CMMSE: A low-dimensional realization algorithm for periodic input/output behavioral systems

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Abstract

The state-space realization of linear systems is of utmost importance in linear systems theory. After the realization problem for the time-invariant case has been solved, particular attention was paid to the case of linear periodic systems (see, e.g. 1,2,3,4,5,6,7,8). Recently, such systems have regained importance, for instance, in the context of coding theory (see9), where periodic convolutional encoders play an important role, 10. The majority of the contributions within this area concern the realization of transfer functions as well as impulse responses, thus excluding the case of input/output linear systems without coprime representations. By the end of the eighties of the last century, Jan C. Willems (see11,12) suggested an approach (nowadays known as the behavioral approach) that considers a wider class of systems and allows to overcome this drawback. According to this approach, the central object in a system is its behavior which consists of all the signals that satisfy the system laws (also called system trajectories). Consequently, the behavior of a system with an input/output representation that is not coprime, contains more trajectories than the set of input/output signals defined by the system transfer function. Our work takes this fact into account. Based on results already obtained in 13,14, we revisit the problem of the realization of linear periodic MIMO behaviors and give further insight into this problem, which allows setting up an algorithm to compute a low-dimensional state-space realization of a periodic behavior. The proposed algorithm is based on a chain decomposition of suitable matrices.

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The state-space realization of linear systems is of utmost importance in linear systems theory. After the realization problem for the time-invariant case has been solved, particular attention was paid to the case of linear periodic systems (see, e.g. [1,2,3,4,5,6,7,8]). Recently, such systems have regained importance, for instance, in the context of coding theory (see[9]), where periodic convolutional encoders play an important role,[10]. The majority of the contributions within this area concern the realization of transfer functions as well as impulse responses, thus excluding the case of input/output linear systems without coprime representations. By the end of the eighties of the last century, Jan C. Willems (see[11,12]) suggested an approach (nowadays known as the behavioral approach) that considers a wider class of systems and allows to overcome this drawback. According to this approach, the central object in a system is its behavior which consists of all the signals that satisfy the system laws (also called system trajectories). Consequently, the behavior of a system with an input/output representation that is not coprime, contains more trajectories than the set of input/output signals defined by the system transfer function. Our work takes this fact into account. Based on results already obtained in[13,14], we revisit the problem of the realization of linear periodic MIMO behaviors and give further insight into this problem, which allows setting up an algorithm to compute a low-dimensional state-space realization of a periodic behavior. The proposed algorithm is based on a chain decomposition of suitable matrices.

KEYWORDS:
Discrete-time systems, input/output systems, periodic behaviors, system realization

1 | INTRODUCTION

The minimal realization problem for linear time-varying systems is a fundamental topic in linear systems theory that has received several contributions as, e.g.,[1,2,3,4,5,6,7,8], in the last decades. More recently this issue has been addressed in the context of the behavioral approach where a strategy was proposed in order to achieve a state-space realization for periodic MIMO behaviors, see[14]. A first step towards the questions this work addresses was initially treated in[13], where we have analyzed the issue of the state-space representation of periodic SISO behaviors only for the particular case of period equal to two. Taking this last two works into account, as a starting point, we present some further results on the matrix chain used in[14] to produce such realizations and provide an algorithm that allows to obtain low-dimensional realizations for periodic MIMO behaviors.

[0]**Abbreviations:** SISO, Single-Input Single-Output; MIMO, Multi-Input Multi-Output

The paper is organized as follows. Section 2 contains some background material, Section 3 is devoted to the construction of the realizations, Section 4 is devoted to the presentation of a numerical example, and the conclusions are left to Section 5.

2 | PRELIMINARIES/BACKGROUND

In the behavioral framework, see[11] and[12], the notion of a dynamical system Σ has the *behavior* as a basic concept. More concretely, a system Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ the time set, $\mathbb{W}$ the signal space, and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}} := \{w : \mathbb{T} \to \mathbb{W}\}$ the system behavior, i.e., the set of all "legal" trajectories according to the system laws. In this paper we consider the discrete-time case, i.e., $\mathbb{T} = \mathbb{Z}$ and, moreover, assume that the signal space is $\mathbb{W} = \mathbb{R}^q$, with $q \in \mathbb{N}$.

For $\tau \in \mathbb{Z}$, define the τ-shift as $\sigma^\tau : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}}$, by

$$(\sigma^\tau w)(k) := w(k+\tau).$$

The notion of time-invariance relies on the invariance of the behavior with respect to the time shifts, i.e., $\sigma\mathfrak{B} = \mathfrak{B}$ (see[11,12]), while periodicity relies on the property $\sigma^{\mathtt{P}}$ invariance, for a given $\mathtt{P} \in \mathbb{N}$, defined next.

Definition 1 ([15])**.** A system Σ is said to be $\mathtt{P}$-periodic, with $\mathtt{P} \in \mathbb{N}$, if its behavior $\mathfrak{B}$ satisfies $\sigma^{\mathtt{P}}\mathfrak{B} = \mathfrak{B}$, and, moreover, $\mathtt{P}$ is the smallest value for which this equality holds.

Observe that, time-invariant behaviors are also periodic behaviors, with period 1. For a deeper insight into the notions of time-invariance and periodicity in the scope of the behavioral approach, as well as into the lifting technique that will be used here, a careful reading of works such[12,16,15,17,18] is strongly encouraged.

Here we start from periodic MIMO behaviors where the system signal w is partitioned into inputs and outputs, i.e., $w = (u, y)$ where the sub-vector u contains the inputs and the sub-vector y contains the outputs. Moreover, we assume that such behaviors are described by input/output difference equations with time-varying periodic coefficients:

$$\left(P_t\left(\sigma,\sigma^{-1}\right) y\right)(t+\mathtt{P}k) = \left(Q_t\left(\sigma,\sigma^{-1}\right) u\right)(t+\mathtt{P}k),\ t = 0,\dots,\mathtt{P}-1, k \in \mathbb{Z} \tag{1}$$

where, for each $t = 0,\dots,\mathtt{P}-1$, $P_t\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{\bullet\times p}\left[\xi,\xi^{-1}\right]$ and $Q_t\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{\bullet\times m}\left[\xi,\xi^{-1}\right]$, $m, p \in \mathbb{N}$, are matrices having as entries Laurent polynomials in the indeterminate ξ.

Note that (1) can also be written as

$$\left(P\left(\sigma,\sigma^{-1}\right)y\right)(\mathtt{P}k) = \left(Q\left(\sigma,\sigma^{-1}\right)u\right)(\mathtt{P}k),\ k\in\mathbb{Z}, \tag{2}$$

where

$$P\left(\xi,\xi^{-1}\right) := \begin{bmatrix} P_0\left(\xi,\xi^{-1}\right) \\ \xi P_1\left(\xi,\xi^{-1}\right) \\ \vdots \\ \xi^{\mathtt{P}-1}P_{\mathtt{P}-1}\left(\xi,\xi^{-1}\right) \end{bmatrix} \quad \text{and} \quad Q\left(\xi,\xi^{-1}\right) := \begin{bmatrix} Q_0\left(\xi,\xi^{-1}\right) \\ \xi Q_1\left(\xi,\xi^{-1}\right) \\ \vdots \\ \xi^{\mathtt{P}-1}Q_{\mathtt{P}-1}\left(\xi,\xi^{-1}\right) \end{bmatrix}.$$

From now on, such systems will simply be called $\mathtt{P}$-periodic MIMO behaviors.

By factoring P and Q as, see[18],

$$P\left(\xi,\xi^{-1}\right) = P^L\left(\xi^{\mathtt{P}},\xi^{-\mathtt{P}}\right)\Omega_{\mathtt{P},p}(\xi), \qquad Q\left(\xi,\xi^{-1}\right) = Q^L\left(\xi^{\mathtt{P}},\xi^{-\mathtt{P}}\right)\Omega_{\mathtt{P},m}(\xi)$$

where

$$\Omega_{\mathtt{P},p}(\xi) := \begin{bmatrix} I_p & \xi I_p & \cdots & \xi^{\mathtt{P}-1}I_p \end{bmatrix}^T, \qquad \Omega_{\mathtt{P},m}(\xi) := \begin{bmatrix} I_m & \xi I_m & \cdots & \xi^{\mathtt{P}-1}I_m \end{bmatrix}^T,$$

we write down relation (2) as

$$\left(P^L\left(\sigma^{\mathtt{P}},\sigma^{-\mathtt{P}}\right)\Omega_{\mathtt{P},p}(\sigma)\, y\right)(\mathtt{P}k) = \left(Q^L\left(\sigma^{\mathtt{P}},\sigma^{-\mathtt{P}}\right)\Omega_{\mathtt{P},m}(\sigma)\, u\right)(\mathtt{P}k), k \in \mathbb{Z}\,. \tag{3}$$

Define the lifted input and output trajectories

$$u^L(k) := (Lu)(k) := \begin{bmatrix} u(\mathtt{P}k) \\ \vdots \\ u(\mathtt{P}k+\mathtt{P}-1) \end{bmatrix}, \qquad y^L(k) := (Ly)(k) := \begin{bmatrix} y(\mathtt{P}k) \\ \vdots \\ y(\mathtt{P}k+\mathtt{P}-1) \end{bmatrix}$$

see [19,20,15,18], and note that $L\left(\sigma^{\mathrm{P}} v\right) = \sigma\left(Lv\right)$. Then (3) can be written as

$$\left(P^L\left(\sigma,\sigma^{-1}\right)y^L\right)(k) = \left(Q^L\left(\sigma,\sigma^{-1}\right)u^L\right)(k),\ k\in\mathbb{Z}. \tag{4}$$

The behavior $\mathfrak{B}^L$, defined by $L\left(\mathfrak{B}\right) := \{(Lu, Ly), (u,y)\in\mathfrak{B}\}$, called the *lifted behavior* associated with $\mathfrak{B}$, is time-invariant, and equals the set of trajectories

$$\left\{\left(u^L, y^L\right)\in\left(\mathbb{R}^{\mathrm{P}m}\right)^{\mathbb{Z}}\times\left(\mathbb{R}^{\mathrm{P}p}\right)^{\mathbb{Z}} \mid (4)\text{ holds}\right\},$$

that is,

$$\mathfrak{B}^L = \ker\left[P^L\left(\sigma,\sigma^{-1}\right) \quad -Q^L\left(\sigma,\sigma^{-1}\right)\right].$$

In [17,18] it is shown that the lifted of the i/o P-periodic behavior keeps an i/o structure.

A P-periodic state-space system $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$

$$\begin{cases}(\sigma x)(k) = A(k)x(k) + B(k)u(k)\\ \\ y(k) = C(k)x(k) + D(k)u(k)\end{cases} \qquad k\in\mathbb{Z}, \tag{5}$$

where the matrices $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are periodic functions with period P, is said to be a *(P-periodic) state-space realization* of a P-periodic input/output behavior $\mathfrak{B}$ if

$$\mathfrak{B} = \{(u,y) \mid \exists x \text{ such that } (u,x,y) \text{ satisfies (5)}\}.$$

The definition of a (time-invariant) state-space realization $\Sigma = (A,B,C,D)$ for a time-invariant behavior is analogous, see [21]. A state-space realization of a behavior is called minimal if the dimension of the state vector is the smallest among all the realizations of the same behavior. This holds both in the time-invariant and in the periodic cases.

