Differentiability on time and continuity on fractional order of solutions for Caputo fractional evolution equations

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August 16, 2022

Abstract

In this paper, the differentiability on time and continuity on fractional order of solutions for a class of Caputo fractional evolution equations are studied. Under appropriate assumptions, the existence and differentiability on time of solutions for linear as well as semilinear Caputo fractional evolution equations are analyzed, the continuity of solutions on fractional order for linear and semilinear Caputo fractional evolution equations are discussed. In addition, if the fractional order converges to \$1\$, then the solutions of the Caputo fractional differential equations become the solutions of classic evolution equations. The continuity of solutions on fractional order for some fractional systems is numerically studied, and the results are basically consistent with the theoretical results.

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Abstract

In this paper, the differentiability on time and continuity on fractional order of solutions for a class of Caputo fractional evolution equations are studied. Under appropriate assumptions, the existence and differentiability on time of solutions for linear as well as semilinear Caputo fractional evolution equations are analyzed, the continuity of solutions on fractional order for linear and semilinear Caputo fractional evolution equations are discussed. In addition, if the fractional order converges to 1, then the solutions of the Caputo fractional differential equations become the solutions of classic evolution equations. The continuity of solutions on fractional order for some fractional systems is numerically studied, and the results are basically consistent with the theoretical results.

Keywords: Existence, Differentiability, Continuity, Caputo fractional derivative, Fractional evolution equation

2010 MSC: Primary 34A08, secondary 35B30

Jia Wei He, Yong Zhou, Hölder regularity for non-autonomous fractional evolution equations, Fractional Calculus and Applied Analysis, 2022, https://doi.org/10.1007/s13540-022-00019-1.

1 Introduction

Starting with some conjectures of Leibniz and Euler, followed by the works of other famous mathematicians, including Laplace, Fourier, Abel, Liouville and Riemann, fractional calculus which allows us to consider integration and differentiation of any order has been applied to various branches of science and engineering, such as control systems [1], physics [2], electrical engineering [3], viscoelastic mechanics [4], signal processing [5], bioengineering [6] and so on. From the definition of Caputo fractional derivative, the memory effect or non-local property of

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Caputo fractional derivative is represented by a convolution integral with power-law memory kernel, which makes Caputo fractional differential equations an excellent tool in complex systems, which is one of the main advantages of Caputo fractional differential equation (nonlocal) compared with classical (local) model [7,8].

In recent years, Caputo fractional differential equations have become more and more important in theory and application, which has attracted great attention of researchers. Many authors pay attention to the Caputo fractional differential equations, such as the existence and uniqueness of solutions [9–13], the continuous dependence of solutions on initial values [14,15], dynamic behavior [16–21] and so on. Very recently, there are a few researchers pay attention to the continuity of solutions on fractional order of Caputo fractional differential equations. For example, Dang, Nane, Nguyen and Tuan studied the continuity of the solutions with respect to the fractional parameters as well as the initial value of a class of equations including the Abel equations of the first and second kind, and time fractional diffusion type equations [22]. Tuan, O'Regan and Ngoc studied the continuity of the solution of both the initial problem and the inverse initial value problems with respect to the fractional order [23]. Binh, Hoang, Baleanu and Van considered a problem of continuity fractional-order for pseudo-parabolic equations with the fractional derivative of Caputo [24]. Note that these papers do not study the relationship between the solutions of integer order differential equations and fractional ones. In practice, many problems of time fractional equations depend on fractional parameters, that is, fractional orders. However, these fractional parameters are unknown in the modeling processes. Therefore, the continuity of solutions on these parameters is very important for modeling. Moreover, if this continuity is not tenable, numerical calculations are not allowed. On the other hand, from the definition of Caputo fractional derivative, it can be seen that the differentiability on time of the solutions for Caputo fractional differential equations is a basic and necessary property. Therefore, the existence and differentiability on time of solutions are two basic and key problems in the qualitative theory of Caputo fractional differential equations.

