(t), \\ R_t - R_{xx} = cI - aR, & t > 0, 0 < x < h(t), \\ S_x = I_x = R_x = 0, & t \geq 0, x = 0, \\ I = R = 0, & t \geq 0, x \geq h(t), \\ h'(t) = -\mu I_x(t,h(t)), & t \geq 0, \\ h(t) = h_0, S(t,x) = S_0(x), & t = 0, x \geq 0, \\ R(t,x) = R_0(x), I(t,x) = I_0(x), & t = 0, 0 \leq x \leq h_0. \end{cases}$$

The existence, uniqueness and some estimates of the global solution were also discussed first. Then, with the help of studying the long-time behavior of the solution to a Cauchy problem for a nonhomogeneous heat equation, the long-time behavior of the solution was obtained for the disease vanishing case. At last, some sufficient conditions for the disease vanishing were established. Then in [8], Huang and Wang considered the following nonlocal case with double free boundaries

$$\begin{cases} S_t - S_{xx} = \gamma I - S(K*I), & t>0, x\in\mathbb{R},\\ I_t - I_{xx} = bI - S(K*I), & t>0, x\in(g(t),h(t)),\\ I(t,x)=0, & t\geq 0, x\in\mathbb{R}\backslash(g(t),h(t)),\\ g'(t) = -\mu I_x(t,g(t)), h'(t) = -\mu I_x(t,h(t)), & t>0,\\ S(0,x)=S_0(x), I(0,x)=I_0(x), & x\in\mathbb{R},\\ -g(0)=h_0=h(0). \end{cases}$$

However, it has been increasingly recognized that the dispersal of many species is better described by "nonlocal diffusion" such as

$$J*u-u=\int_{\mathbb{R}} J(x-y)u(t,y)dy - u$$

rather than "local diffusion" Δu, where $J:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is nonnegative even function and satisfies $2\int_0^\infty J(x)dx = 1$.

In [2], Cao et al. considered the dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, which can be viewed the nonlocal version of the well known local diffusion model $u_t - du_{xx} = f(u)$ in [3]. Their results showed that their spreading-vanishing criteria are significantly different. The result in [3] indicated that no matter how small is the diffusion coefficient d relative to the initial growth rate $f'(0)$, vanishing can always happen if h_0 and μ are both sufficiently small. However, in [2], as long as $d\leq f'(0)$, spreading always happens.

A nonlocal version of the free boundary problem studied by Zhao et al. ([11]) is as follows:

$$\begin{cases} S_t = d\int_{\mathbb{R}} J(x-y)S(t,y)dy - dS - \beta SI - \mu_1 S + b, & t>0,\ x\in\mathbb{R},\\ I_t = d\int_{g(t)}^{h(t)} J(x-y)I(t,y)dy - dI + \beta SI - \mu_2 I - \alpha I, & t>0,\ x\in(g(t),h(t)),\\ R_t = d\int_{g(t)}^{h(t)} J(x-y)R(t,y)dy - dR + \alpha I - \mu_3 R, & t>0,\ x\in(g(t),h(t)),\\ I(t,x)=R(t,x)=0, & t\geq 0, x\in\mathbb{R}\setminus(g(t),h(t)),\\ g'(t) = -\mu\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J(x-y)I(t,x)dydx, & t>0,\\ h'(t) = \mu\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J(x-y)I(t,x)dydx, & t>0,\\ -g(0)=h(0)=h_0>0, S(0,x)=S_0(x), & x\in\mathbb{R},\\ I(0,x)=I_0(x),\ R(0,x)=R_0(x), & x\in(-h_0,h_0). \end{cases} \tag{1.3}$$

In [11], it was showed that the disease will not spread to the whole area if the basic reproduction number $R_0 := \frac{b\beta}{\mu_1(\mu_2+\alpha)} < 1$ or the initial infected radius h_0, expanding ability μ, and the initial datum S_0 are all sufficiently small when $1 < R_0 < 1 + \frac{d}{\mu_2+\alpha}$. Moreover, the disease will spread to the whole area if $R_0 > 1$ and the initial infected radius h_0 is suitably large or h_0 is small but μ is large.

Motivated by above, in this paper, we consider the following SIR epidemic problem with double free boundaries,

$$
\begin{cases}
S_t = d\displaystyle\int_{\mathbb{R}} J(x-y)S(t,y)dy - dS - \beta S \int_{g(t)}^{h(t)} K(x,y)I(t,y)dy - \mu_1 S + b, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R}, \\
I_t = d\displaystyle\int_{g(t)}^{h(t)} J(x-y)I(t,y)dy - dI + \beta S \int_{g(t)}^{h(t)} K(x,y)I(t,y)dy - \mu_2 I - \alpha I, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in(g(t),h(t)), \\
R_t = d\displaystyle\int_{g(t)}^{h(t)} J(x-y)R(t,y)dy - dR + \alpha I - \mu_3 R, & t>0,\ x\in(g(t),h(t)), \\
I(t,x) = R(t,x) = 0, & t\ge 0,\ x\in\mathbb{R}\setminus(g(t),h(t)), \\
g'(t) = -\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J(x-y)I(t,x)dydx, & t>0, \\
h'(t) = \mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J(x-y)I(t,x)dydx, & t>0, \\
-g(0) = h(0) = h_0, \\
S(0,x) = S_0(x), & x\in\mathbb{R}, \\
I(0,x) = I_0(x),\ R(0,x) = R_0(x), & x\in(-h_0,h_0),
\end{cases}
\tag{1.4}
$$

where nonlocal diffusion and nonlocal incidence rate appear together. Here the parameters d, b, μ and h_0 are positive constants, μ_1, μ_2 and μ_3 are the death rate of susceptible, infectious and recovered individuals respectively, α is the recovery rate of the infectives.

We are interested in, compared with [10] and [11], whether the critical value of the spreading and vanishing of the epidemic disease has changed, and what difference it will make under the interaction of nonlocal diffusion and nonlocal incidence rate.

Throughout this paper, we always assume that the initial functions I_0, S_0 and R_0 satisfy

(H) $S_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $S_0 > 0$ in $\mathbb{R}$; $I_0, R_0 \in C([-h_0, h_0])$, $I_0, R_0 > 0$ in $(-h_0, h_0)$ and $I_0 = R_0 = 0$ in $\mathbb{R}\setminus(-h_0, h_0)$.

The kernel function K satisfies

(K) K is bounded and locally Lipschitz continuous in $\mathbb{R}^2$, $K(x,y) = K(y,x) \ge 0$ and $\displaystyle\int_{\mathbb{R}} K(x,y)dx = 1$ for any $y \in \mathbb{R}$.