According to this definition, a state-space realization describes the complete system behavior rather than the input/output trajectories that are obtained by the corresponding transfer function, in the time-invariant case. This is an important issue for the realization of non-controllable behaviors, [11,12].

Given a periodic n-dimensional state-space model $\Sigma(k)$ with period P, one can obtain a time-invariant formulation (see, e.g. [22]) that preserves the state-space dimension while using the lifting technique for the input and the output signals leading to a time-invariant (lifted) version, Σ^L, of $\Sigma(k)$. For that, let

$$z(k) = x(\mathrm{P}k), \qquad u^L(k) = \begin{bmatrix} u(\mathrm{P}k)\\ u(\mathrm{P}k+1)\\ \vdots\\ u(\mathrm{P}k+\mathrm{P}-1)\end{bmatrix}, \qquad y^L(k) = \begin{bmatrix} y(\mathrm{P}k)\\ y(\mathrm{P}k+1)\\ \vdots\\ y(\mathrm{P}k+\mathrm{P}-1)\end{bmatrix},$$

and, for $i,j = 1,\ldots,\mathrm{P}$, define

$$\phi_A(i,j) := \begin{cases} A(i-1)\cdots A(j), & \text{if } j < i-1\\ I_n, & \text{if } j = i-1\end{cases}.$$

Then, the evolution of $z(k)$ and $y^L(k)$ driven by $u^L(k)$ is described by the following time-invariant n-dimensional state-space model $\Sigma^L = (F,G,H,J)$:

$$\begin{cases}(\sigma z)(k) = Fz(k) + Gu^L(k)\\ y^L(k) = Hz(k) + Ju^L(k)\end{cases}, \tag{6}$$

with, for $i,j = 1,\ldots,\mathrm{P}$,

$$F = \phi_A(\mathrm{P},0) \qquad G = \left[G_1 G_2 \cdots G_{\mathrm{P}}\right]$$

$$H = \begin{bmatrix} H_1\\ H_2\\ \vdots\\ H_{\mathrm{P}}\end{bmatrix} \qquad J = \left[J_{ij}\right],$$

where

$$\begin{aligned}
\phi_A(\mathtt{P},0) &:= A(\mathtt{P}-1)\cdots A(0)\\
G_j &:= \phi_A(\mathtt{P},j)\,B(j-1)\\
H_i &:= C(i-1)\,\phi_A(i-1,0)\\
J_{ij} &:= \begin{cases} 0_{p\times m} & \text{if } i<j\\ D(i-1) & \text{if } i=j\\ C(i-1)\phi_A(i-1,j)B(j-1) & \text{if } i>j \end{cases} .
\end{aligned} \tag{7}$$

The state-space system Σ^L obtained in this way from the periodic state-space system $\Sigma(k)$ is said to be *induced* by system $\Sigma(k)$. If $\Sigma(k)$ is a periodic state-space realization of the periodic behavior $\mathfrak{B}$ and Σ^L is induced by $\Sigma(k)$, then Σ^L is a time-invariant state-space realization of the lifted (time-invariant) version of $\mathfrak{B}$:

$$\mathfrak{B}^L := \left\{\left(u^L, y^L\right) | (u,y)\in\mathfrak{B}\right\}.$$

3 | PERIODIC STATE-SPACE REPRESENTATIONS

In the sequel, we provide constructive necessary and sufficient conditions for a time-invariant state-space system to be induced by a periodic one with period $\mathtt{P}$. However, before proceeding, we give an example where we analyze the structure of an induced time-invariant system and show how to recover the matrices $A(k)$, $B(k)$, $C(k)$, and $D(k)$ of the original periodic system, from the induced one. This provides a deeper insight into the general case.

Example 1. Let $\Sigma(k) = (A(k), B(k), C(k), D(k))$ be a periodic n-dimensional state-space model with period 3. Then, taking into account that

$$\begin{aligned}
x(3k+1) &= A(0)\,x(3k) + B(0)\,u(3k)\\
x(3k+2) &= A(1)\,x(3k+1) + B(1)\,u(3k+1)\\
&= A(1)\,[A(0)\,x(3k) + B(0)\,u(3k)]\\
&\quad + B(1)\,u(3k+1)\\
x(3k+3) &= A(2)\,x(3k+2) + B(2)\,u(3k+2)\\
&= A(2)\,\{A(1)\,[A(0)\,x(3k) + B(0)\,u(3k)]\\
&\quad + B(1)\,u(3k+1)\} + B(2)\,u(3k+2)\,,
\end{aligned}$$

and

$$\begin{aligned}
y(3k) &= C(0)\,x(3k) + D(0)\,u(3k)\\
y(3k+1) &= C(1)\,[A(0)\,x(3k) + B(0)\,u(3k)]\\
&\quad + D(1)\,u(3k+1)\\
y(3k+2) &= C(2)\,\{A(1)\,[A(0)\,x(3k) + B(0)\,u(3k)]\\
&\quad + B(1)\,u(3k+1)\} + D(2)\,u(3k+2)\,,
\end{aligned}$$

we can conclude that:

$$\begin{aligned}
x(3k+3) &= A(2)\,A(1)\,A(0)\,x(3k)\\
&\quad + A(2)\,A(1)\,B(0)\,u(3k) + A(2)\,B(1)\,u(3k+1)\\
&\quad + B(2)\,u(3k+2)\,,
\end{aligned}$$

and

$$\begin{bmatrix} y(3k)\\ y(3k+1)\\ y(3k+2)\end{bmatrix} = \begin{bmatrix} C(0)\\ C(1)\,A(0)\\ C(2)\,A(1)\,A(0)\end{bmatrix} x(3k) + \begin{bmatrix} D(0) & 0 & 0\\ C(1)\,B(0) & D(1) & 0\\ C(2)\,A(1)\,B(0) & C(2)\,B(1) & D(2)\end{bmatrix}\begin{bmatrix} u(3k)\\ u(3k+1)\\ u(3k+2)\end{bmatrix},$$

which, by putting

$$z(k) = x(3k)\,, \qquad u^L(k) = \begin{bmatrix} u(3k) \\ u(3k+1) \\ u(3k+2) \end{bmatrix}, \qquad y^L(k) = \begin{bmatrix} y(3k) \\ y(3k+1) \\ y(3k+2) \end{bmatrix},$$

leads to

$$\begin{aligned} z(k+1) &= F z(k) + \overbrace{\begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix}}^{G} u^L(k) \\ y^L(k) &= \underbrace{\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}}_{H} z(k) + \underbrace{\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}}_{J} u^L(k)\,, \end{aligned}$$

with

$$\begin{aligned} F &= A(2)\,A(1)\,A(0) = \phi_A(3,0) & G_1 &= A(2)\,A(1)\,B(0) = \phi_A(3,1)\,B(0) \\ & & G_2 &= A(2)\,B(1) = \phi_A(3,2)\,B(1) \\ & & G_3 &= B(2) = \phi_A(3,3)\,B(2) \\ H_1 &= C(0) = C(0)\,\phi_A(0,0) & J_{12} &= J_{13} = J_{23} = 0 \\ H_2 &= C(1)\,A(0) = C(1)\,\phi_A(1,0) & J_{11} &= D(0)\,,\ J_{22} = D(1)\,,\ J_{33} = D(2) \\ H_3 &= C(2)\,A(1)\,A(0) = C(2)\,\phi_A(2,0) & J_{21} &= C(1)\,B(0) = C(1)\,\phi_A(1,1)\,B(0) \\ & & J_{31} &= C(2)\,A(1)\,B(0) = C(2)\,\phi_A(2,1)\,B(0) \\ & & J_{32} &= C(2)\,B(1) = C(2)\,\phi_A(2,2)\,B(1). \end{aligned}$$

$\Sigma^L = (F, G, H, J)$ is the time-invariant state-space system *induced* by $\Sigma(k)$. Note that this system has $3m$ inputs and $3p$ outputs. Moreover, its J matrix is a lower block triangular matrix with J_{ij} blocks of size $p \times m$.

Now, let us go backwards and recover the matrices of the original periodic state-space system based on the blocks of the matrices F, G, H, and J of Σ^L. For this purpose, construct the following matrix:

$${}^1M = \left[\begin{array}{c|c} F & G_1 \\ H_3 & J_{31} \\ H_2 & J_{21} \end{array}\right].$$

Due to the special form of these blocks,

$${}^1M = \left[\begin{array}{c|c} A(2)\,A(1)\,A(0) & A(2)\,A(1)\,B(0) \\ C(2)\,A(1)\,A(0) & C(2)\,A(1)\,B(0) \\ C(1)\,A(0) & C(1)\,B(0) \end{array}\right]$$

can be factored as

$${}^1M = \underbrace{\left[\begin{array}{c} A(2)\,A(1) \\ C(2)\,A(1) \\ \hline C(1) \end{array}\right]}_{n \text{ columns}} \left[\begin{array}{c|c} A(0) & B(0) \end{array}\right]$$

(implying that it has rank less than or equal to n). Based on the factors of ${}^1M$, construct the matrices

$$\begin{aligned} {}^1Q &:= \begin{bmatrix} A(2)\,A(1) \\ C(2)\,A(1) \end{bmatrix} & {}^1S &:= A(0) \\ {}^1R &:= C(1) & {}^1T &:= B(0)\,. \end{aligned}$$

Further, define a new matrix ${}^2M$ as ${}^2M = \left[{}^2M_1 | {}^2M_2\right]$, with

$${}^2M_1 = {}^1Q, \qquad \text{and} \qquad {}^2M_2 = \begin{bmatrix} G_2 \\ J_{32} \end{bmatrix},$$

i.e.,

$$
{}^{2}M = \left[\begin{array}{c|c} A(2)\,A(1) & A(2)\,B(1) \\ C(2)\,A(1) & C(2)\,B(1) \end{array}\right].
$$

Clearly, ${}^{2}M$ can be factored as

$$
{}^{2}M = \underbrace{\left[\begin{array}{c} A(2) \\ \hline C(2) \end{array}\right]}_{n \text{ columns}} \left[\begin{array}{c|c} A(1) & B(1) \end{array}\right],
$$

implying that, like ${}^{1}M$, this matrix also has rank less than or equal to n. Define ${}^{2}Q := A(2)$, ${}^{2}R := C(2)$, ${}^{2}S := A(1)$, and ${}^{2}T := B(1)$. The matrices of the original periodic state-space system that induced Σ^{L} are given as follows in terms of the matrices resulting from the previous factorizations:

$$
\begin{array}{llll}
A(0) = {}^{1}S & B(0) = {}^{1}T & C(0) = H_1 & D(0) = J_{11} \\
A(1) = {}^{2}S & B(1) = {}^{2}T & C(1) = {}^{1}R & D(1) = J_{22} \\
A(2) = {}^{2}Q & B(2) = G_3 & C(2) = {}^{2}R & D(2) = J_{33}.
\end{array}
$$

In order to investigate whether a linear time-invariant state-space system $\Sigma = (F, G, H, J)$ of dimension n with $m\mathtt{P}$ inputs and $p\mathtt{P}$ outputs is induced by a periodic state-space system with period $\mathtt{P}$, consider for $i, j = 1, \ldots, \mathtt{P}$ the following partitions of G, H and J:

$$
G = \begin{bmatrix} G_1 & G_2 & \cdots & G_{\mathtt{P}} \end{bmatrix}, \qquad H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_{\mathtt{P}} \end{bmatrix}, \qquad \text{and} \qquad J = \begin{bmatrix} J_{ij} \end{bmatrix}, \tag{8}
$$

with G_j of size $n \times m$, H_i of size $p \times n$, and J_{ij} of size $p \times m$.