In this work, motivated by the above consideration, we are eager to study the existence, differentiability on time and continuity on fractional order of solutions for the following linear Caputo fractional evolution equations

$$\begin{cases} {}^C D_t^{\alpha} x(t) = Ax(t) + F(t), \ 0 < t \leq T, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

as well as the semilinear Caputo fractional evolution equations

$$\begin{cases} {}^C D_t^{\alpha} x(t) = Ax(t) + F(t, x(t)), \ 0 < t \leq T, \\ x(0) = x_0, \end{cases} \tag{1.2}$$

in a Banach space X, where ${}^C D_t^{\alpha}$, $0 < \alpha < 1$, is the regularized Caputo fractional derivative of order α, A is a bounded linear operator on X, $T \in (0, +\infty)$ is a constant and x_0 is given belonging to a subset of X. Under appropriate assumptions, the existence and differentiability on time of solutions for (1.1) and (1.2) are analyzed, the continuity of solutions on fractional order for (1.1) and (1.2) are discussed. Furthermore, if the fractional order α converges to 1, the solutions of (1.1) and (1.2) become the solutions of their corresponding classical linear and semilinear evolution equations. Numerical studies are performed to explore the continuity of solutions on fractional order for some fractional systems, which show reasonable agreement with the theoretic results.

The rest of this paper is organized as follows. In Section 2, we review some symbols, definitions, lemmas and preliminary facts. In Section 3, we analyze the existence and differentiability on time of solutions for (1.1) and (1.2). In Section 4, we discuss the continuity of solutions on fractional order for (1.1). We firstly study the continuity of solutions on fractional order α for $0 < \alpha < 1$, and then the continuity of solutions as $\alpha \to 1^-$. In Section 5, the corresponding semilinear equations (1.2) are studied. Then, we give some examples and their numerical studies in Section 6 to illustrate our theoretical results. Last but not least, some conclusions are drawn in Section 7.

2 Preliminaries

This section is concerned with some notations, definitions, lemmas and preliminary facts which are used in what follows.

From now on, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ stand for the set of positive integral numbers, real numbers, positive real numbers and complex numbers respectively. Let $(X, \|\cdot\|)$ be a Banach space. For $p \in [1, +\infty)$ and set $I = [0, T]$ for some $T \in (0, +\infty)$, we denote by $L^p(I, X)$ the Bochner space of all equivalence classes of strongly measurable functions $x : I \to X$, such that

$$\|x\|_{L^p} := \left(\int_I \|x(t)\|^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Denote by $C(I, X)$ the Banach space of all continuous functions from I to X with the norm

$$\|x\|_\infty := \max\{\|x(t)\| : t \in I\},$$

and $C^n(I, X)$ $(n \in \mathbb{Z}^+)$ the set of all n order continuous differentiable functions from I to X. Let also $\mathbb{L}(X)$ be the Banach space of all bounded linear operators from X into itself endowed with the norm

$$\|T\|_{\mathbb{L}(X)} = \sup\{\|Tx\| : x \in X,\ \|x\| = 1\},$$

and write $\|T\|_{L(X)}$ as $\|T\|$ for every $T \in L(X)$ when it has no loss of the clarity.

Firstly, we recall some basic definitions and results on Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative.

Definition 2.1. *[7, 8, 21, 27] The Riemann-Liouville fractional integral of order $\alpha > 0$ with lower limit zero for a continuous function $g : \mathbb{R}^+ \to X$ is defined as*

$$I_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds, \ t > 0,$$

where I_t^α denotes the Riemann-Liouville fractional integral of order α, $\Gamma(\cdot)$ is the Euler's Gamma function and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Definition 2.2. *[7, 8, 21, 27] The Riemann-Liouville fractional derivative of order $\alpha > 0$ with lower limit zero for a function $g \in C^{n+1}(\mathbb{R}^+, X)$ is defined as*

$${}^{RL}D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} g(s)ds, \ t > 0,$$

where ${}^{RL}D_t^\alpha$ denotes the Riemann-Liouville fractional derivative of order α.

Definition 2.3. *[7, 8, 21, 27] The Caputo fractional derivative of order $\alpha > 0$ with lower limit zero for a function $g \in C^{n+1}(\mathbb{R}^+, X)$ is defined as*

$${}^{C}D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds, \ t > 0,$$

where ${}^{C}D_t^\alpha$ denotes the Caputo fractional derivative of order α.

Lemma 2.1. *[7, 8, 21, 27] If $g(t) \in C^n(\mathbb{R}^+, X)$ and $n-1 < \alpha < n \in \mathbb{Z}^+$, then*

(1) $I_t^\alpha[{}^{C}D_t^\alpha g(t)] = g(t) - \sum\limits_{k=0}^{n-1} g^{(k)}(0)\frac{t^k}{k!}$,

(2) ${}^{RL}D_t^\alpha[I_t^\alpha g(t)] = g(t)$,

(3) ${}^{RL}D_t^\alpha g(t) = {}^{C}D_t^\alpha g(t) + \sum\limits_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} g^{(k)}(0)$.