The another kernel funciton $J : \mathbb{R} \to \mathbb{R}$ satisfies

(J) J is continuous and symmetric, $J(0) > 0$, $J(x) \ge 0$, $\displaystyle\int_{\mathbb{R}} J(x)dx = 1$, $\sup_{\mathbb{R}} J < \infty$.

With the same argument as in [4, 11], we can get the existence and uniqueness of the solution to (1.4).

Theorem 1.1. *The problem (1.4) has a unique positive solution (S, I, R, g, h) defined for all $t > 0$, and there exists a constant C such that*

$$
0 < S \le C \quad in\ \mathbb{R}_+ \times \mathbb{R}, \quad 0 < I, R \le C \quad in\ D_\infty, \quad 0 < -g', h' \le C \ \in\ \mathbb{R}_+,
$$

where $\mathbb{R}_+ = (0, \infty)$ and $D_\infty = \{(t,x) \in \mathbb{R}^2 : t \in (0,\infty),\ x \in (g(t), h(t))\}$.

It is easily seen that $h(t)$ is monotonically increasing and that $g(t)$ is monotonically decreasing. Therefore there exists $g_\infty \in [-\infty, 0)$ and $h_\infty \in (0, \infty]$ such that $g_\infty = \lim_{t\to\infty} g(t)$ and $h_\infty = \lim_{t\to\infty} h(t)$.

We define the basic reproduction number R_0 as

$$R_0 := \frac{b\beta}{\mu_1(\mu_2 + \alpha)},$$

the main result is as follows.

Theorem 1.2. *Let (S, I, R, g, h) be the solution of (1.4). Assume further that $J(x) > 0$ in $\mathbb{R}$. Then we have*

(i) If $R_0 < 1$, then $h_\infty - g_\infty < \infty$.

(ii) If $R_0 > 1$, then there exists $\ell^* > 0$ (ℓ^* is determined by (3.6)) such that

(a) if $2h_0 \geq \ell^*$, then $h_\infty - g_\infty = \infty$.

(b) if $2h_0 < \ell^*$, then there exists $0 < \mu_* \leq \mu^*$ such that $h_\infty - g_\infty = \infty$ for $\mu > \mu^*$, and $h_\infty - g_\infty < \infty$ for $0 < \mu \leq \mu_*$ and $S_0(x) \leq \frac{b}{\mu_1}$.

(iii) If $h_\infty - g_\infty < \infty$, then

$$\lim_{t\to\infty} \max_{[g(t),h(t)]} I(t, \cdot) = 0, \quad \lim_{t\to\infty} \max_{[g(t),h(t)]} R(t, \cdot) = 0,$$

$$\lim_{t\to\infty} S(t, \cdot) = \frac{a - r}{b} \text{ locally uniformly in } \mathbb{R}.$$

Remark 1.1. *Theorem 1.2 (i) indicates that vanishing always happens when $R_0 < 1$, there is no requirement for the initial value and the initial area. However, in [7], for SIS model, when $R_0^F(0) < 1$, vanishing can always happen provided that $\|I_0\|_{L^\infty}$ and h_0 and N_0 are sufficiently small.*

Theorem 1.2 (ii) shows that if $R_0 > 1$, the disease will not spread to the whole area if h_0, S_0 and μ are sufficiently small. The result is the same as in [10] for local diffusion case, but is different with that in [11] where apart from $R_0 > 1$, the condition $R_0 < 1 + \frac{d}{\mu_2+\alpha}$ is required. This is one of the differences between the local and nonlocal incidence rate.

The organization of this paper is as follows. In Section 2, we firstly analyze an eigenvalue problem and give the properties of its principal eigenvalue λ_p. Then we propose two maximum principles and a comparison principle for our free boundary problem which will be frequently used in this paper. Section 3 is concerned with some sufficient conditions that ensure the disease vanishing and spreading, and give the long-time asymptotic limit of the solution when vanishing happens.

2 Eigenvalue problem and comparison principle

Define the linear operator $\mathcal{L}_\Omega + c : C(\bar{\Omega}) \to C(\bar{\Omega})$ as follows

$$(\mathcal{L}_\Omega + c)[\phi](x) = -d \left(\int_\Omega J(x-y)\phi(y)dy - \phi(x) \right) - c \int_\Omega K(x,y)\phi(y)dy + (\mu_2 + \alpha)\phi(x),$$

where Ω is an open interval in $\mathbb{R}$, possibly unbounded, $d, c, \mu_2, \alpha > 0$, and the kernel J and K satisfy **(J)** and **(K)** respectively. Define

$$\lambda_p(\mathcal{L}_\Omega + c) := \sup\{\lambda \in \mathbb{R} : (\mathcal{L}_\Omega + c)\phi \geq \lambda\phi \text{ in } \Omega \text{ for some } \phi \in C(\bar{\Omega}), \phi > 0\}.$$

As usual, if $\lambda_p(\mathcal{L}_\Omega + c)$ is an eigenvalue of the operator $\mathcal{L}_\Omega + c$ with a continuous and positive eigenfunction, we call it a principal eigenvalue. In fact, when $\Omega = (\ell_1, \ell_2)$ with $-\infty \leq \ell_1 < \ell_2 \leq \infty$, we can get that $\lambda_p(\mathcal{L}_{(\ell_1,\ell_2)} + c)$ is a principle eigenvalue([1, 2]).

By the variational characterization of $\lambda_p(\mathcal{L}_\Omega + c)$ (see, e.g., [1]), we have

$$\lambda_p(\mathcal{L}_\Omega + c) = \inf_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_2 = 1}} \left\{ -d \int_\Omega \int_\Omega J(x-y)\phi(y)\phi(x)dydx + d - c \int_\Omega \int_\Omega K(x,y)\phi(y)\phi(x)dydx + \alpha + \mu_2 \right\}.$$

Now we give its properties.

Proposition 2.1. *Assume that* **(J)** *and* **(K)** *hold, c is a positive constant and $\Omega = (\ell_1, \ell_2)$ with $-\infty \leq \ell_1 < \ell_2 \leq \infty$. Then the following hold true:*

(i) $\lambda_p(\mathcal{L}_{(\ell_1,\ell_2)} + c)$ *is strictly decreasing and continuous in $\ell = \ell_2 - \ell_1$,*

(ii) $\lim_{\ell_2 - \ell_1 \to +\infty} \lambda_p(\mathcal{L}_{(\ell_1,\ell_2)} + c) = -c + \alpha + \mu_2$,

(iii) $\lim_{\ell_2 - \ell_1 \to 0} \lambda_p(\mathcal{L}_{(\ell_1,\ell_2)} + c) = d + \alpha + \mu_2$.