Note that a necessary, but not sufficient, condition for Σ to be induced by a periodic state-space system of period $\mathtt{P}$ is that J is a block lower triangular matrix. This allows us to quickly discard some non-induced realizations. However, if J is block lower triangular, one must perform a deeper analysis. With this aim, using the previous notation, the following definition is introduced.

Definition 2 ([14]). Let $\Sigma = (F, G, H, J)$ be a linear time-invariant n-dimensional state-space system with $m\mathtt{P}$ inputs and $p\mathtt{P}$ outputs, for a given positive integer $\mathtt{P}$. Define an n-chain of size s generated by Σ as a sequence of matrices ${}^{1}M, \ldots, {}^{s}M$, each one of rank less than or equal to n, such that:

- ${}^{1}M := \begin{bmatrix} {}^{1}M_1 | {}^{1}M_2 \end{bmatrix}$ with

$$
{}^{1}M_1 := {}^{0}Q = \begin{bmatrix} F \\ H_{\mathtt{P}} \\ \vdots \\ H_2 \end{bmatrix}, \qquad \text{and} \qquad {}^{1}M_2 := \begin{bmatrix} G_1 \\ J_{\mathtt{P}1} \\ \vdots \\ J_{21} \end{bmatrix}
$$

- ${}^{\ell+1}M := \begin{bmatrix} {}^{\ell+1}M_1 | {}^{\ell+1}M_2 \end{bmatrix}$ with ${}^{\ell+1}M_1 := {}^{\ell}Q$ where ${}^{\ell}Q$ is a $(n + (\mathtt{P} - (\ell + 1))\, p) \times n$ matrix such that $\exists\, {}^{\ell}R, {}^{\ell}S$, and ${}^{\ell}T$ satisfying

$$
{}^{\ell}M = \begin{bmatrix} {}^{\ell}Q \\ {}^{\ell}R \end{bmatrix} \begin{bmatrix} {}^{\ell}S & {}^{\ell}T \end{bmatrix}, \qquad \text{and} \qquad {}^{\ell+1}M_2 := \begin{bmatrix} G_{\ell+1} \\ J_{\mathtt{P},\ell+1} \\ \vdots \\ J_{\ell+2,\ell+1} \end{bmatrix}, \quad \ell = 1, \ldots, s-1.
$$

It follows from Definition 2 that each matrix ${}^{\ell}M$, with $\ell = 1, \ldots, s$, has $n + (\mathtt{P} - \ell)\, p$ rows and $n + m$ columns; the size of the chain, s, has maximum value $\mathtt{P} - 1$, and clearly, if Σ generates an n-chain of size s it also generates an n-chain of size smaller than s.

This latter definition, where the concept of n-chain is introduced, provides the support to establish necessary and sufficient conditions for a linear time-invariant n-dimensional state-space system (with $m\mathtt{P}$ inputs and $p\mathtt{P}$ outputs) to be induced by a m-input and p-output periodic state-space system with the same state dimension n.

Theorem 1 ([14]). Let $\Sigma = (F, G, H, J)$ be a linear time-invariant n-dimensional state-space system with $m\mathrm{P}$ inputs and $p\mathrm{P}$ outputs, for a given positive integer P. Then, Σ is induced by a P-periodic state-space system of dimension n if, and only if:

(i) J is a lower block triangular matrix with $\mathrm{P} \times \mathrm{P}$ blocks of size $p \times m$;

(ii) Σ generates an n-chain of size $\mathrm{P} - 1$.

It has been shown in [14], that a linear time-invariant system $\Sigma = (F, G, H, J)$ that generates an n-chain of size $\mathrm{P} - 1$ is induced by a P-periodic state-space system $\Sigma(k) = (A(k), B(k), C(k), D(k))$ such that:

$$
\begin{array}{llll}
A(0) = {}^{1}S & B(0) = {}^{1}T & C(0) = H_1 & D(0) = J_{11} \\
A(1) = {}^{2}S & B(1) = {}^{2}T & C(1) = {}^{1}R & D(1) = J_{22} \\
A(2) = {}^{3}S & B(2) = {}^{3}T & C(2) = {}^{2}R & D(2) = J_{33} \\
\vdots & \vdots & \vdots & \vdots \\
A(\mathrm{P}-2) = {}^{\mathrm{P}-1}S & B(\mathrm{P}-2) = {}^{\mathrm{P}-1}T & C(\mathrm{P}-2) = {}^{\mathrm{P}-2}R & D(\mathrm{P}-2) = J_{\mathrm{P}-1,\mathrm{P}-1} \\
A(\mathrm{P}-1) = {}^{\mathrm{P}-1}Q & B(\mathrm{P}-1) = G_{\mathrm{P}} & C(\mathrm{P}-1) = {}^{\mathrm{P}-1}R & D(\mathrm{P}-1) = J_{\mathrm{PP}},
\end{array}
\tag{9}
$$

where the matrices on the right-hand side are obtained from (8) and the corresponding chain decomposition.

Remark 1. Note that if $\Sigma = (F, G, H, J)$ is *induced* by a P-periodic state-space system, then so are all its algebraic equivalent realizations $\Sigma_S = \left(SFS^{-1}, SG, HS^{-1}, J\right)$, where S is an invertible matrix.

Since an n-dimensional periodic realization $\Sigma(k)$ of $\mathfrak{B}$ induces an invariant realization Σ of $\mathfrak{B}^L$ also with dimension n, the minimal state-space dimension of the realizations Σ^L of $\mathfrak{B}^L$ does not exceed the minimal state dimension of the periodic realizations $\Sigma(k)$ of $\mathfrak{B}$. Assume now that Σ is a minimal n-dimensional realization of $\mathfrak{B}^L$ which is not induced by any P-periodic realization of $\mathfrak{B}$ with dimension n. Then, taking into account Remark 1 together with the fact that the minimal realizations of time-invariant behaviors are all algebraically equivalent, one can conclude that, if periodic realizations of $\mathfrak{B}$ exist at all, the minimal ones have a state-space dimension greater than n.
In order to give some insight into how this issue appears and may be treated, consider the following example.

Example 2. Let $\mathfrak{B}$ be a SISO 3-periodic behavior and consider the corresponding lifted behavior $\mathfrak{B}^L$. Let further $\Sigma = (F, G, H, J)$ of dimension n, with 3 inputs and 3 outputs, be a linear time-invariant state-space realization of $\mathfrak{B}^L$. Assume that Σ is not induced by a SISO linear 3-periodic state-space system of dimension n, due to the failure of the rank condition on the matrix ${}^{1}M$ (cf. Definition 2), meaning that

$$\operatorname{rank} {}^{1}M = n + 1,$$

since ${}^{1}M$ is a$(n+2)\times(n+1)$ matrix.

Define a new time-invariant state-space system ${}_{1}\Sigma = \left({}_{1}F, {}_{1}G, {}_{1}H, {}_{1}J\right)$ of dimension $n + 1$, where

$$
{}_{1}F := \left[\begin{array}{c|c} 0_{1\times1} & 0_{1\times n} \\ \hline 0_{n\times1} & F \end{array}\right] \qquad
{}_{1}G := \left[\begin{array}{c} 0_{1\times3} \\ \hline G \end{array}\right] = \left[\begin{array}{c|c|c} 0_{1\times1} & 0_{1\times1} & 0_{1\times1} \\ \hline G_1 & G_2 & G_3 \end{array}\right] =: \left[\, {}_{1}G_1 \,\middle|\, {}_{1}G_2 \,\middle|\, {}_{1}G_3 \,\right]
$$

$$
{}_{1}H := \left[\, 0_{3\times1} \,\middle|\, H \,\right] = \left[\begin{array}{c|c} 0_{1\times1} & H_1 \\ \hline 0_{1\times1} & H_2 \\ \hline 0_{1\times1} & H_3 \end{array}\right] =: \left[\begin{array}{c} {}_{1}H_1 \\ \hline {}_{1}H_2 \\ \hline {}_{1}H_3 \end{array}\right] \qquad {}_{1}J = \left[{}_{1}J_{ij}\right] := \left[J_{ij}\right] = J,\ i, j = 1, 2, 3.
$$

Note that ${}_{1}\Sigma$ has the same input/output behavior as Σ, since the introduction of the zero rows and columns corresponds to adding a new state variable which does not influence the dynamics of Σ. Therefore, ${}_{1}\Sigma$ is also a realization of $\mathfrak{B}^L$.

Now, for this new system ${}_1\Sigma$, construct the matrix

$$ {}_1^1M = \left[\begin{array}{c|c} {}_1F & {}_1G_1 \\ {}_1H_3 & {}_1J_{31} \\ {}_1H_2 & {}_1J_{21} \end{array}\right] = \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times n} & 0_{1\times 1} \\ \hline & F & G_1 \\ 0_{(n+2)\times 1} & H_3 & J_{31} \\ & H_2 & J_{21} \end{array}\right] = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times (n+1)} \\ \hline 0_{(n+2)\times 1} & {}^1M \end{array}\right]. \tag{10} $$

Clearly rank ${}_1^1M$ = rank ${}^1M = n+1$, and hence ${}_1^1M$ can be factored as

$$ {}_1^1M = \underbrace{\left[\begin{array}{c} {}_1^1Q \\ \hline {}_1^1R \end{array}\right]}_{n+1 \text{ columns}} \left[\begin{array}{c|c} {}_1^1S & {}_1^1T \end{array}\right], \tag{11} $$

where ${}_1^1R$ has 1 row whereas ${}_1^1S$ is a square matrix of size $n+1$. Similarly to what was done in Example 1, define a new matrix ${}_1^2M$ as

$$ {}_1^2M = \left[{}_1^2M_1 | {}_1^2M_2\right], \tag{12} $$

with

$$ {}_1^2M_1 = {}_1^1Q, \qquad \text{and} \qquad {}_1^2M_2 = \begin{bmatrix} {}_1G_2 \\ {}_1J_{32} \end{bmatrix}, \tag{13} $$

i.e.,

$$ {}_1^2M = \left[\begin{array}{c|c} {}_1^1Q & \begin{array}{c} {}_1G_2 \\ {}_1J_{32} \end{array} \end{array}\right] = \left[\begin{array}{c|c} {}_1^1Q & \begin{array}{c} 0_{1\times 1} \\ G_2 \\ J_{32} \end{array} \end{array}\right], $$

which is a square matrix of size $n+2$.