Lemma 2.2. *[7, 8, 21, 27] Let $\alpha, \beta \in \mathbb{R}^+$. Then for all $f \in L^1(\mathbb{R}^+, X)$ and $\forall \mathrm{t} > 0$,*

$$I_t^\alpha[I_t^\beta f(t)] = I_t^{\alpha+\beta} f(t).$$

To deal with fractional differential equations, we need the following special functions and generalization of Gronwall's inequality.

Denote by $E_{\alpha,\beta}$ the generalized Mittag-Leffler special function defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int\limits_{\Upsilon} \frac{\lambda^{\alpha-\beta} e^\lambda}{\lambda^\alpha - z} \mathrm{d}\lambda, \ \alpha, \beta > 0, z \in \mathbb{C},$$

where Υ is a contour which starts and ends at $-\infty$ and encircles the disc $|\lambda| \leq |z|^{1/\alpha}$ counter-clockwise. For short, set $E_\alpha(z) := E_{\alpha,1}(z)$.

Lemma 2.3. *[26] Suppose $\alpha > 0$, $k(t)$ is a nonnegative function locally integrable on $0 \le t \le T$ (some $T < +\infty$), and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t \le T$, $g(t) \le M = Const$, and suppose $f(t)$ is nonnegative and locally integrable on $0 \le t \le T$ with*

$$f(t) \le k(t) + g(t) \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Then

$$f(t) \le k(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} k(s) \right] ds, \ 0 \le t \le T.$$

Lemma 2.4. *[26] Under the hypothesis of Lemma 2.3, let $k(t)$ be a nondecreasing function on $0 \le t \le T$. Then*

$$f(t) \le k(t) E_\alpha[g(t)\Gamma(\alpha)t^\alpha], \ 0 \le t \le T.$$

3 Existence and differentiability on time of solutions

In this section, we study the existence and differentiability on time of solutions for the following linear Caputo fractional evolution equations

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + F(t), \ 0 < t \le T, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

as well as the semilinear Caputo fractional evolution equations

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + F(t, x(t)), \ 0 < t \le T, \\ x(0) = x_0, \end{cases} \tag{3.2}$$

in a Banach space X, where ${}^C D_t^\alpha$, $0 < \alpha < 1$, is the regularized Caputo fractional derivative of order α, A is a bounded linear operator on X, $T \in (0, +\infty)$ is a constant and x_0 is given belonging to a subset of X.

Firstly, we give two useful lemmas.

Lemma 3.1. *If $x : [0, T] \to X$ is a continuously differentiable function, $x(t) \in D(A)$ for $t \in [0, T]$. Then $x(t)$ is a solution to (3.1) if and only if*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s) ds, \ 0 \le t \le T. \tag{3.3}$$

Proof. The sufficiency of (3.3).

For $t = 0$, one has $x(t) = x_0$, thus (3.1) holds.

For $0 < t \le T$, applying the Riemann-Liouville fractional derivative ${}^{RL}D_t^\alpha$ on both sides of (3.3) and using properties of the fractional derivative (See Lemma 2.1 (2)), we have

$${}^{RL}D_t^\alpha x(t) = x_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + Ax(t) + F(t).$$

Considering $0 < \alpha < 1$ and using the relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative (See Lemma 2.1 (3)), one can easily obtain

$$ {}^{RL}D_t^\alpha x(t) = {}^{C}D_t^\alpha x(t) + x_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. $$

Thus, we get

$$ {}^{C}D_t^\alpha x(t) = Ax(t) + F(t),\ 0 < t \leq T, $$

which implies (3.1) holds.

The necessity of (3.3).

Note that $0 < \alpha < 1$, $1 - \alpha > 0$. Applying $I_t^{1-\alpha}$ to (3.1), we have

$$ I_t^\alpha {}^{C}D_t^\alpha x(t) = I_t^\alpha Ax(t) + I_t^\alpha F(t). $$

According to Definition 2.3, one has

$$ I_t^\alpha {}^{C}D_t^\alpha x(t) = I_t^\alpha [I_t^{1-\alpha}[D_t^1 x(t)]]. $$

By using Lemma 2.2, we have

$$ I_t^\alpha {}^{C}D_t^\alpha x(t) = I_t^\alpha [I_t^{1-\alpha}[D_t^1 x(t)]] = I_t^1[D_t^1 x(t)]. $$

Then

$$ I_t^\alpha {}^{C}D_t^\alpha x(t) = I_t^1[D_t^1 x(t)] = \int_0^t x'(s)ds = x(t) - x(0). $$

So

$$ x(t) - x(0) = I_t^\alpha Ax(t) + I_t^\alpha F(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s)\mathrm{d}s, $$

which implies

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s)\mathrm{d}s, $$

i.e. (3.3) holds.