Proof. We can refer to [2, Proposition 3.4] and [8, Proposition B.1] to prove it. We omit the details here. □

To give the maximum principles and comparison principle, we first introduce some notations. For given $h_0, T > 0$ we define

$$H_{h_0,T} := \left\{ h \in C([0,T]) : h(0) = h_0, \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\},$$
$$G_{h_0,T} := \{ g \in C([0,T]) : -g \in H_{h_0,T} \}.$$

For $g \in G_{h_0,T}$, $h \in H_{h_0,T}$, we define

$$\Omega_T^{g,h} := \left\{ (t,x) \in \mathbb{R}^2 : 0 < t \leq T,\ g(t) < x < h(t) \right\},$$
$$\Omega_T^{\infty} := \left\{ (t,x) \in \mathbb{R}^2 : 0 < t \leq T,\ x \in \mathbb{R} \right\}.$$

The following is the maximum principle for nonlocal diffusion and nonlocal incidence rate, which is critical in this paper and can be proved by the similar argument as in [2, Lemma 2.2] and [13, Lemma 2.1].

Lemma 2.1. *(Maximum principle) Assume that* **(J)** *and* **(K)** *hold, $g \in G_{h_0,T}, h \in H_{h_0,T}$ for some $h_0, T > 0$. Suppose that $u(t,x)$ as well as $u_t(t,x)$ is continuous in $\bar{\Omega}_T^{g,h}$ and satisfies, for some c_1, $c_2 \in L^\infty(\Omega_T^{g,h})$, $c_1 \geq 0$,*

$$\begin{cases} u_t \geq d \displaystyle\int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du(t,x) + c_1 \int_{g(t)}^{h(t)} K(x,y)u(t,y)dy + c_2 u(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ g(t) < x < h(t), \\ u(t, g(t)) \geq 0,\ u(t,h(t)) \geq 0, \qquad\qquad\qquad\qquad t > 0, \\ u(0,x) \geq 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad |x| \leq h_0. \end{cases} \tag{2.1}$$

Then $u(t,x) \geq 0$ for all $0 \leq t \leq T$ and $x \in [g(t), h(t)]$. Moreover, if $u(0,x) \not\equiv 0$ in $[-h_0, h_0]$, then $u(t,x) > 0$ in $\Omega_T^{g,h}$.

By Lemma 2.2 of [12] and Lemma 2.1, we can get the following maximum principle.

Lemma 2.2. *(Maximum principle) Assume that* **(J)** *and* **(K)** *hold,* $g \in G_{h_0,T}$, $h \in H_{h_0,T}$ *for* h_0, $T > 0$. *Suppose that* $S(t,x)$ *and* $S_t(t,x)$ *are continuous in* $\bar{\Omega}_T^{\infty}$, $I(t,x)$, $R(t,x)$, $I_t(t,x)$ *and* $R_t(t,x)$ *are continuous in* $\bar{\Omega}_T^{g,h}$ *and satisfy, for* $a_1 \in L^{\infty}(\Omega_T^{\infty})$, $a_{21}, a_{22}, a_{31}, a_{32} \in L^{\infty}(\Omega_T^{g,h})$, *and* $a_{21}, a_{31} \geq 0$,

$$
\begin{cases}
S_t \geq d\displaystyle\int_{\mathbb{R}} J(x-y)S(t,y)dy - dS + b + a_1 S, & 0 < t \leq T, x \in \mathbb{R},\\
I_t \geq d\displaystyle\int_{g(t)}^{h(t)} J(x-y)I(t,y)dy - dI + a_{21}\int_{g(t)}^{h(t)} K(x,y)I(t,y)dy + a_{22}I, & \\
 & 0 < t \leq T, x \in (g(t),h(t)),\\
R_t \geq d\displaystyle\int_{g(t)}^{h(t)} J(x-y)R(t,y)dy - dR + a_{31}I + a_{32}R, & 0 < t \leq T, x \in (g(t),h(t)),\\
I(t,x) \geq 0, R(t,x) \geq 0, & 0 < t \leq T, x \in \mathbb{R}\backslash(g(t),h(t)),\\
S(0,x) \geq 0, & x \in \mathbb{R},\\
I(0,x) \geq 0, R(0,x) \geq 0, & x \in [-h_0,h_0].
\end{cases}
$$

Then $S(t,x) \geq 0$ *for all* $0 \leq t \leq T$ *and* $x \in \mathbb{R}$, $I(t,x) \geq 0, R(t,x) \geq 0$ *for all* $0 \leq t \leq T$ *and* $x \in [g(t),h(t)]$. *Moreover, if* $S(0,x) \not\equiv 0$ *in* $\mathbb{R}$, *and* $I(0,x) \not\equiv 0, R(0,x) \not\equiv 0$ *in* $[-h_0,h_0]$, *then* $S(t,x) > 0$ *in* Ω_T^{∞}, *and* $I(t,x) > 0, R(t,x) > 0$ *in* $\Omega_T^{g,h}$.

To research the long-time behavior of the solution to the SIR epidemic problem, we propose the following Comparison principle.

Lemma 2.3. *(Comparison principle) Assume that* **(J)**, **(K)** *and* **(H)** *hold,* $\bar{g} \in G_{\bar{h}_0,T}$, $\bar{h} \in H_{\bar{h}_0,T}$, *for* h_0, $T > 0$, *suppose that* $\bar{S}$, $\bar{S}_t \in C(\bar{\Omega}_T^{\infty})$, $\bar{I}$, $\bar{I}_t$, $\bar{R}$, $\bar{R}_t \in C(\bar{\Omega}_T^{\bar{g},\bar{h}})$ *and satisfy,*