If rank ${}_1^2M \leqslant n+1$, then ${}_1^2M$ can be also factored similarly to ${}_1^1M$, namely:

$$ {}_1^2M = \underbrace{\left[\begin{array}{c} {}_1^2Q \\ \hline {}_1^2R \end{array}\right]}_{n+1 \text{ columns}} \left[\begin{array}{c|c} {}_1^2S & {}_1^2T \end{array}\right], \tag{14} $$

where ${}_1^2R$ has 1 row whereas ${}_1^2S$ and ${}_1^2Q$ are both square matrices of size $n+1$. Now, defining

$$ \begin{array}{llll} A(0) = {}_1^1S & B(0) = {}_1^1T & C(0) = {}_1H_1 & D(0) = {}_1J_{11} \\ A(1) = {}_1^2S & B(1) = {}_1^2T & C(1) = {}_1^1R & D(1) = {}_1J_{22} \\ A(2) = {}_1^2Q & B(2) = {}_1G_3 & C(2) = {}_1^2R & D(2) = {}_1J_{33} \end{array} $$

it follows that ${}_1\Sigma(k) = (A(k), B(k), C(k), D(k))$ is a 3-periodic state-space system of dimension $n+1$ that induces the time-invariant system ${}_1\Sigma$. Because ${}_1\Sigma$ is a realization of $\mathfrak{B}^L$, it turns out that ${}_1\Sigma(k)$ is a $(n+1)$-dimensional 3-periodic state-space realization of the 3-periodic behavior $\mathfrak{B}$.

Suppose now that matrix ${}_1^2M$ is full rank, i.e., rank ${}_1^2M = n+2$, thus not allowing the decomposition made in (14). In this case, define a new state-space system ${}_2\Sigma = \left({}_2F, {}_2G, {}_2H, {}_2J\right)$ of dimension $n+2$, where the matrices ${}_2F$, ${}_2G$, ${}_2H$, and ${}_2J$ are obtained from ${}_1F$, ${}_1G$, ${}_1H$, and ${}_1J$ similarly as ${}_1F$, ${}_1G$, ${}_1H$, and ${}_1J$ were obtained from F, G, H, and J, i.e.,

$$ {}_2F := \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times (n+1)} \\ \hline 0_{(n+1)\times 1} & {}_1F \end{array}\right] = \left[\begin{array}{c|c} 0_{2\times 2} & 0_{2\times n} \\ \hline 0_{n\times 2} & F \end{array}\right] \qquad {}_2G := \left[\begin{array}{c} 0_{1\times 3} \\ \hline {}_1G \end{array}\right] = \left[\begin{array}{c} 0_{2\times 3} \\ \hline G \end{array}\right] = \left[\begin{array}{c|c|c} \begin{array}{c} 0_{2\times 1} \\ \hline G_1 \end{array} & \begin{array}{c} 0_{2\times 1} \\ \hline G_2 \end{array} & \begin{array}{c} 0_{2\times 1} \\ \hline G_3 \end{array} \end{array}\right] $$

$$ =: \left[\begin{array}{c|c|c} {}_2G_1 & {}_2G_2 & {}_2G_3 \end{array}\right] $$

$$ {}_2H := \left[\begin{array}{c|c} 0_{3\times 1} & {}_1H \end{array}\right] = \left[\begin{array}{c|c} 0_{3\times 2} & H \end{array}\right] $$

$$ = \left[\begin{array}{c} 0_{1\times 2} \mid H_1 \\ \hline 0_{1\times 2} \mid H_2 \\ \hline 0_{1\times 2} \mid H_3 \end{array}\right] =: \left[\begin{array}{c} {}_2H_1 \\ \hline {}_2H_2 \\ \hline {}_2H_3 \end{array}\right] \qquad \text{and } {}_2J = \left[{}_2J_{ij}\right] := \left[{}_1J_{ij}\right] = {}_1J = J, \; i,j = 1,2,3. $$

Note that, ${}_2\Sigma$ is again a realization of $\mathfrak{B}^L$, but with dimension $n+2$. As previously, based on ${}_2\Sigma$, construct the matrix

$$ {}^1_2M = \left[\begin{array}{c|c} {}_2F & {}_2G_1 \\ {}_2H_3 & {}_2J_{31} \\ {}_2H_2 & {}_2J_{21} \end{array}\right] = \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times(n+1)} & 0_{1\times 1} \\ \hline & {}_1F & {}_1G_1 \\ 0_{(n+3)\times 1} & {}_1H_3 & {}_1J_{31} \\ & {}_1H_2 & {}_1J_{21} \end{array}\right] = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+2)} \\ \hline 0_{(n+3)\times 1} & {}^1_1M \end{array}\right], $$

which, taking (10) into account, is also given by

$$ {}^1_2M = \left[\begin{array}{c|c|c} 0_{2\times 2} & 0_{2\times n} & 0_{2\times 1} \\ \hline & F & G_1 \\ 0_{(n+2)\times 2} & H_3 & J_{31} \\ & H_2 & J_{21} \end{array}\right] = \left[\begin{array}{c|c} 0_{2\times 2} & 0_{2\times(n+1)} \\ \hline 0_{(n+2)\times 2} & {}^1M \end{array}\right]. $$

Taking (11) into account:

$$ {}^1_2M = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+2)} \\ \hline 0_{(n+3)\times 1} & \begin{bmatrix} {}^1_1Q \\ {}^1_1R \end{bmatrix} \begin{bmatrix} {}^1_1S & {}^1_1T \end{bmatrix} \end{array}\right] = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+1)} \\ \hline 0_{(n+2)\times 1} & {}^1_1Q \\ \hline 0_{1\times 1} & {}^1_1R \end{array}\right] \times \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times(n+1)} & 0_{1\times 1} \\ \hline 0_{(n+1)\times 1} & {}^1_1S & {}^1_1T \end{array}\right] =: \left[\begin{array}{c} {}^1_2Q \\ \hline {}^1_2R \end{array}\right] \left[\begin{array}{c|c} {}^1_2S & {}^1_2T \end{array}\right]. $$

Define the matrix ${}^2_2M$ as ${}^2_2M = \left[{}^2_2M_1 | {}^2_2M_2\right]$, with

$$ {}^2_2M_1 = {}^1_2Q, \qquad \text{and} \qquad {}^2_2M_2 = \begin{bmatrix} {}_2G_2 \\ {}_2J_{32} \end{bmatrix}, $$

i.e.,

$$ {}^2_2M = \left[\begin{array}{c|c} {}^1_2Q & \begin{matrix} {}_2G_2 \\ {}_2J_{32} \end{matrix} \end{array}\right] = \left[\begin{array}{c|c} {}^1_2Q & \begin{matrix} 0_{1\times 1} \\ {}_1G_2 \\ {}_1J_{32} \end{matrix} \end{array}\right] = \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times(n+1)} & 0_{1\times 1} \\ \hline 0_{(n+2)\times 1} & {}^1_1Q & {}^2_1M_2 \end{array}\right] \overset{\text{(cf. (12),(13))}}{=} \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+2)} \\ \hline 0_{(n+2)\times 1} & {}^2_1M \end{array}\right], \tag{15} $$

then

$$ \operatorname{rank} {}^2_2M = \operatorname{rank} {}^2_1M = n+2, $$

and, consequently, ${}^2_2M$ can be factored as:

$$ {}^2_2M = \underbrace{\left[\begin{array}{c} {}^2_2Q \\ \hline {}^2_2R \end{array}\right]}_{n+2 \text{ columns}} \left[\begin{array}{c|c} {}^2_2S & {}^2_2T \end{array}\right], $$

where ${}^2_2R$ has 1 row whereas ${}^2_2S$ and ${}^2_2Q$ are both square of size $n+2$. Now, defining

$$ \begin{array}{llll} A(0) = {}^1_2S & B(0) = {}^1_2T & C(0) = {}_2H_1 & D(0) = {}_2J_{11} \\ A(1) = {}^2_2S & B(1) = {}^2_2T & C(1) = {}^1_2R & D(1) = {}_2J_{22} \\ A(2) = {}^2_2Q & B(2) = {}_2G_3 & C(2) = {}^2_2R & D(2) = {}_2J_{33} \end{array} $$

it follows that ${}_2\Sigma(k) = (A(k), B(k), C(k), D(k))$ is a 3-periodic state-space system of dimension $n+2$ that induces the time-invariant system ${}_2\Sigma$. Because ${}_2\Sigma$ is a realization of $\mathfrak{B}^L$, it turns out that ${}_2\Sigma(k)$ is a $(n+2)$-dimensional 3-periodic state-space realization of the 3-periodic behavior $\mathfrak{B}$.

In the first part of the previous example, we have seen that, in case the n-dimensional time-invariant state-space system Σ (obtained as a realization of the lifted behavior $\mathfrak{B}^L$) does not generate an n-chain of size $\mathtt{P}-1$ $(=2)$, but allows the construction of a suitable sequence of $\mathtt{P}-1$ matrices of rank less than or equal to $n+1$. In this case it is possible to construct a $\mathtt{P}$-periodic state-space system ${}_1\Sigma(k)$ of dimension $n+1$ that is a $(n+1)$-dimensional periodic state-space realization of $\mathfrak{B}$.

On the other hand, if Σ allows the construction of a suitable sequence of $\mathtt{P}-1$ matrices of rank less than or equal to $n+2$, then it is possible to construct a $\mathtt{P}$-periodic state-space system ${}_2\Sigma(k)$ of dimension $n+2$ that is a $(n+2)$-dimensional periodic state-space realization of $\mathfrak{B}$.

Thus, the possibility of defining suitable sequences of matrices of a certain dimension and the length of such sequences seems to play an important role in obtaining periodic state-space realizations for a periodic behavior $\mathfrak{B}$ starting from a invariant state-space realization of the corresponding lifted behavior $\mathfrak{B}^L$.
This suggests the introduction of the following definition.