A similar argument enables us to give the following lemma.

Lemma 3.2. *If $x : [0, T] \to X$ is a continuously differentiable function, $x(t) \in D(A)$ for $t \in [0, T]$. Then $x(t)$ is a solution to the equation (3.2) if and only if*

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x(s))ds,\ 0 \leq t \leq T. \quad (3.4) $$

Proof. By a similar argument with Lemma 3.1, one can prove this lemma.

Then, we consider the following Volterra equations

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x(s)) \mathrm{d}s, \ 0 \le t \le T, \tag{3.5}$$

where $0 < \alpha < 1$, $x_0 \in X$ and $F(s, v) : [0, T] \times X \to X$.

We have the following result.

Lemma 3.3. *If $F(t, x)$ is continuous with respect to t on $[0, T]$ and Lipschitz continuous with respect to x, namely, there exists a positive constant L such that*

$$\|F(t, x) - F(t, y)\| \le L\|x - y\|, \quad \forall t \in [0, T], \quad x, y \in X.$$

Then there exists a unique solution $x(t)$ for (3.5), moreover $x(t) \in C([0, T], X)$.

Proof. Denote

$$Gx(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x(s)) \mathrm{d}s.$$

Firstly, we show that if $x(t) \in C([0, T], X)$, then $Gx(t) \in C([0, T], X)$, i.e ,

$$G : C([0, T], X) \to C([0, T], X).$$

In fact

$$Gx(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x(s)) \mathrm{d}s = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} F(t-s, x(t-s)) \mathrm{d}s.$$

Then for any $t_1, t_2 \in [0, T]$

$$\begin{aligned}
&\|Gx(t_1) - Gx(t_2)\| \\
=&\left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} F(t_1 - s, x(t_1 - s)) \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} s^{\alpha-1} F(t_2 - s, x(t_2 - s)) \mathrm{d}s \right\| \\
\le&\left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} F(t_1 - s, x(t_1 - s)) \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} F(t_1 - s, x(t_2 - s)) \mathrm{d}s \right\| \\
&+ \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} F(t_1 - s, x(t_2 - s)) \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} F(t_2 - s, x(t_2 - s)) \mathrm{d}s \right\| \\
&+ \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} F(t_2 - s, x(t_2 - s)) \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} s^{\alpha-1} F(t_2 - s, x(t_2 - s)) \mathrm{d}s \right\| \\
:=&\omega_1 + \omega_2 + \omega_3.
\end{aligned} \tag{3.6}$$

For ω_1, we have

$$\begin{aligned}
\omega_1 \le& \frac{L}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} \|x(t_1 - s) - x(t_2 - s)\| \mathrm{d}s \\
\le& \frac{L}{\Gamma(\alpha)} \sup_{s \in [0, t_1]} \{\|x(t_1 - s) - x(t_2 - s)\|\} \int_0^{t_1} s^{\alpha-1} \mathrm{d}s \\
=& \frac{L}{\Gamma(\alpha + 1)} \sup_{s \in [0, t_1]} \{\|x(t_1 - s) - x(t_2 - s)\|\} T^\alpha.
\end{aligned} \tag{3.7}$$

Note that $x(t) \in C([0,T],X)$, then

$$\sup_{s\in[0,t_1]} \{\|x(t_1-s)-x(t_2-s)\|\} \to 0 \text{ as } t_1 \to t_2,$$

which implies $\omega_1 \to 0$.

For ω_2, we have

$$\begin{aligned}\omega_2 \le & \frac{L}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1} \|F(t_1-s,x(t_2-s))-F(t_2-s,x(t_2-s))\| \mathrm{d}s \\ \le & \frac{L}{\Gamma(\alpha+1)} \sup_{s\in[0,t_1]} \{\|F(t_1-s,x(t_2-s))-F(t_2-s,x(t_2-s))\|\} T^{\alpha}.\end{aligned} \tag{3.8}$$

Note that $F(t,x)$ is continuous with respect to t on $[0,T]$. Then

$$\sup_{s\in[0,t_1]} \{\|F(t_1-s,x(t_2-s))-F(t_2-s,x(t_2-s))\|\} \to 0 \text{ as } t_1 \to t_2,$$

which implies $\omega_2 \to 0$.