$$
\begin{cases}
\bar{S}_t \geq d\displaystyle\int_{\mathbb{R}} J(x-y)\bar{S}(t,y)dy - d\bar{S} - \mu_1\bar{S} + b, & 0 < t \leq T,\ x \in \mathbb{R},\\
\bar{I}_t \geq d\displaystyle\int_{\bar{g}(t)}^{\bar{h}(t)} J(x-y)\bar{I}(t,y)dy - d\bar{I} + \beta\bar{S}\int_{\bar{g}(t)}^{\bar{h}(t)} K(x,y)\bar{I}(t,y)dy - \mu_2\bar{I} - \alpha\bar{I}, & \\
 & 0 < t \leq T,\ x \in (\bar{g}(t),\bar{h}(t)),\\
\bar{R}_t \geq d\displaystyle\int_{\bar{g}(t)}^{\bar{h}(t)} J(x-y)\bar{R}(t,y)dy - d\bar{R} + \alpha\bar{I} - \mu_3\bar{R}, & 0 < t \leq T,\ x \in (\bar{g}(t),\bar{h}(t)),\\
\bar{I}(t,x) \geq 0, \bar{R}(t,x) \geq 0, & 0 < t \leq T,\ x \in \mathbb{R} \setminus (\bar{g}(t),\bar{h}(t)),\\
\bar{g}'(t) \leq -\mu\displaystyle\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{-\infty}^{\bar{g}(t)} J(x-y)\bar{I}(t,x)dydx, & 0 < t \leq T,\\
\bar{h}'(t) \geq \mu\displaystyle\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{I}(t,x)dydx, & 0 < t \leq T,\\
\bar{g}(0) \leq -h_0, \bar{h}_0 \geq h_0, \bar{S}(0,x) \geq S_0(x), & x \in \mathbb{R},\\
\bar{I}(0,x) \geq I_0(x),\ \bar{R}(0,x) \geq R_0(x), & x \in (-h_0,h_0),
\end{cases}
$$

then the unique solution (S,I,R,g,h) *of (1.4) satisfies*

$$S(t,x) \leq \bar{S}(t,x) \ \text{ for } \ 0 < t \leq T,\ x \in \mathbb{R},$$

$$I(t,x) \leq \bar{I}(t,x),\ R(t,x) \leq \bar{R}(t,x),\ g(t) \geq \bar{g}(t),\ h(t) \leq \bar{h}(t) \ \text{ for } \ 0 < t \leq T,\ x \in [g(t),h(t)].$$

Proof. We can prove it by references to [2, Theorem 3.1] and [13, Lemma 2.3]. First of all, thanks to **(H)** we know that $\bar{I}(0,x) \geq I_0(x) > 0$ in $(-h_0, h_0)$, thus $\bar{I}(0,x) \not\equiv 0$ in $[-\bar{h}_0, \bar{h}_0]$, by Lemma 2.1, we have $\bar{I}(t,x) > 0$ for $0 < t \leq T,\ \bar{g}(t) < x < \bar{h}(t)$, and thus both $\bar{h}$ and $-\bar{g}$ are strictly increasing. For small $\varepsilon > 0$, let $(S_\varepsilon, I_\varepsilon, R_\varepsilon, g_\varepsilon, h_\varepsilon)$ denote the unique solution of (1.4) with h_0 replaced by $h_0^\varepsilon := h_0(1-\varepsilon)$, μ replaced by $\mu_\varepsilon := \mu(1-\varepsilon)$, and S_0, I_0, R_0 replaced by S_0^ε, I_0^ε, R_0^ε respectively which satisfy

$$0 \leq S_0^\varepsilon < S_0(x) \text{ in } \mathbb{R},$$

$$0 \leq I_0^\varepsilon < I_0(x),\ 0 \leq R_0^\varepsilon < R_0(x) \text{ in } [-h_0^\varepsilon,\ h_0^\varepsilon],\ I_0^\varepsilon(\pm h_0^\varepsilon) = R_0^\varepsilon(\pm h_0^\varepsilon) = 0,$$

and $I_0^\varepsilon(\frac{h_0}{h_0^\varepsilon}) \to I_0(x)$, $R_0^\varepsilon(\frac{h_0}{h_0^\varepsilon}) \to R_0(x)$ as $\varepsilon \to 0$ in the $C([-h_0, h_0])$ norm. We claim that $h_\varepsilon(t) < \bar{h}(t)$ and $g_\varepsilon(t) > \bar{g}(t)$ for all $t \in (0, T]$. Clearly, these hold true for small $t > 0$. Suppose that there exists $t_1 \leq T$ such that

$$h_\varepsilon(t) < \bar{h}(t),\ g_\varepsilon(t) > \bar{g}(t) \text{ for } t \in (0, t_1) \text{ and } [h_\varepsilon(t_1) - \bar{h}(t_1)][g_\varepsilon(t_1) - \bar{g}(t_1)] = 0.$$

Without loss of generality, we may assume that $h_\varepsilon(t_1) = \bar{h}(t_1)$, $g_\varepsilon(t_1) \geq \bar{g}(t_1)$.

Firstly, let $u(t,x) = \bar{S} - S_\varepsilon$, then u satisfies

$$\begin{cases} u_t \geq d \displaystyle\int_{\mathbb{R}} J(x-y)u(t,y)dy - du - \mu_1 u, & 0 < t \leq T, x \in \mathbb{R}, \\ u(0,x) > 0, & x \in \mathbb{R}. \end{cases}$$

It follows from [12, Lemma 2.2] that $u(t,x) > 0$, thus $\bar{S}(t,x) > S_\varepsilon(t,x)$ for all $0 \leq t \leq T,\ x \in \mathbb{R}$.

Next, we compare I_ε and $\bar{I}$, R_ε and $\bar{R}$ over the region

$$\Omega_{t_1}^\varepsilon := \left\{(t,x) \in R^2 : 0 < t \leq t_1, g_\varepsilon(t) < x < h_\varepsilon(t)\right\}.$$

Let $w(t,x) = \bar{I} - I_\varepsilon$, then for all $(t,x) \in \Omega_{t_1}^\varepsilon$, there is

$$\begin{aligned} w_t \ &\geq d \int_{g_\varepsilon(t)}^{h_\varepsilon(t)} J(x-y)w(t,y)dy - (d+\alpha+\mu_2)w(t,x) \\ &+\beta\bar{S}\int_{\bar{g}(t)}^{\bar{h}(t)} K(x,y)\bar{I}(t,y)dy - \beta S_\varepsilon \int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)I_\varepsilon(t,y)dy, \end{aligned}$$

where

$$\begin{aligned} &\beta\bar{S}\int_{\bar{g}(t)}^{\bar{h}(t)} K(x,y)\bar{I}(t,y)dy - \beta S_\varepsilon \int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)I_\varepsilon(t,y)dy \\ \geq\ &\beta\bar{S}\int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)\bar{I}(t,y)dy - \beta S_\varepsilon \int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)I_\varepsilon(t,y)dy \\ =\ &\beta(\bar{S} - S_\varepsilon)\int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)\bar{I}(t,y)dy + \beta S_\varepsilon \int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)[\bar{I}(t,y) - I_\varepsilon(t,y)]dy \\ =\ &\beta u(t,x)\int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)\bar{I}(t,y)dy + \beta S_\varepsilon \int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)w(t,y)dy. \end{aligned}$$