Definition 3. Let $\Sigma = (F, G, H, J)$ be a linear time-invariant n-dimensional state-space system with $m\mathtt{P}$ inputs and $p\mathtt{P}$ outputs, for a given positive integer $\mathtt{P}$. For $t = 0, \dots, \bar{t}$, where $\bar{t} := \max\{(\mathtt{P}-1)p, m\}$, define an (n,t)-chain of size s_t generated by Σ as a sequence of matrices ${}^{1}_{t}M, \dots, {}^{s_t}_{t}M$, each one of rank less than or equal to $n+t$, such that:

- ${}^{1}_{t}M := \left[{}^{1}_{t}M_1 | {}^{1}_{t}M_2\right]$ with

$$
{}^{1}_{t}M_1 := {}^{0}_{t}Q = \left[\begin{array}{c|c} 0_{t\times t} & 0_{t\times n} \\ \hline 0_{(n+(\mathtt{P}-1)p)\times t} & \begin{array}{c} F \\ H_{\mathtt{P}} \\ \vdots \\ H_2 \end{array} \end{array}\right] = \left[\begin{array}{c|c} 0_{t\times t} & 0_{t\times n} \\ \hline 0_{(n+(\mathtt{P}-1)p)\times t} & {}^{1}M_1 \end{array}\right], \tag{16}
$$

$$
{}^{1}_{t}M_2 = \left[\begin{array}{c} 0_{t\times m} \\ \hline G_1 \\ J_{\mathtt{P}1} \\ \vdots \\ J_{21} \end{array}\right] = \left[\begin{array}{c} 0_{t\times m} \\ \hline {}^{1}M_2 \end{array}\right] \tag{17}
$$

and

- ${}^{\ell+1}_{t}M := \left[{}^{\ell+1}_{t}M_1 | {}^{\ell+1}_{t}M_2\right]$ with ${}^{\ell+1}_{t}M_1 := {}^{\ell}_{t}Q$ where ${}^{\ell}_{t}Q$ is a $(n + (\mathtt{P} - (\ell + 1))\, p + t) \times (n + t)$ matrix such that $\exists\, {}^{\ell}_{t}R, {}^{\ell}_{t}S$, and ${}^{\ell}_{t}T$ satisfying

$$
{}^{\ell}_{t}M = \begin{bmatrix} {}^{\ell}_{t}Q \\ {}^{\ell}_{t}R \end{bmatrix} \left[\, {}^{\ell}_{t}S \,\middle|\, {}^{\ell}_{t}T \,\right], \tag{18}
$$

$$
{}^{\ell+1}_{t}M_2 := \left[\begin{array}{c} 0_{t\times m} \\ \hline G_{\ell+1} \\ J_{\mathtt{P},\ell+1} \\ \vdots \\ J_{\ell+2,\ell+1} \end{array}\right], \quad \ell = 1, \dots, s_t - 1.
$$

It is straightforward to see that Definition 2 is a particular case of Definition 3, for $t = 0$. We shall call t the *offset* of the (n,t)-chain. For the sake of simplicity, whenever the *offset* t is 0, we drop the t.
Note that, if there exists an (n,t)-chain of size $s_t < \mathtt{P} - 1$, then there exists an $\left(n, t^\star\right)$-chain of size $s_{t^\star} > s_t$, for some $t^\star > t$. In the particular case where $m = p = 1$, there exists an $(n, t+1)$-chain of size $s_{t+1} \geqslant s_t + 1$. In this case, for $\mathtt{P} = 2$, the existence of a chain of size $s_{t+1} = s_t + 1$ follows from a procedure similar to the one introduced in [13].

Remark 2. From equations eqs. (16) and (17), it immediately follows that, for $t = 1, \dots, \bar{t}$,

$$
{}^{1}_{t}M = \left[\begin{array}{c|c} 0_{t\times t} & 0_{t\times(n+m)} \\ \hline 0_{(n+(\mathtt{P}-1)p)\times t} & {}^{1}M \end{array}\right] = \cdots = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathtt{P}-1)p+t-1)\times 1} & {}^{1}_{t-1}M \end{array}\right], \tag{19}
$$

and

$$
{}^{\ell+1}_{t}M_2 = \left[\begin{array}{c} 0_{t\times m} \\ \hline {}^{\ell+1}M_2 \end{array}\right] = \cdots = \left[\begin{array}{c} 0_{1\times m} \\ \hline {}^{\ell+1}_{t-1}M_2 \end{array}\right], \quad \ell = 1, \dots, s_t - 1, \tag{20}
$$

allowing us to conclude that

$$
\operatorname{rank} {}^{1}M = \cdots = \operatorname{rank} {}^{1}_{t}M.
$$

Furthermore, if, for some value of t, $\operatorname{rank} {}^{1}_{t}M \leqslant n + t$, then

$$
\operatorname{rank} {}^{1}_{i}M < n + i, \text{ for } i = t + 1, \dots, \bar{t}\,. \tag{21}
$$

Proposition 1. If rank ${}^{\ell}_{t-1}M \leqslant n+t-1$, with $t = 1,\dots,\bar{t}$, then

$$ {}^{\ell+1}_{t}M = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathrm{P}-(\ell+1))p+t-1)\times 1} & {}^{\ell+1}_{t-1}M \end{array}\right], \quad \ell = 1,\dots,s_t-1. $$

Proof. We first prove that, for $t = 1,\dots,\bar{t}$,

$$ {}^{2}_{t}M = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathrm{P}-2)p+t-1)\times 1} & {}^{2}_{t-1}M \end{array}\right] $$

if rank ${}^{1}_{t-1}M \leqslant n+t-1$. From (21), we conclude that if rank ${}^{1}_{t-1}M \leqslant n+t-1$, then rank ${}^{1}_{t}M < n+t$ allowing us to rewrite (19) as

$$ {}^{1}_{t}M = \begin{bmatrix} {}^{1}_{t}Q \\ {}^{1}_{t}R \end{bmatrix} \begin{bmatrix} {}^{1}_{t}S & {}^{1}_{t}T \end{bmatrix} = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathrm{P}-1)p+t-1)\times 1} & \begin{bmatrix} {}^{1}_{t-1}Q \\ {}^{1}_{t-1}R \end{bmatrix} \begin{bmatrix} {}^{1}_{t-1}S & {}^{1}_{t-1}T \end{bmatrix} \end{array}\right] $$

$$ = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} \\ \hline 0_{(n+(\mathrm{P}-2)p+t-1)\times 1} & {}^{1}_{t-1}Q \\ \hline 0_{p\times 1} & {}^{1}_{t-1}R \end{array}\right] \times \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} & 0_{1\times m} \\ \hline 0_{(n+t-1)\times 1} & {}^{1}_{t-1}S & {}^{1}_{t-1}T \end{array}\right]. $$

Taking (20) into account, we obtain

$$ {}^{2}_{t}M = \left[\, {}^{1}_{t}Q \mid {}^{2}_{t}M_2 \,\right] = \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} & 0_{1\times m} \\ \hline 0_{(n+(\mathrm{P}-2)p+t-1)\times 1} & {}^{1}_{t-1}Q & {}^{2}_{t-1}M_2 \end{array}\right] = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathrm{P}-2)p+t-1)\times 1} & {}^{2}_{t-1}M \end{array}\right] $$

and, therefore, rank ${}^{2}_{t}M =$ rank ${}^{2}_{t-1}M$. If, moreover, rank ${}^{2}_{t-1}M \leqslant n+t-1$, then rank ${}^{2}_{t}M < n+t$.

This reasoning can be repeated for any value of $\ell = 2,\dots,s_t-1$ allowing us to conclude that if, for some $t = 1,\dots,\bar{t}$, rank ${}^{\ell}_{t-1}M \leqslant n+t-1$ (which in turn, by definition, implies that rank ${}^{\ell^\star}_{t-1}M \leqslant n+t-1$, for all $\ell^\star = 1,\dots,\ell-1$ and, consequently, rank ${}^{\ell^\star}_{t}M =$ rank ${}^{\ell^\star}_{t-1}M$), then rank ${}^{\ell}_{t}M < n+t$. Thus, the matrix ${}^{\ell}_{t}M$ can be written as

$$ {}^{\ell}_{t}M = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathrm{P}-\ell)p+t-1)\times 1} & {}^{\ell}_{t-1}M \end{array}\right], $$

and decomposed as:

$$ {}^{\ell}_{t}M = \begin{bmatrix} {}^{\ell}_{t}Q \\ {}^{\ell}_{t}R \end{bmatrix} \begin{bmatrix} {}^{\ell}_{t}S & {}^{\ell}_{t}T \end{bmatrix} = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathrm{P}-\ell)p+t-1)\times 1} & \begin{bmatrix} {}^{\ell}_{t-1}Q \\ {}^{\ell}_{t-1}R \end{bmatrix} \begin{bmatrix} {}^{\ell}_{t-1}S & {}^{\ell}_{t-1}T \end{bmatrix} \end{array}\right] $$

$$ = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} \\ \hline 0_{(n+(\mathrm{P}-(\ell+1))p+t-1)\times 1} & {}^{\ell}_{t-1}Q \\ \hline 0_{p\times 1} & {}^{\ell}_{t-1}R \end{array}\right] \times \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} & 0_{1\times m} \\ \hline 0_{(n+t-1)\times 1} & {}^{\ell}_{t-1}S & {}^{\ell}_{t-1}T \end{array}\right]. $$

Therefore, taking (20) into account, we obtain

$$ {}^{\ell+1}_{t}M = \left[{}^{\ell}_{t}Q \middle| {}^{\ell+1}_{t}M_2\right] = \left[\begin{array}{c|c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} & 0_{1\times m} \\ \hline 0_{(n+(\mathrm{P}-(\ell+1))p+t-1)\times 1} & {}^{\ell}_{t-1}Q & {}^{\ell+1}_{t-1}M_2 \end{array}\right] = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+m+t-1)} \\ \hline 0_{(n+(\mathrm{P}-(\ell+1))p+t-1)\times 1} & {}^{\ell+1}_{t-1}M \end{array}\right]. $$

□

Note that, if $\operatorname{rank}\,{}^{\ell}_{t-1}M \leqslant n+t-1$, with $t = 1, \ldots, \bar{t}$, then it follows, from Proposition 1, that, for $\ell = 1, \ldots, s_t$,

$$
{}^{\ell}_{t}Q = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} \\ \hline 0_{(n+(\mathtt{P}-(\ell+1))p+t-1)\times 1} & {}^{\ell}_{t-1}Q \end{array}\right] \qquad {}^{\ell}_{t}S = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+t-1)} \\ \hline 0_{(n+t-1)\times 1} & {}^{\ell}_{t-1}S \end{array}\right]
$$

$$
{}^{\ell}_{t}R = \left[\begin{array}{c|c} 0_{p\times 1} & {}^{\ell}_{t-1}R \end{array}\right] \qquad {}^{\ell}_{t}T = \left[\begin{array}{c} 0_{1\times m} \\ \hline {}^{\ell}_{t-1}T \end{array}\right].
$$

Remark 3. An immediate consequence of Proposition 1 is that $\operatorname{rank}\,{}^{\ell+1}_{t}M = \operatorname{rank}\,{}^{\ell+1}_{t-1}M$, if $\operatorname{rank}\,{}^{\ell}_{t-1}M \leqslant n+t-1$. Furthermore, if $\operatorname{rank}\,{}^{\ell+1}_{t-1}M \leqslant n+t-1$, then $\operatorname{rank}\,{}^{\ell+1}_{t}M < n+t$.