For ω_3, we have

$$\omega_3 = \left\| \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t_1} s^{\alpha-1} F(t_2-s,x(t_2-s))\mathrm{d}s \right\| = \frac{1}{\Gamma(\alpha)} \theta^{\alpha-1} F(t_2-\theta,x(t_2-\theta)) \left|t_1-t_2\right|,$$

where $\min\{t_1,t_2\} < \theta < \max\{t_1,t_2\}$. When $t_1 \to t_2$, $\omega_3 \to 0$.

Then when $t_1 \to t_2$,

$$\|Gx(t_1)-Gx(t_2)\| \le \omega_1+\omega_2+\omega_3 \to 0,$$

which implies $Gx(t) \in C([0,T],X)$.

Secondly, we show that for $n \ge 1$,

$$\|G^n u - G^n v\|_\infty \le \frac{(Lt^\alpha)^n}{\Gamma(n\alpha+1)} \|u-v\|_\infty .$$

When $n=1$,

$$\begin{aligned}\|Gu-Gv\|_\infty = & \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s,u(s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s,v(s))\mathrm{d}s \right\|_\infty \\ \le & \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u-v\|_\infty \mathrm{d}s \le \frac{L\|u-v\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \\ = & \frac{L\|u-v\|_\infty}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} \le \frac{Lt^\alpha}{\Gamma(\alpha+1)} \|u-v\|_\infty .\end{aligned} \tag{3.9}$$

Assume that for $k \le n$, we already have

$$\left\|G^k u - G^k v\right\|_\infty \le \frac{(Lt^\alpha)^k}{\Gamma(k\alpha+1)} \|u-v\|_\infty .$$

Then for $k = n + 1$,

$$
\begin{aligned}
&\left\|G^{n+1}u - G^{n+1}v\right\|_\infty \\
=&\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, G^n u(s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, G^n v(s))\mathrm{d}s\right\|_\infty \\
\leq&\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|G^n u(s) - G^n v(s)\right\|_\infty \mathrm{d}s \leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{(Ls^\alpha)^n \|u-v\|_\infty}{\Gamma(n\alpha+1)}\mathrm{d}s \\
\leq&\frac{L^{n+1}\|u-v\|_\infty}{\Gamma(\alpha)\Gamma(n\alpha+1)}\int_0^t (t-s)^{\alpha-1}s^{n\alpha}ds = \frac{L^{n+1}\|u-v\|_\infty}{\Gamma(\alpha)\Gamma(n\alpha+1)}\int_0^1 (t-st)^{\alpha-1}(st)^{n\alpha}tds \\
=&\frac{t^{(n+1)\alpha}L^{n+1}\|u-v\|_\infty}{\Gamma(\alpha)\Gamma(n\alpha+1)}\int_0^1 (1-s)^{\alpha-1}s^{n\alpha}ds = \frac{t^{(n+1)\alpha}L^{n+1}\|u-v\|_\infty}{\Gamma(\alpha)\Gamma(n\alpha+1)}B(\alpha, n\alpha+1) \\
=&\frac{t^{(n+1)\alpha}L^{n+1}}{\Gamma((n+1)\alpha+1)}\|u-v\|_\infty,
\end{aligned}
$$

this implies for $n \geq 1$,

$$\|G^n u - G^n v\|_\infty \leq \frac{(Lt^\alpha)^n}{\Gamma(n\alpha+1)}\|u-v\|_\infty.$$

So when $t \in [0, T]$

$$\|G^n u - G^n v\|_\infty \leq \frac{(LT^\alpha)^n}{\Gamma(n\alpha+1)}\|u-v\|_\infty.$$

Note that

$$\lim_{n\to\infty}\frac{(Lt^\alpha)^n}{\Gamma(n\alpha+1)} = 0.$$

Then there exist $N \in \mathbb{Z}^+$, when $n > N$,

$$\|G^n u - G^n v\|_\infty \leq \frac{1}{2}\|u-v\|_\infty.$$

By Banach fixed point theorem, there exists a unique function $x(t) \in C([0, T], X)$ such that $Gx = x$, i.e., there exists a unique solution of (3.5) on the interval $[0, T]$.

From Lemma 3.3, we have the following results.