Since $u(t,x) > 0$, then so w satisfies

$$w_t \geq d\int_{g_\varepsilon(t)}^{h_\varepsilon(t)} J(x-y)w(t,y)dy - (d+\alpha+\mu_2)w(t,x) + \beta S_\varepsilon(t,x)\int_{g_\varepsilon(t)}^{h_\varepsilon(t)} K(x,y)w(t,y)dy.$$

By Theorem 1.1 we have $0 < S_\varepsilon(t,x) \le C$ for all $0 \le t \le T$ and $x \in \mathbb{R}$, thus it follows from Lemma 2.1 that $\bar{I} > I_\varepsilon$ in $\Omega^\varepsilon_{t_1}$. Similarly, we have $\bar{R} > R_\varepsilon$ in $\Omega^\varepsilon_{t_1}$. Furthermore, according to the definition of t_1, we have $h'_\varepsilon(t_1) \ge \bar{h}'(t_1)$. Thus

$$\begin{aligned} 0 &\ge \bar{h}'(t_1) - h'_\varepsilon(t_1) \\ &\ge \mu \int_{\bar{g}(t_1)}^{\bar{h}(t_1)} \int_{\bar{h}(t_1)}^{+\infty} J(x-y)\bar{I}(t_1,x)dydx - \mu_\varepsilon \int_{g_\varepsilon(t_1)}^{h_\varepsilon(t_1)} \int_{h_\varepsilon(t_1)}^{+\infty} J(x-y)I_\varepsilon(t_1,x)dydx \\ &> \mu_\varepsilon \int_{g_\varepsilon(t_1)}^{h_\varepsilon(t_1)} \int_{h_\varepsilon(t_1)}^{+\infty} J(x-y)[\bar{I}(t_1,x) - I_\varepsilon(t_1,x)]dydx > 0, \end{aligned}$$

which is a contradiction. Similarly, we can prove $g_\varepsilon(t) > \bar{g}(t)$ for all $t \in (0,T]$. The claim is thus proved. Then the above arguments yield that $\bar{S}(t,x) > S_\varepsilon(t,x)$ for all $0 \le t \le T,\ x \in \mathbb{R}$, and $\bar{I}(t,x) > I_\varepsilon(t,x),\ \bar{R}(t,x) > R_\varepsilon(t,x)$ for all $0 \le t \le T,\ g_\varepsilon(t) < x < h_\varepsilon(t)$.

Since the unique solution of (1.4) depends continuously on the parameters in (1.4), the desired result then follows by letting $\varepsilon \to 0$.

$\square$

3 Spreading and vanishing

Lemma 3.1. *If $R_0 < 1$, then*

$$h_\infty - g_\infty < \infty.$$

Proof. From the first equation of (1.4), we can see that $S(t,x)$ satisfies

$$\begin{cases} S_t \le d\displaystyle\int_{\mathbb{R}} J(x-y)S(t,y)dy - dS + b - \mu_1 S, & t > 0, x \in \mathbb{R}, \\ S(0,x) = S_0(x), & x \in \mathbb{R}. \end{cases}$$

Let $\bar{S}(t)$ be the solution of

$$\begin{cases} \bar{S}_t = b - \mu_1 \bar{S}, \quad t > 0, \\ \bar{S}(0) = \|S_0\|_\infty. \end{cases}$$

It follows from [12, Lemma 2.2] that $S(t,x) \le \bar{S}(t)$ for $t > 0$ and $x \in \mathbb{R}$. Since $\lim_{t\to\infty} \bar{S}(t) = \frac{b}{\mu_1}$, we have

$$\limsup_{t\to\infty} S(t,x) \le \frac{b}{\mu_1} \text{ for } x \in \mathbb{R}. \tag{3.1}$$

Now we begin to show that $h_\infty - g_\infty < \infty$. Direct calculations yield

$$\begin{aligned} \frac{d}{dt}\int_{g(t)}^{h(t)} I(t,x)dx &= \int_{g(t)}^{h(t)} I_t(t,x)dx + h'(t)I(t,h(t)) - g'(t)I(t,g(t)) \\ &\le \int_{g(t)}^{h(t)} \left[d\int_{g(t)}^{h(t)} J(x-y)I(t,y)dy - d\int_{\mathbb{R}} J(x-y)I(t,x)dy \right. \\ &\quad \left. + \beta S(t,x)\int_{g(t)}^{h(t)} K(x,y)I(t,y)dy - \alpha I(t,x) - \mu_2 I(t,x) \right] dx \\ &= -\tfrac{d}{\mu_1}(h'(t) - g'(t)) + \int_{g(t)}^{h(t)} \beta S(t,x) \int_{g(t)}^{h(t)} K(x,y)I(t,y)dydx \\ &\quad - \int_{g(t)}^{h(t)} (\alpha + \mu_2)I(t,x)dx. \end{aligned}$$

For any $\varepsilon \in (0, \frac{(1-R_0)(\alpha+\mu_2)}{\beta})$, there exists $T(\varepsilon)$ such that $S(t,x) \le \frac{b}{\mu_1} + \varepsilon$ for $t \ge T$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}\int_{g(t)}^{h(t)} \beta S(t,x) \int_{g(t)}^{h(t)} K(x,y) I(t,y) dy dx &= \int_{g(t)}^{h(t)} I(t,y) \int_{g(t)}^{h(t)} \beta S(t,x) K(x,y) dx dy \\ &\le \int_{g(t)}^{h(t)} \beta(\frac{b}{\mu_1} + \varepsilon) I(t,y) dy.\end{aligned}$$

Hence

$$\frac{d}{dt} \int_{g(t)}^{h(t)} I(t,x) dx \le -\frac{d}{\mu_1}(h'(t) - g'(t)) + \int_{g(t)}^{h(t)} (\beta(\frac{b}{\mu_1} + \varepsilon) - \alpha - \mu_2) I(t,x) dx. \tag{3.2}$$

Integrating (3.2) from T to $t(> T)$ gives

$$\begin{aligned}0 \le \int_{g(t)}^{h(t)} I(t,x) dx &\le \int_{g(T)}^{h(T)} I(T,x) dx + \frac{d}{\mu}(h(T) - g(T)) - \frac{d}{\mu}(h(t) - g(t)) \\ &+ \int_T^t \left[\int_{g(s)}^{h(s)} (\beta(\frac{b}{\mu_1} + \varepsilon) - \alpha - \mu_2) I(s,x) dx \right] ds,\end{aligned}$$

from $R_0 < 1$ and the definition of ε, we know that $\beta(\frac{b}{\mu_1} + \varepsilon) - \alpha - \mu_2 < 0$ for $s \in [T,t]$ and $x \in [g(s), h(s)]$, then we have

$$\frac{d}{\mu}(h(t) - g(t)) \le \int_{g(T)}^{h(T)} I(T,x) dx + \frac{d}{\mu}(h(T) - g(T)).$$