Given a matrix ${}^{\ell}_{t}M$, for some ℓ and t, and defining the *offset gap* $\delta^{\star}$ as the minimum value of $\delta \geqslant 1$, for which it holds that

$$
\operatorname{rank}\,{}^{\ell}_{t}M > n+t \qquad \text{and} \qquad \operatorname{rank}\,{}^{\ell}_{t+\delta}M \leqslant n+t+\delta,
$$

it follows that $\delta^{\star} = 1$ for SISO systems. In order to see that, define $\underline{t} := \min\{(\mathtt{P}-1)\,p, m\}$ and observe that $\underline{t} = 1$ in SISO systems. Thus, since a matrix ${}^{\ell}_{t}M$ has $n+(\mathtt{P}-\ell)\,p+t$ rows and $n+m+t$ columns, if $\operatorname{rank}\,{}^{\ell}_{t}M > n+t$, then $\operatorname{rank}\,{}^{\ell}_{t}M = n+t+1$. By definition, the premise that the matrix ${}^{\ell}_{t}M$ exists requires that $\operatorname{rank}\,{}^{\ell-1}_{t}M \leqslant n+t$ which, in turn, implies (by Proposition 1) that $\operatorname{rank}\,{}^{\ell}_{t+1}M = n+t+1$.

Example 3. Now, recall Example 2. In this particular case, $\bar{t} = \max\{(3-1)\times 1, 1\} = 2$, meaning that Σ possibly generates an $(n,0)$-chain (or only n-chain, for the sake of simplicity), an $(n,1)$-chain, and/or an $(n,2)$-chain, of size $\mathtt{P}-1\ (=2)$. Since $\underline{t} = \min\{(3-1)\times 1, 1\} = 1$ and, by hypothesis, $\operatorname{rank}\,{}^{1}M = n+1$, it follows that $\operatorname{rank}\,{}^{1}_{1}M \leqslant n+1$, which means that Σ may only generate an $(n,1)$-chain and/or an $(n,2)$-chain, both of size equal to 2, depending on the rank condition of matrix ${}^{2}_{1}M$. Taking the result stated in Proposition 1 into account, regardless of the rank of matrix ${}^{2}_{1}M$, observe that matrix ${}^{2}_{2}M$ can always be expressed as in equation (15), i.e.,

$$
{}^{2}_{2}M = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times(n+2)} \\ \hline 0_{(n+2)\times 1} & {}^{2}_{1}M \end{array}\right].
$$

The ideas pointed out in Example 3 give us insight into how we can overcome the possible nonexistence of an n-chain of size $\mathtt{P}-1$ (generated by a linear time-invariant state-space system of dimension n). A conservative solution consists in noticing that an $\left(n,\bar{t}\right)$-chain of size $\mathtt{P}-1$ is always generated.

Theorem 2 ([14])**.** Let $\mathfrak{B}$ be a linear $\mathtt{P}$-periodic i/o behavior with m inputs and p outputs described by equations (1). Let $\mathfrak{B}^{L}$ be the corresponding time-invariant lifted behavior with minimal n-dimensional state-space realization $\Sigma = (F, G, H, J)$. Assume that the direct feedthrough matrix J in the realizations of $\mathfrak{B}^{L}$ is a lower block triangular matrix with $\mathtt{P}\times\mathtt{P}$ blocks of size $p\times m$. Then $\mathfrak{B}$ has a $\mathtt{P}$-periodic realization of dimension $n+\bar{t}$, where $\bar{t} := \max\{(\mathtt{P}-1)\,p, m\}$.

Proof. Assume that $\Sigma = (F, G, H, J)$ is a time-invariant state-space linear system, of dimension n that does not generate an n-chain of size $\mathtt{P}-1$. Taking a worst-case scenario approach, suppose that this is due to the fact that the rank condition of matrix ${}^{1}M$ does not hold, i.e.,

$$
\operatorname{rank}\,{}^{1}M > n.
$$

In this scenario, at least one matrix ${}^{1}_{t}M$ verifies the rank condition $\operatorname{rank}\,{}^{1}_{t}M \leqslant n+t$. In particular this happens for ${}^{1}_{\underline{t}}M$, i.e.,

$$
\operatorname{rank}\,{}^{1}_{\underline{t}}M \leqslant n+\underline{t},
$$

with $\underline{t} = \min\{(\mathtt{P}-1)\,p, m\}$.
Furthermore, from Remark 2, we conclude that

$$
\operatorname{rank}\,{}^{1}_{t}M \leqslant n+t,\ t = \underline{t}, \ldots, \bar{t}. \tag{22}
$$

Thus, by Remark 3, we conclude that

$$
\operatorname{rank}\,{}^{2}_{\underline{t}}M = \operatorname{rank}\,{}^{2}_{\underline{t}+1}M = \cdots = \operatorname{rank}\,{}^{2}_{\bar{t}}M.
$$

Now, analogously, suppose that the rank condition of matrix ${}^{2}_{\underline{t}}M$ fails, meaning that

$$
\operatorname{rank}\,{}^{2}_{\underline{t}}M > n+\underline{t}.
$$

Since the matrix ${}^{2}_{\underline{t}}M$ has size $\left(n+(\mathrm{P}-2)\,p+\underline{t}\right)\times\left(n+m+\underline{t}\right)$ and $\underline{t}>\min\{(\mathrm{P}-2)\,p,m\}$, we may state that

$$\operatorname{rank}{}^{2}_{t}M\leqslant n+t,\ t=\underline{t}+\underline{t},\ldots,\bar{t}. \tag{23}$$

This reasoning can be repeated until we have

$$\operatorname{rank}{}^{\mathrm{P}-1}_{t^{\star}}M=\cdots=\operatorname{rank}{}^{\mathrm{P}-1}_{\bar{t}}M,$$

with

$$t^{\star}:=\begin{cases}\left(\left\lfloor\dfrac{\bar{t}}{\underline{t}}\right\rfloor-1\right)\underline{t} & \text{if}\quad \left\lfloor\dfrac{\bar{t}}{\underline{t}}\right\rfloor\underline{t}=\bar{t}\\[2ex] \left\lfloor\dfrac{\bar{t}}{\underline{t}}\right\rfloor\underline{t} & \text{if}\quad \left\lfloor\dfrac{\bar{t}}{\underline{t}}\right\rfloor\underline{t}<\bar{t}\end{cases},$$

allowing us to conclude that

$$\operatorname{rank}{}^{\mathrm{P}-1}_{\bar{t}}M\leqslant n+\bar{t}. \tag{24}$$

Therefore, from the relations in eqs. (22) to (24), we conclude that

$$\operatorname{rank}{}^{1}_{\bar{t}}M\leqslant n+\bar{t},\operatorname{rank}{}^{2}_{\bar{t}}M\leqslant n+\bar{t},\ldots,\operatorname{rank}{}^{\mathrm{P}-1}_{\bar{t}}M\leqslant n+\bar{t},$$

i.e., the sequence of matrices ${}^{1}_{\bar{t}}M,\ldots,{}^{s_{\bar{t}}}_{\bar{t}}M$ has size $s_{\bar{t}}=\mathrm{P}-1$, meaning that $\Sigma=(F,G,H,J)$ does generate an $\left(n,\bar{t}\right)$-chain of size $\mathrm{P}-1$.

Thus, there exist matrices ${}^{1}_{\bar{t}}Q$, ${}^{1}_{\bar{t}}R$, ${}^{1}_{\bar{t}}S$, and ${}^{1}_{\bar{t}}T$ such that

$${}^{1}_{\bar{t}}M=\left[{}^{1}_{\bar{t}}M_{1}|{}^{1}_{\bar{t}}M_{2}\right]=\left[\begin{array}{c|c|c} 0_{\bar{t}\times\bar{t}} & 0_{\bar{t}\times n} & 0_{\bar{t}\times m}\\ \hline 0_{n\times\bar{t}} & F & G_{1}\\ 0_{p\times\bar{t}} & H_{\mathrm{P}} & J_{\mathrm{P}1}\\ \vdots & \vdots & \vdots\\ 0_{p\times\bar{t}} & H_{2} & J_{21}\end{array}\right]=\underbrace{\left[\begin{array}{c|c} 0_{\bar{t}\times\bar{t}} & 0_{\bar{t}\times(n+m)}\\ \hline 0_{(n+(\mathrm{P}-1)p)\times\bar{t}} & {}^{1}M\end{array}\right]}_{={}^{1}\overline{M}}=\underbrace{\left[\begin{array}{c}{}^{1}_{\bar{t}}Q\\ \hline {}^{1}_{\bar{t}}R\end{array}\right]}_{n+\bar{t}\text{ columns}}\left[\begin{array}{c|c}{}^{1}_{\bar{t}}S & {}^{1}_{\bar{t}}T\end{array}\right],$$

where ${}^{1}_{\bar{t}}R$ has p rows whereas ${}^{1}_{\bar{t}}S$ has $n+\bar{t}$ columns. Analogously, there exist matrices ${}^{2}_{\bar{t}}Q$, ${}^{2}_{\bar{t}}R$, ${}^{2}_{\bar{t}}S$, and ${}^{2}_{\bar{t}}T$ such that

$${}^{2}_{\bar{t}}M=\left[{}^{1}_{\bar{t}}Q|{}^{2}_{\bar{t}}M_{2}\right]=\left[\begin{array}{c|c} & 0_{\bar{t}\times m}\\ & G_{2}\\ {}^{1}_{\bar{t}}Q & J_{\mathrm{P}2}\\ & \vdots\\ & J_{32}\end{array}\right]=\underbrace{\left[\begin{array}{c}{}^{2}_{\bar{t}}Q\\ \hline {}^{2}_{\bar{t}}R\end{array}\right]}_{n+\bar{t}\text{ columns}}\left[\begin{array}{c|c}{}^{2}_{\bar{t}}S & {}^{2}_{\bar{t}}T\end{array}\right],$$

where ${}^{2}_{\bar{t}}R$ has p rows whereas ${}^{2}_{\bar{t}}S$ has $n+\bar{t}$ columns. Since Σ generates an $\left(n,\bar{t}\right)$-chain of size $\mathrm{P}-1$ this procedure can be repeated until we obtain the decomposition of matrix ${}^{\mathrm{P}-1}_{\bar{t}}M$:

$${}^{\mathrm{P}-1}_{\bar{t}}M=\left[{}^{\mathrm{P}-2}_{\bar{t}}Q|{}^{\mathrm{P}-1}_{\bar{t}}M_{2}\right]=\left[\begin{array}{c|c} & 0_{\bar{t}\times m}\\ {}^{\mathrm{P}-2}_{\bar{t}}Q & G_{\mathrm{P}-1}\\ & J_{\mathrm{P},\mathrm{P}-1}\end{array}\right]=\underbrace{\left[\begin{array}{c}{}^{\mathrm{P}-1}_{\bar{t}}Q\\ \hline {}^{\mathrm{P}-1}_{\bar{t}}R\end{array}\right]}_{n+\bar{t}\text{ columns}}\left[\begin{array}{c|c}{}^{\mathrm{P}-1}_{\bar{t}}S & {}^{\mathrm{P}-1}_{\bar{t}}T\end{array}\right],$$