Lemma 3.4. *Let $A \in \mathbb{L}(X)$ be a closed operator and $F(t) \in C([0, T], X)$. Then (3.3) exists a unique solution $x(t)$, moreover $x(t) \in C([0, T], X)$.*

Proof. Set $\widetilde{F}(t, x) = Ax + F(t)$. Then $\widetilde{F}(t, x)$ is continuous with respect to t on $[0, T]$, and

$$\|\widetilde{F}(t, x) - \widetilde{F}(t, y)\| \leq \|A\|\,\|x-y\|, \quad \forall t \in [0, T], \quad x, y \in X.$$

By Lemma 3.3, there exists a unique solution $x(t) \in C([0, T], X)$ for

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\mathrm{d}s,\ 0 \leq t \leq T,$$

which implies (3.3) exists a unique solution $x(t) \in C([0, T], X)$.

Lemma 3.5. *Let $A \in \mathbb{L}(X)$ be a closed operator, $F(t,x)$ is continuous with respect to t on $[0,T]$ and Lipschitz continuous with respect to x, namely, there exists a positive constant L such that*

$$\|F(t,x) - F(t,y)\| \le L\|x-y\|, \quad \forall t \in [0,T], \quad x, y \in X.$$

Then (3.4) exists a unique solution $x(t)$, moreover $x(t) \in C([0,T],X)$.

Proof. Set $\widetilde{F}(t,x) = Ax + F(t,x)$. Then $\widetilde{F}(t,x)$ continuous with respect to t on $[0,T]$, and

$$\|\widetilde{F}(t,x) - \widetilde{F}(t,y)\| \le (\|A\| + L)\,\|x-y\|\,, \quad \forall t \in [0,T], \quad x, y \in X.$$

By Lemma 3.3, there exists a unique solution $x(t) \in C([0,T],X)$ for

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} Ax(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s,x(s))\mathrm{d}s, \ 0 \le t \le T,$$

which implies that (3.4) exists a unique solution $x(t)$, moreover $x(t) \in C([0,T],X)$.

In the following, we proof the existence and differentiability on time of solutions for (3.1).

Theorem 3.1. *Let $A \in \mathbb{L}(X)$ be a closed operator and $F(t) \in C^1([0,T],X)$. Then (3.1) exists a unique solution $x(t)$, moreover $x(t) \in C^1([0,T],X)$.*

Proof. Firstly, we show that (3.3) exists a unique solution $x(t) \in C^1([0,T],X)$.

By Lemma 3.4, integral equation (3.3) exists a unique solution $x(t) \in C([0,T],X)$, such that

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} Ax(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s)\mathrm{d}s.$$

Note that $x(t) \in C([0,T],X)$, which allow applying the Riemann-Liouville fractional derivative ${}^{RL}D_t^{\alpha}$ on both sides. Then we get

$${}^{RL}D_t^{\alpha}x(t) = x_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + Ax(t) + F(t),$$

which implies ${}^{RL}D_t^{\alpha}x(t) \in X$ exists for any $0 < \alpha < 1$. Then consider ${}^{RL}D_t^{1-\alpha}Ax(t)$. Since $A \in \mathbb{L}(X)$ and A is a closed operator, we get

$$\frac{d}{dt}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(s)\right) =^{RL} D_t^{1-\alpha}Ax(t) = A({}^{RL}D_t^{1-\alpha}x(t)) \in X.$$

On the other hand,

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\mathrm{d}s = I_t^{\alpha}F(t) = I_t^{\alpha}(I_t^1F'(t) - F(0)) = I_t^1 I_t^{\alpha}F'(t) - \frac{t^{\alpha}}{\Gamma(\alpha+1)}F(0).$$

Then we get

$$\frac{d}{dt}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\mathrm{d}s\right) = I_t^{\alpha}F'(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}F(0) \in X.$$

So

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{d}{dt}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(s)\right) + \frac{d}{dt}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\mathrm{d}s\right)\\ &= A({}^{RL}D_t^{1-\alpha}x(t)) + I_t^{\alpha}F'(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}F(0),\end{aligned}$$

which implies (3.3) exists a unique solution $x(t) \in C^1([0,T],X)$.

Since $A \in \mathbb{L}(X)$, $D(A) = X$. Then by Lemma 3.1, (3.1) also exists a unique solution $x(t) \in C^1([0,T],X)$.

In order to study (3.2), we need the following lemma.