Thus we can get $h_\infty - g_\infty < \infty$ by letting $t \to \infty$. □

Lemma 3.2. *Assume that* **(J)** *and* **(K)** *hold,* $J(x) > 0$ *in* $\mathbb{R}$*. Let* (S, I, R, g, h) *be the solution of (1.4). If* $h_\infty - g_\infty < \infty$*, then*

$$\lim_{t\to\infty} \max_{[g(t),h(t)]} I(t,\cdot) = 0, \quad \lim_{t\to\infty} \max_{[g(t),h(t)]} R(t,\cdot) = 0, \tag{3.3}$$

$$\lim_{t\to\infty} S(t,x) = \frac{b}{\mu_1} \quad \textit{locally uniformly in } \mathbb{R}.$$

Moreover,

$$\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)} + \beta\frac{b}{\mu_1}) \ge 0. \tag{3.4}$$

Proof. We first prove $\lim_{t\to\infty} \max_{[g(t),h(t)]} I(t,\cdot) = 0$. By virtue of [4, Lemma 3.2], we can have that

$$\lim_{t\to\infty} I(t,x) = 0 \text{ for almost every } x \in [-h_0, h_0].$$

We define $M(t) := \max_{x\in[g(t),h(t)]} I(t,x)$ and $X(t) := \{x \in (g(t), h(t)) : I(t,x) = M(t)\}$. We want to prove $\lim_{t\to\infty} \max_{x\in[g(t),h(t)]} I(t,x) = 0$, that is, we need to show that

$$\lim_{t\to\infty} M(t) = 0. \tag{3.5}$$

Noting $M(t)$ is continuous and $X(t)$ is a compact set for each $t>0$. Therefore, we can find $\underline{\xi}(t), \bar{\xi}(t) \in X(t)$ such that

$$I_t(t, \underline{\xi}(t)) = \min_{x\in X(t)} I_t(t,x),\ I_t(t,\bar{\xi}(t)) = \max_{x\in X(t)} I_t(t,x).$$

Now we are ready to prove (3.5). Arguing indirectly, we assume $\sigma^* := \limsup_{t\to\infty} M(t) \in (0,\infty)$. By the argument in [4, Theorem 3.3], there exists a sequence $t_n > 0$ increasing to ∞ as $n\to\infty$, and $\xi_n \in \{\underline{\xi}(t_n), \bar{\xi}(t_n)\}$ such that $\lim_{n\to\infty} M(t_n,\xi_n) = \sigma^*$ and $\lim_{n\to\infty} \partial_t I(t_n,\xi_n) = 0$. Thanks to (3.1), for any $\varepsilon_1 > 0$, there exists $N > 0$ such that, for $n \geq N$, $S(t_n,\xi_n) \leq \frac{b}{\mu_1} + \varepsilon_1$. By [4, Lemma 3.1], we have $\lim_{t\to\infty} h'(t) = 0$. Due to $J(x) > 0$ in $\mathbb{R}$, in view of [4, Lemma 3.2], we can get

$$\lim_{t\to\infty} \int_{g(t)}^{h(t)} I(t,y)dy = 0.$$

Since $\sup_{x\in\mathbb{R}} J(x) < \infty$ and $\sup_{x,y\in\mathbb{R}} K(x,y) < \infty$, we have

$$\lim_{t\to\infty} \int_{g(t)}^{h(t)} J(x-y)I(t,y)dy = 0 \text{ uniformly for } x\in\mathbb{R}$$

and

$$\lim_{t\to\infty} \int_{g(t)}^{h(t)} K(x,y)I(t,y)dy = 0 \text{ uniformly for } x\in\mathbb{R}.$$

Taking

$$I_t = d\int_{g(t)}^{h(t)} J(x-y)I(t,y)dy - dI + \beta S\int_{g(t)}^{h(t)} K(x,y)I(t,y)dy - \alpha I - \mu_2 I,$$

with $(t,x) = (t_n,\xi_n)$ and $n\to\infty$ we obtain

$$0 = (-d-\alpha-\mu_2)\sigma^* < 0.$$

Then we get a contradiction. Hence $\lim_{t\to\infty} I(t,x) = 0$ for $x\in[g(t),h(t)]$.

Next we prove $\lim_{t\to\infty} \max_{x\in[g(t),h(t)]} R(t,x) = 0$. For any small $\varepsilon_2 > 0$, there exists $T_2(\varepsilon_2)$ such that $I(t,x) \leq \varepsilon_2$ for $t \geq T_2$ and $x\in[g(t),h(t)]$. Then

$$\begin{cases} R_t \leq d\displaystyle\int_{g(t)}^{h(t)} J(x-y)R(t,y)dy - dR + \alpha\varepsilon_2 - \mu_3 R, & t > T_2, x\in(g(t),h(t)),\\ R(t,x) = 0, & t\geq T_2, x\in\mathbb{R}\backslash(g(t),h(t)),\\ R(T_2,x) = R(T_2,x), & x\in(g(T_2),h(T_2)). \end{cases}$$

Let $\bar{R}(t)$ be the solution of

$$\begin{cases} \bar{R}_t = \alpha\varepsilon_2 - \mu_3\bar{R}, & t > T_2,\\ \bar{R}(T_2) = \|R(T_2,\cdot)\|_\infty. \end{cases}$$

It follows from [2, Lemma 2.2] that $R(t,x) \leq \bar{R}(t)$ for $t > T_2$ and $x\in[g(t),h(t)]$. Since $\lim_{t\to\infty} \bar{R}(t) = \frac{\alpha\varepsilon_2}{\mu_3}$, we have $\limsup_{t\to\infty} R(t,x) \leq \frac{\alpha\varepsilon_2}{\mu_3}$ for $x\in[g(t),h(t)]$. By letting $\varepsilon_2\to\infty$, we can obtain $\lim_{t\to\infty} R(t,x) = 0$ for $x\in[g(t),h(t)]$.