where again ${}^{\mathrm{P}-1}_{\bar{t}}R$ has p rows whereas ${}^{\mathrm{P}-1}_{\bar{t}}S$ has $n+\bar{t}$ columns. Now, defining

$$\begin{array}{cccc}
\overline{A}(0)={}^{1}_{\bar{t}}S & \overline{B}(0)={}^{1}_{\bar{t}}T & \overline{C}(0)=\overline{H}_{1} & \overline{D}(0)=J_{11}\\
\overline{A}(1)={}^{2}_{\bar{t}}S & \overline{B}(1)={}^{2}_{\bar{t}}T & \overline{C}(1)={}^{1}_{\bar{t}}R & \overline{D}(1)=J_{22}\\
\overline{A}(2)={}^{3}_{\bar{t}}S & \overline{B}(2)={}^{3}_{\bar{t}}T & \overline{C}(2)={}^{2}_{\bar{t}}R & \overline{D}(2)=J_{33}\\
\vdots & \vdots & \vdots & \vdots\\
\overline{A}(\mathrm{P}-2)={}^{\mathrm{P}-1}_{\bar{t}}S & \overline{B}(\mathrm{P}-2)={}^{\mathrm{P}-1}_{\bar{t}}T & \overline{C}(\mathrm{P}-2)={}^{\mathrm{P}-2}_{\bar{t}}R & \overline{D}(\mathrm{P}-2)=J_{\mathrm{P}-1,\mathrm{P}-1}\\
\overline{A}(\mathrm{P}-1)={}^{\mathrm{P}-1}_{\bar{t}}Q & \overline{B}(\mathrm{P}-1)=\overline{G}_{\mathrm{P}} & \overline{C}(\mathrm{P}-1)={}^{\mathrm{P}-1}_{\bar{t}}R & \overline{D}(\mathrm{P}-1)=J_{\mathrm{PP}}
\end{array}$$

it follows that $\overline{\Sigma}(k) = \left(\overline{A}(k), \overline{B}(k), \overline{C}(k), \overline{D}(k)\right)$, of dimension $n + \bar{t}$, is a P-periodic state-space system that induces a linear time-invariant system $\overline{\Sigma}^L = \left(\overline{F}, \overline{G}, \overline{H}, \overline{J}\right)$, with mP inputs, pP outputs, and dimension $n + \bar{t}$, with

$$\overline{F} := \left[\begin{array}{c|c} 0_{\bar{t}\times\bar{t}} & 0_{\bar{t}\times n} \\ \hline 0_{n\times\bar{t}} & F \end{array}\right] \qquad \overline{G} := \left[\begin{array}{c|c|c|c} 0_{\bar{t}\times m} & 0_{\bar{t}\times m} & \cdots & 0_{\bar{t}\times m} \\ \hline G_1 & G_2 & \cdots & G_{\mathrm{P}} \end{array}\right]$$

$$\overline{H} := \left[\begin{array}{c|c} 0_{p\times\bar{t}} & H_1 \\ \hline 0_{p\times\bar{t}} & H_2 \\ \hline \vdots & \vdots \\ \hline 0_{p\times\bar{t}} & H_{\mathrm{P}} \end{array}\right] \qquad \text{and } \overline{J} := J.$$

This system has clearly the same input/output behavior as Σ. $\square$

This shows that every periodic input/output behavior $\mathfrak{B}$ has a periodic state-space realization. The procedure presented here in order to obtain such realization consists in considering the time-invariant lifted version $\mathfrak{B}^L$ of $\mathfrak{B}$, constructing a linear time-invariant state-space realization Σ^L of $\mathfrak{B}^L$, and, using Σ^L to define a periodic state-space realization of $\mathfrak{B}$ as shown in the proof of Theorem 2.

Corollary 1 ([14]). Let $\mathfrak{B}$ be a P-periodic input/output behavior, for some positive integer P. Then, $\mathfrak{B}$ has a P-periodic state-space realization.

Another consequence of Theorem 2 is the establishment of bounds for the dimension of the periodic state-space realizations of a periodic input/output behavior.

Corollary 2 ([14]). Let $\mathfrak{B}$ be a P-periodic input/output behavior for some positive integer P with m inputs and p outputs. Let further $\mathfrak{B}^L$ be the corresponding lifted time-invariant behavior, and assume that the minimal dimension of the time-invariant state-space realizations of $\mathfrak{B}^L$ is $n_{\mathfrak{B}^L}$. Then, the minimal dimension $n_{\mathfrak{B}}$ of the periodic state-space realizations of $\mathfrak{B}$ satisfies the inequality

$$n_{\mathfrak{B}^L} \leqslant n_{\mathfrak{B}} \leqslant n_{\mathfrak{B}^L} + \bar{t},$$

where $\bar{t} := \max\{(\mathrm{P}-1)\,p, m\}$.

Although these results answer positively to the question of existence of a periodic state-space realization of a periodic i/o behavior, they do not answer the question of minimality. Indeed, as shown in Example 2, there may exist periodic state-space realizations with dimension lower than $n + \bar{t}$.

In order to look for such realizations, we apply a step by step procedure as follows. We start with the matrix ${}^{1}M$ and investigate whether this matrix is associated with an n-chain of size $\mathrm{P}-1$. If this is the case, the periodic state-space realization of dimension n corresponding to this chain (according to 9) is a minimal realization of the i/o periodic behavior. In case this does not happen we construct the matrix ${}^{1}_{1}M$ according to Definition 3 and investigate whether this matrix is associated with an $(n,1)$-chain of size $\mathrm{P}-1$ and proceed as before in order to see whether it is possible to obtain a periodic state-space realization of dimension $n+1$. This process is repeated until a matrix ${}^{1}_{t^\star}M$ is obtained which is associated with an $\left(n, t^\star\right)$-chain of size $\mathrm{P}-1$. Due to Theorem 2 this procedure stops at most for $t^\star = \bar{t} := \max\{(\mathrm{P}-1)\,p, m\}$.

The $\left(n + t^\star\right)$-dimensional periodic state-space realization obtained in this way is not guaranteed to be a minimal realization of the original periodic i/o behavior. This is due to the fact that our strategy corresponds to increasing the dimension of the time-invariant realizations of the lifted behavior in a particular way. This leads us to non-minimal realizations, which, as is well-known, are not necessarily algebraically equivalent. Therefore, the fact that a given non-minimal time-invariant realization is not induced by a periodic state-space realization does not mean that the same happens for all the other time-invariant realizations of equal dimension.

Nevertheless, in many cases, our procedure allows finding periodic state-space lower-dimensional realizations, i.e., with dimension smaller than $n + \bar{t}$.

The following algorithm presents the procedure in a systematic way.

This algorithm can be synthesized in a flowchart, see Figure 1 .

Algorithm 1 Matrix chain algorithm

Input: MIMO P-periodic behavior $\mathfrak{B}$

Output: Minimal P-periodic state-space representation of $\mathfrak{B}$. $\triangleright$ $\Sigma(k) = (A(k), B(k), C(k), D(k))$

Step 1 Construct the lifted behavior $\mathfrak{B}^L$

Step 2 Compute a minimal representation $\Sigma = (F, G, H, J)$ of $\mathfrak{B}^L$ and its dimension n

Step 3 Let $t = 0$ and $\ell = 1$

Step 4 Construct matrix ${}^{\ell}_{t}M$ as defined in eqs. (16) and (17)

Step 5 If rank ${}^{\ell}_{t}M \leqslant n + t$ **then**

  Step 5.1 Factorize ${}^{\ell}_{t}M$ as in (18)

  Step 5.2 If $\ell < P - 1$ **then**

    Step 5.2.1 Let $\ell = \ell + 1$

    Step 5.2.2 Go to **Step 4**

    Else Define $t^\star := t$ and go to **Step 8**

  Else Continue

Step 6 Let $t = t + 1$ and $\ell = 1$

Step 7 Go to **Step 4**

Step 8 Define matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ accordingly to the factorizations of matrices ${}^{1}_{t^\star}M, \dots, {}^{P-1}_{t^\star}M$ $\triangleright$ The matrices that result from the $(n, t^\star)$-chain

Step 9 Stop

4 | EXAMPLE

Consider the linear 3-periodic input/output behavior described by the following equations:

$$\begin{cases} \left(\left(\sigma^2 + 3\sigma - 1\right) y\right)(3k) = \left((\sigma + 19)\, u_1\right)(3k) + u_2(3k) \\ \left(\left(\sigma^2 + \sigma + 3\right) y\right)(3k+1) = \left(\left(-\sigma^2 + 1 + 18\sigma^{-1}\right) u_1\right)(3k+1) + u_2(3k+1), \quad k \in \mathbb{Z}. \\ \left(\left(\sigma^2 + 2\sigma + 1\right) y\right)(3k+2) = \left(\left(\sigma + 1 + 18\sigma^{-2}\right) u_1\right)(3k+2) + \left(\sigma^{-1} u_2\right)(3k+2) \end{cases} \tag{25}$$

Noticing that $u = \begin{bmatrix} u_1 \; u_2 \end{bmatrix}^T$, these equations can be written as:

$$\left(\begin{bmatrix} -1 & 3 & 1 \\ \sigma^3 & 3 & 1 \\ 2\sigma^3 & \sigma^3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sigma \\ \sigma^2 \end{bmatrix} y \right)(3k) = \left(\begin{bmatrix} 19 & 1 & 1 & 0 & 0 & 0 \\ 18 - \sigma^3 & 0 & 1 & 1 & 0 & 0 \\ 18 + \sigma^3 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_2 \\ \sigma I_2 \\ \sigma^2 I_2 \end{bmatrix} u \right)(3k), \; k \in \mathbb{Z},$$

which, applying the lifting technique of Section 2, yields the linear time-invariant input/output behavior:

$$\left(\underbrace{\begin{bmatrix} -1 & 3 & 1 \\ \sigma & 3 & 1 \\ 2\sigma & \sigma & 1 \end{bmatrix}}_{P^L(\sigma,\sigma^{-1})} y^L \right)(k) = \left(\underbrace{\begin{bmatrix} 19 & 1 & 1 & 0 & 0 & 0 \\ 18 - \sigma & 1 & 1 & 0 & 0 & 0 \\ 18 + \sigma & 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{Q^L(\sigma,\sigma^{-1})} u^L \right)(k), \; k \in \mathbb{Z}. \tag{26}$$

Note that $P^L\left(\xi, \xi^{-1}\right)$ and $Q^L\left(\xi, \xi^{-1}\right)$ are not left coprime, since the matrix:

$$R^L\left(\xi, \xi^{-1}\right) := \left[P^L\left(\xi, \xi^{-1}\right) \quad -Q^L\left(\xi, \xi^{-1}\right) \right], \tag{27}$$

with normal rank 3, has a rank drop for $\xi = -1$.