Lemma 3.6. *Let $A \in \mathbb{L}(X)$ be a closed operator and $F(t,\lambda) : [0,T] \times \mathbb{R} \to X$. If $F(t,\lambda)$ is continuous with respect to t on $[0,T]$ and λ on $[a,b] \subset \mathbb{R}$. Then for any $\lambda \in [a,b]$ and $x_0 \in X$, there exists a unique solution $x(t,\lambda,x_0) \in C([0,T],X)$ for the following integral equations*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,\lambda)x(s)\,ds,\ 0 \le t \le T. \quad (3.10)$$

Moreover, $x(t,\lambda,x_0)$ is continuous with respect to $\lambda \in [a,b]$ and $x_0 \in X$.

Proof. Fixing $\lambda \in [a,b]$ and $x_0 \in X$, set $\widetilde{F}(t,x) = Ax + F(t,\lambda)x$. Then

$$\left\|\widetilde{F}(t,x_1) - \widetilde{F}(t,x_2)\right\| \le \left(\|A\| + \sup_{(t,\lambda)\in[0,T]\times[a,b]} \|F(t,\lambda)\|\right)\|x_1 - x_2\| = L\,\|x_1 - x_2\|,$$

where

$$L = \|A\| + \sup_{(t,\lambda)\in[0,T]\times[a,b]} \|F(t,\lambda)\|.$$

By Lemma 3.3, (3.10) exists a unique solution $x(t,\lambda,x_0) \in C([0,T],X)$. Then

$$\begin{aligned}\|x(t,\lambda,x_{0,1}) - x(t,\lambda)x_{0,2}\| =& \|x_{0,1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{F}(s,\lambda,x(s,\lambda,x_{0,1}))\mathrm{d}s\\ &- x_{0,2} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{F}(s,\lambda,x(s,\lambda,x_{0,2}))\mathrm{d}s\|\\ \le& \|x_{0,1} - x_{0,2}\| + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s,\lambda,x_{0,1}) - x(s,\lambda,x_{0,2})\|.\end{aligned}$$

By Lemma 2.4, we get

$$\|x(t,\lambda,x_{0,1}) - x(t,\lambda,x_{0,2})\| \le \|x_{0,1} - x_{0,2}\|\,E_\alpha(Lt^\alpha) \le \|x_{0,1} - x_{0,2}\|\,E_\alpha(LT^\alpha).$$

Thus

$$\|x(t,\lambda,x_{0,1}) - x(t,\lambda,x_{0,2})\| \to 0, \quad \text{as} \quad x_{0,1} \to x_{0,2},$$

which implies $x(t,\lambda,x_0)$ is continuous with respect to $x_0 \in X$.

On the other hand

$$
\begin{aligned}
&\|x(t,\lambda_1,x_0)-x(t,\lambda_2,x_0)\|\\
=&\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{F}(s,\lambda_1,x(s,\lambda_1,x_0))\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{F}(s,\lambda_2),x(s,\lambda_2,x_0))\mathrm{d}s\right\|\\
\le&\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,\lambda_1)x(s,\lambda_1,x_0)\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,\lambda_1)x(s,\lambda_2,x_0)\mathrm{d}s\right\|\\
&+\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,\lambda_1)x(s,\lambda_2,x_0)\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,\lambda_2)x(s,\lambda_2,x_0)\mathrm{d}s\right\|\\
&+\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(t,\lambda_1,x_0)\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(t,\lambda_2,x_0)\mathrm{d}s\right\|\\
\le&\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|x(t,\lambda_1,x_0)-x(t,\lambda_2,x_0)\right\|\mathrm{d}s+\frac{M_1M_2(\lambda_1-\lambda_2)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{d}s\\
=&\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|x(t,\lambda_1,x_0)-x(t,\lambda_2,x_0)\right\|\mathrm{d}s+\frac{M_1t^\alpha}{\alpha\Gamma(\alpha)}M_2(\lambda_1-\lambda_2),
\end{aligned}
$$

where

$$
M_1=\sup_{t\in[0,T]}x(t,\lambda_2,x_0),\ M_2=\sup_{t\in[0,T]}\left\|F(t,\lambda_1)-F(t,\lambda_2)\right\|.
$$

By Lemma 2.4, we get

$$
\|x(t,\lambda_1,x_0)-x(t,\lambda_2,x_0)\|\le\frac{M_1t^\alpha}{\alpha\Gamma(\alpha)}E_\alpha(L_1t^\alpha)M_2(\lambda_1-\lambda_2)\le\frac{M_1T^\alpha}{\alpha\Gamma(\alpha)}E_\alpha(L_1T^\alpha)M_2(\lambda_1-\lambda_2).
$$

Thus

$$
\|x(t,\lambda_1,x_0)-x(t,\lambda_2,x_0)\|\to 0,\quad \text{as}\quad \lambda_1\to\lambda_2,
$$

which implies $x(t,\lambda,x_0)$ is continuous with respect to $\lambda\in\mathbb{R}$.