Then we prove $\lim\limits_{t\to\infty} S(t,x)=\frac{b}{\mu_1}$. For any small $\varepsilon_3>0$, there exists $T_3(\varepsilon_3)$ such that $I(t,x)\le\varepsilon_3$ for $t\ge T_3$ and $x\in[g(t),h(t)]$. Then

$$\begin{cases} S_t\ge d\displaystyle\int_{\mathbb{R}} J(x-y)S(t,y)dy-dS+b-\beta S\varepsilon_3-\mu_1 S, & t>0,\ x\in\mathbb{R},\\ S(T_3,x)=S(T_3,x), & x\in\mathbb{R}. \end{cases}$$

Let $\underline{S}(t)$ be solution of

$$\begin{cases} \underline{S}_t=b-\beta\underline{S}\varepsilon_3-\mu_1\underline{S}, & t>T_3,\\ \underline{S}(T_3,x)=\inf\limits_{x\in\mathbb{R}}\underline{S}(T_3,x), & x\in\mathbb{R}. \end{cases}$$

It follows from [12, Lemma 2.2] that $S(t,x)\ge\underline{S}(t)$ for $t\ge T_3$ and $x\in\mathbb{R}$. Since $\lim\limits_{t\to\infty}\underline{S}(t)=\frac{b}{\beta\varepsilon_3+\mu_1}$, we have $\liminf\limits_{t\to\infty} S(t,x)\ge\frac{b}{\beta\varepsilon_3+\mu_1}$ for $x\in\mathbb{R}$. By the arbitrariness of ε_3, we have $\liminf\limits_{t\to\infty} S(t,x)\ge\frac{b}{\mu_1}$ for $x\in\mathbb{R}$. Then combining this with (3.1), we get $\lim\limits_{t\to\infty} S(t,x)=\frac{b}{\mu_1}$.

In what follows, we prove (3.4). On the contrary, assume that $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)}+\beta\frac{b}{\mu_1})<0$. Then there exists $\varepsilon_4\in(0,\frac{b}{\mu_1})$ such that $\lambda_p(\mathcal{L}_{(g_\infty+\varepsilon_4,h_\infty-\varepsilon_4)}+\beta(\frac{b}{\mu_1}-\varepsilon_4))<0$. Moreover, for such ε_4, according to conditions $h_\infty-g_\infty<\infty$ and $\lim\limits_{t\to\infty} S(t,x)=\frac{b}{\mu_1}$ for $x\in\mathbb{R}$, there exists $T_4(\varepsilon_4)$ such that $g(t)<g_\infty+\varepsilon_4,\ h(t)>h_\infty-\varepsilon_4$ for $t>T_4$, and $S(t,x)\ge\frac{b}{\mu_1}-\varepsilon_4$ for $t>T_4$ and $x\in\mathbb{R}$. Then for $t>T_4, x\in[g_\infty+\varepsilon_4,h_\infty-\varepsilon_4]$,

$$I_t\ge d\int_{g_\infty+\varepsilon_4}^{h_\infty-\varepsilon_4} J(x-y)I(t,y)dy-dI+\beta(\frac{b}{\mu_1}-\varepsilon_4)\int_{g_\infty+\varepsilon_4}^{h_\infty-\varepsilon_4} K(x,y)I(t,y)dy-\alpha I-\mu_2 I.$$

Let $0<\phi_\varepsilon(x)\le 1$ be the corresponding normalized eigenfunction of $\lambda_p(\mathcal{L}_{(g_\infty+\varepsilon_4,h_\infty-\varepsilon_4)}+\beta(\frac{b}{\mu_1}-\varepsilon_4))$, namely

$$-d\int_{g_\infty-\varepsilon_4}^{h_\infty-\varepsilon_4} J(x-y)\phi_\varepsilon(y)dy-\beta(\frac{b}{\mu_1}-\varepsilon_4)\int_{g_\infty-\varepsilon_4}^{h_\infty-\varepsilon_4} K(x,y)\phi_\varepsilon(y)dy+(d+\alpha+\mu_2)\phi_\varepsilon(x)=\lambda_p\phi_\varepsilon(x).$$

Thus $\lambda_p\phi_\varepsilon(x)<0$, for any $\delta>0$,

$$-d\int_{g_\infty+\varepsilon_4}^{h_\infty-\varepsilon_4} J(x-y)\delta\phi_\varepsilon(y)dy-\beta(\frac{b}{\mu_1}-\varepsilon_4)\int_{g_\infty-\varepsilon_4}^{h_\infty-\varepsilon_4} K(x,y)\delta\phi_\varepsilon(y)dy+(d+\alpha+\mu_2)\delta\phi_\varepsilon(x)=\lambda_p\delta\phi_\varepsilon(x),$$

obviously that $\lambda_p\delta\phi_\varepsilon(x)<0$.

Then we can choose δ small enough such that $\delta\phi_\varepsilon(x)<I(T_4,x)$ for $x\in[g_\infty+\varepsilon_4,h_\infty-\varepsilon_4]$. By using the comparison principle we can obtain

$$\liminf_{t\to\infty} I(t,x)>\liminf_{t\to\infty}\underline{I}(t,x)=\delta\phi_\varepsilon(x)>0 \text{ in } (g_\infty+\varepsilon_4,h_\infty-\varepsilon_4).$$

This contradicts with the first limit of (3.3). Therefore we get (3.4). □

It follows from Proposition 2.1 that when $R_0>1$ there exists $\ell^*>0$ such that

$$\begin{cases} \lambda_p(\mathcal{L}_{(\ell_1,\ell_2)}+\beta\frac{b}{\mu_1})=0, & \ell_2-\ell_1=\ell^*,\\ \lambda_p(\mathcal{L}_{(\ell_1,\ell_2)}+\beta\frac{b}{\mu_1})<0, & \ell_2-\ell_1>\ell^*,\\ \lambda_p(\mathcal{L}_{(\ell_1,\ell_2)}+\beta\frac{b}{\mu_1})>0, & \ell_2-\ell_1<\ell^*. \end{cases} \tag{3.6}$$

On the basis of (3.6), we get the following result.

Lemma 3.3. *Assume that* **(J)** *and* **(K)** *hold, and $J(x) > 0$ in $\mathbb{R}$. Let (S, I, R, g, h) be the solution of (1.4). If $R_0 > 1$, then*

(i) if $h_\infty - g_\infty < \infty$, then $h_\infty - g_\infty \le \ell^*$;

(ii) if $2h_0 \ge \ell^*$, then $h_\infty - g_\infty = \infty$;

(iii) if $2h_0 < \ell^*$, then there exists $0 < \mu_* \le \mu^*$ such that $h_\infty - g_\infty = \infty$ (spreading) for $\mu > \mu^*$, and $h_\infty - g_\infty < \infty$ (vanishing) for $0 < \mu \le \mu_*$ and $S_0(x) \le \frac{b}{\mu_1}$ in $x \in \mathbb{R}$.