Thus, the transfer function:

$$G^L\left(\xi, \xi^{-1}\right) = \left(P^L\left(\xi, \xi^{-1}\right)\right)^{-1} \left(Q^L\left(\xi, \xi^{-1}\right)\right) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{3\xi}{\xi-3} & \frac{-1}{\xi-3} & \frac{-1}{\xi-3} & \frac{1}{\xi-3} & \frac{1}{\xi-3} & 0 \\ \frac{9(\xi-6)}{\xi-3} & \frac{\xi}{\xi-3} & \frac{\xi}{\xi-3} & \frac{-3}{\xi-3} & \frac{-3}{\xi-3} & 0 \end{bmatrix}$$

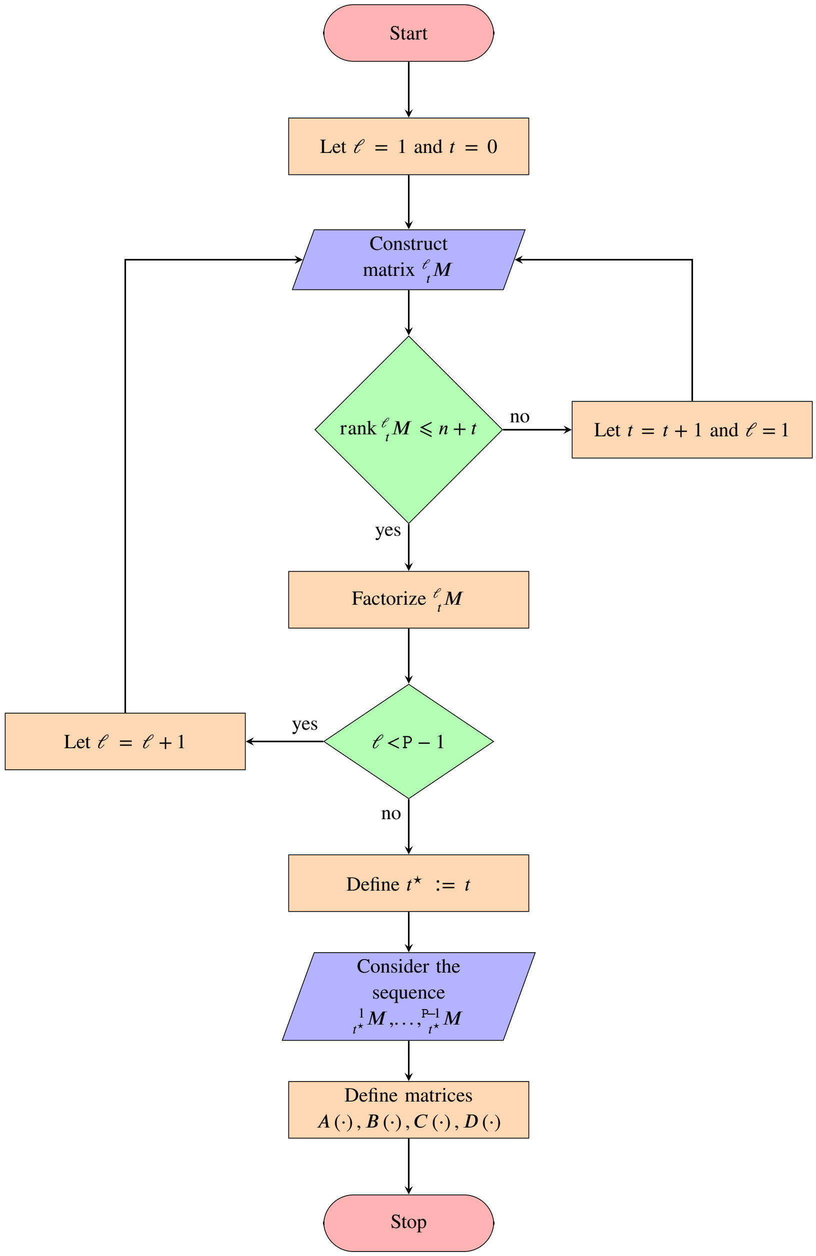

FIGURE 1 Periodic state-space representation flowchart

does not describe the full input/output behavior given by (26).

Therefore, rather than obtaining a state-space realization via G^L, we shall take the input/output equations (26) as a starting point, and apply the realization procedure proposed in [21] (for further details see [23,24,25,26]). For that purpose, following the reasoning of

Algorithm 2 (cf.[25, p. 1066]), the state vector

$$x := X\left(\sigma, \sigma^{-1}\right) w$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix},$$

where $w_i = y_i^L$ if $i = 1, 2, 3$, and $w_i = -u_i^L$ if $i = 4, \ldots, 9$, corresponds to a minimal state realization for the kernel of R^L (given by (27)). Now, considering the i/o relations (26), and the obtained state equations

$$\begin{aligned} x_1 &= w_1 + w_4 \\ x_2 &= 2w_1 + w_2 - w_4, \end{aligned}$$

it is an easy task to reach to the following minimal (time-invariant) state-space realization of the lifted input/output behavior (26):

$$\begin{cases} (\sigma x)(k) = \overbrace{\begin{bmatrix} -1 & 0 \\ -7 & 3 \end{bmatrix}}^{F} x(k) + \overbrace{\left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & -1 & -1 & 1 & 1 & 0 \end{array}\right]}^{G_1 \;|\; G_2 \;|\; G_3} u^L(k) \\[2em] y^L(k) = \begin{array}{c} H_1 \\ \hline H_2 \\ \hline H_3 \end{array} \left\{ \left[\begin{array}{cc} 1 & 0 \\ \hline -2 & 1 \\ \hline 7 & -3 \end{array}\right] \right. x(k) + \underbrace{\left[\begin{array}{cc|cc|cc} -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 3 & 0 & 0 & 0 & 0 & 0 \\ \hline 9 & 1 & 1 & 0 & 0 & 0 \end{array}\right]}_{J=[J_{ij}]} u^L(k) \end{cases} . \tag{28}$$

Note that this realization has dimension $n = 2$.

In order to obtain a periodic state-space realization for the original periodic input/output behavior (25), consider the matrix:

$${}^1M = \left[\begin{array}{c|c} F & G_1 \\ H_3 & J_{31} \\ H_2 & J_{21} \end{array}\right] = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ -7 & 3 & 9 & -1 \\ 7 & -3 & 9 & 1 \\ -2 & 1 & 3 & 0 \end{array}\right].$$

It is easy to check that, rank ${}^1M = 4 > n = 2$. Define a new time-invariant realization of (26) with increased dimension[1] $n + (4-2) = 4$ as follows:

$$
{}_2F := \left[\begin{array}{c|c} 0_{2\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & F \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -7 & 3 \end{bmatrix} \qquad {}_2G := \left[\begin{array}{c|c|c} 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} \\ \hline G_1 & G_2 & G_3 \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}
$$

$$
{}_2H := \left[\begin{array}{c|c} 0_{1\times 2} & H_1 \\ \hline 0_{1\times 2} & H_2 \\ \hline 0_{1\times 2} & H_3 \end{array}\right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 7 & -3 \end{bmatrix} \qquad {}_2J := J.
$$

Further, redefine a new matrix ${}^1_2M$ as:

$$
{}^1_2M = \left[\begin{array}{c|c} {}_2F & {}_2G_1 \\ {}_2H_3 & J_{31} \\ {}_2H_2 & J_{21} \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -7 & 3 & 9 & -1 \\ 0 & 0 & 7 & -3 & 9 & 1 \\ 0 & 0 & -2 & 1 & 3 & 0 \end{array}\right].
$$

Since rank ${}^1_2M = 4 \leqslant n + 2$, it is possible to factor this matrix as:

$$
{}^1_2M = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -7 & 3 & 9 & -1 \\ 7 & -3 & 9 & 1 \\ \hline -2 & 1 & 3 & 0 \end{array}\right] \left[\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] =: \left[\begin{array}{c} {}^1_2Q \\ \hline {}^1_2R \end{array}\right] \left[\begin{array}{c|c} {}^1_2S & {}^1_2T \end{array}\right].
$$

Now, define

$$
{}^2_2M = \left[\begin{array}{c|c} {}^1_2Q & \begin{array}{c} {}_2G_2 \\ J_{32} \end{array} \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -7 & 3 & 9 & -1 & -1 & 1 \\ 7 & -3 & 9 & 1 & 1 & 0 \end{array}\right].
$$

Since rank ${}^2_2M < 4$, one can factor this matrix as:

$$
{}^2_2M = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -7 & 3 & 9 & -1 \\ \hline 7 & -3 & 9 & 1 \end{array}\right] \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{18} \\ 0 & 0 & 0 & 1 & 4 & 1 \end{array}\right] =: \left[\begin{array}{c} {}^2_2Q \\ \hline {}^2_2R \end{array}\right] \left[\begin{array}{c|c} {}^2_2S & {}^2_2T \end{array}\right].
$$

[1] Observe that letting $t = 1$ would not be enough to achieve the desired rank condition.

In this way, we conclude that the 3-periodic input/output system (25) has a 3-periodic state-space realization $\Sigma(k) = (A(k), B(k), C(k), D(k))$ of dimension 4, given by:

$$A(0) = {}^1_2S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B(0) = {}^1_2T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C(0) = {}_2H_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad D(0) = J_{11} = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

$$A(1) = {}^2_2S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B(1) = {}^2_2T = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \\ 0 & \frac{1}{18} \\ 4 & 1 \end{bmatrix}, \quad C(1) = {}^1_2R = \begin{bmatrix} -2 & 1 & 3 & 0 \end{bmatrix}, \quad D(1) = J_{22} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A(2) = {}^2_2Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -7 & 3 & 9 & -1 \end{bmatrix}, \quad B(2) = {}_2G_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C(2) = {}^2_2R = \begin{bmatrix} 7 & -3 & 9 & 1 \end{bmatrix}, \quad D(2) = J_{33} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

5 | CONCLUSION

In this paper, we have shown, in a constructive way, the existence of periodic state-space realizations for periodic behaviors given by linear periodic input/output equations.

In order to obtain such realizations, we started by defining a time-invariant input/output behavior $\mathfrak{B}^L$associated with a given periodic input/output behavior $\mathfrak{B}$ by means of a lifting technique. As a second step, we obtained a state-space realization Σ^L of $\mathfrak{B}^L$ using behavioral methods. Finally, we recovered a periodic state-space realization for $\mathfrak{B}$ from a suitable matrix chain constructed with basis on the matrices of the time-invariant realization Σ^L.

Regarding the minimal dimension of the periodic realizations, we have seen that this may be greater than the minimal dimension of the corresponding time-invariant ones and gave an upper bound for the former in terms of the latter. Moreover, we presented an algorithm that yields periodic realizations of the lower dimension than the given upper bound.

Future work includes pursuing the goal of obtaining minimal realizations for periodic input/output behaviors.

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Conflict of interest

The authors declare no potential conflict of interests.

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