Proof. The proof is same as [14, Lemma 2.10], now we give the details.

(i) Assume that the conclusion is incorrect, then $h_\infty - g_\infty > \ell^*$. From the above (3.6) we can obtain $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)} + \beta\frac{b}{\mu_1}) < 0$. Since $h_\infty - g_\infty < \infty$, then it follows from Lemma 3.2 that we have (3.4) hold, this is a contradiction.

(ii) Suppose that $h_\infty - g_\infty < \infty$, from (i) we can have $h_\infty - g_\infty \le \ell^*$, since the monotonicity of $g(t)$ and $h(t)$, we obtain $h_\infty - g_\infty > 2h_0 \ge \ell^*$, hence this is a contradiction.

(iii) Since $I_t \ge d\displaystyle\int_{g(t)}^{h(t)} J(x-y)I(t,y)dy - dI - \mu_2 I - \alpha I$, by using Lemma 3.9 of [4], we can have that there exists $\mu^* > 0$ such that $h_\infty - g_\infty = \infty$ for $\mu > \mu^*$. Now we begin to prove the conclusion of vanishing. Since $2h_0 < \ell^*$, by (3.6) we have $\lambda_p(\mathcal{L}_{(-h_0,h_0)} + \beta\frac{b}{\mu_1}) > 0$. For some small $\varepsilon > 0$, we define $h^* := h_0 + \varepsilon$ and $\lambda^* := \lambda_p(\mathcal{L}_{(-h^*,h^*)} + \beta\frac{b}{\mu_1}) > 0$. Let ϕ^* be the positive normalized eigenfunction corresponding to λ^*, namely, $\|\phi^*\|_\infty = 1$. For $x \in (-h^*, h^*)$, we have

$$-d\int_{-h^*}^{h^*} J(x-y)\phi^*(y)dy + d\phi^*(x) - \beta\frac{b}{\mu_1}\int_{-h^*}^{h^*} K(x,y)\phi^*(y)dy + (\alpha+\mu_2)\phi^*(x) = \lambda^*\phi^*(x).$$

Then we define

$$\begin{array}{ll} \bar{h}(t) = h_0 + \varepsilon(1-e^{-\delta t}), \bar{g}(t) = -\bar{h}(t), & t > 0, \\ \bar{S}(t,x) = \frac{b}{\mu_1}, & t > 0, x \in \mathbb{R}, \\ \bar{I}(t,x) = K_1 e^{-\delta t}\phi^*(x), & t > 0, \bar{g}(t) < x < \bar{h}(t), \\ \bar{R}(t,x) = K_2 e^{-\delta t}, & t > 0, \bar{g}(t) < x < \bar{h}(t), \end{array}$$

where δ, K_1 and K_2 are positive constants to be choosen later. Noting that $\bar{S}(0,x) = \frac{b}{\mu_1} \ge S_0(x)$ in $\mathbb{R}$, and $h_0 \le \bar{h}(t) < h^*$. It is obvious that

$$\bar{S}_t - d\int_{\mathbb{R}} J(x-y)\bar{S}(t,y)dy + d\bar{S} - b + \mu_1\bar{S} \ge 0$$

for $t > 0$, $x \in \mathbb{R}$. Since $\lambda^* > 0$, we can choose δ small enough and the positive constant K_1 large enough such that $\delta \le \min\left\{\lambda^*, \frac{\mu_3}{2}\right\}$ and $K_1\phi^*(x) \ge I_0(x)$ for $x \in (-h_0, h_0)$. Then we choose K_2 large enough such that $K_2 \ge \max\left\{\|R_0\|_\infty, \frac{\alpha K_1}{\mu_3 - \delta}\right\}$.

Through a series of calculations yield that

$$\begin{aligned} & \bar{I}_t - d\int_{\bar{g}(t)}^{\bar{h}(t)} J(x-y)\bar{I}(t,y)dy + d\bar{I} - \beta\bar{S}\int_{\bar{g}(t)}^{\bar{h}(t)} K(x,y)\bar{I}(t,y)dy + (\alpha+\mu_2)\bar{I} \\ \ge\ & K_1 e^{-\delta t}(-\delta\phi^* - d\int_{-h^*}^{h^*} J(x-y)\phi^*(y)dy + d\phi^* - \beta\bar{S}\int_{-h^*}^{h^*} K(x,y)\phi^*(y)dy + (\alpha+\mu_2)\phi^*) \\ \ge\ & K_1 e^{-\delta t}(-\delta + \lambda^*)\phi^*(x) \ge 0, \end{aligned}$$

$$
\begin{aligned}
\bar{R}_t - d\int_{\bar{g}(t)}^{\bar{h}(t)} J(x-y)\bar{R}(t,y)dy + d\bar{R} - \alpha I + \mu_3\bar{R} &\geq (-\delta K_2 - \alpha K_1\phi^* + \mu_3 K_2)e^{-\delta t} \\
&\geq (-\delta K_2 - \alpha K_1 + \mu_3 K_2)e^{-\delta t} \geq 0
\end{aligned}
$$

for $t>0,\ x\in(\bar{g}(t),\bar{h}(t))$. By the above definition, we have $\bar{I}(0,x) = K_1\phi^*(x) \geq I_0(x)$, $\bar{R}(0,x) = K_2 \geq R_0(x)$ for $x\in(-h_0,h_0)$. Moreover, $\bar{h}'(t) = \varepsilon\delta e^{-\delta t}$, $\mu\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{I}(t,x)dydx \leq 2\mu K_1 h^* e^{-\delta t}$. Hence we define $\mu_* := \frac{\varepsilon\delta}{2K_1h^*}$, if $\mu\leq\mu_*$, then we obtain

$$
\bar{h}'(t) \geq \mu\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{I}(t,x)dydx.
$$

Similarly, one can easily derive that

$$
\bar{g}'(t) \leq -\mu\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{-\infty}^{\bar{g}(t)} J(x-y)\bar{I}(t,x)dydx.
$$

Then we can apply Lemma 2.3 to conclude that

$$
g(t)\geq\bar{g}(t), h(t)\leq\bar{h}(t) \text{ for } t>0.
$$

Thus $\lim\limits_{t\to\infty}(h(t)-g(t)) \leq \lim\limits_{t\to\infty}(\bar{h}(t)-\bar{g}(t)) \leq 2h^* < \infty$, namely, $h_\infty - g_\infty < \infty$.

Therefore, Theorem 1.2 has been proved. □

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