In the following, we give and proof the existence and differentiability on time of solutions for (3.2).

Theorem 3.2. *Let $A\in\mathbb{L}(X)$ be a closed operator, and for the function $F(t,x)$, $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial x}$ exist, moreover $\frac{\partial F}{\partial t}\in C([0,T],X)$ and $\frac{\partial F}{\partial x}\in C([0,T],X)$. Then the equations (3.2) exist a unique solution $x(t)$, moreover $x(t)\in C^1([0,T],X)$.*

Proof. Similar to the proof of Theorem 3.1, we should prove (3.4) exists a unique solution $x(t)\in C^1([0,T],X)$.

Since $\frac{\partial F}{\partial t}$ exist, by Lemma 3.5, (3.4) exists a unique solution $x(t)\in C([0,T],X)$, such that

$$
x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,x(s))\mathrm{d}s.
$$

Rewrite $x(t)$, we get

$$
x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}Ax(t-s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}F(t-s,x(t-s))\mathrm{d}s.
$$

For $h \in \mathbb{R}\backslash\{0\}$,

$$
\begin{aligned}
&\frac{x(t+h)-x(t)}{h} \\
=&\frac{1}{\Gamma(\alpha)h}\int_0^{t+h} s^{\alpha-1}Ax(t+h-s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}Ax(t-s)\mathrm{d}s \\
&+\frac{1}{\Gamma(\alpha)h}\int_0^{t+h} s^{\alpha-1}F(t+h-s,x(t+h-s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t-s,x(t-s))\mathrm{d}s \\
:=&\omega_1+\omega_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\omega_1 =&\frac{1}{\Gamma(\alpha)h}\int_0^{t+h} s^{\alpha-1}Ax(t+h-s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}Ax(t-s)\mathrm{d}s \\
=&\frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}Ax(t+h-s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}Ax(t-s)\mathrm{d}s \\
&+\frac{1}{\Gamma(\alpha)h}\int_t^{t+h} s^{\alpha-1}Ax(t+h-s)\mathrm{d}s \\
=&\frac{1}{\Gamma(\alpha)}\int_0^{t} s^{\alpha-1}\frac{Ax(t+h-s)-Ax(t-s)}{h}\mathrm{d}s + \frac{1}{\Gamma(\alpha)h}\int_t^{t+h} s^{\alpha-1}Ax(t+h-s)\mathrm{d}s \\
=&\frac{1}{\Gamma(\alpha)}\int_0^{t} (t-s)^{\alpha-1}A\left(\frac{x(s+h)-x(s)}{h}\right)\mathrm{d}s + r_1(t,h),
\end{aligned}
$$

where

$$
r_1(t,h) = \frac{1}{\Gamma(\alpha)h}\int_t^{t+h} s^{\alpha-1}Ax(t+h-s)\mathrm{d}s,\ \ h\neq 0.
$$

Since

$$
\lim_{h\to 0} r_1(t,h) = \lim_{h\to 0}\frac{1}{\Gamma(\alpha)h}\int_t^{t+h} s^{\alpha-1}Ax(t+h-s)\mathrm{d}s = \frac{t^{\alpha-1}}{\Gamma(\alpha)}Ax(t).
$$

Define

$$
r_1(t,0) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}Ax(t),
$$

then $r_1(t,h)$ is continuous with respect to h.

For ω_2, we have

$$
\begin{aligned}
\omega_2 =& \frac{1}{\Gamma(\alpha)h}\int_0^{t+h} s^{\alpha-1}F(t+h-s,x(t+h-s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t-s,x(t-s))\mathrm{d}s\\
=& \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t+h-s,x(t+h-s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t-s,x(t-s))\mathrm{d}s\\
&+ \frac{1}{\Gamma(\alpha)h}\int_t^{t+h} s^{\alpha-1}F(t+h-s,x(t+h-s))\mathrm{d}s\\
=& \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t+h-s,x(t+h-s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t+h-s,x(t-s))\mathrm{d}s\\
&+ \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t+h-s,x(t-s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)h}\int_0^{t} s^{\alpha-1}F(t-s,x(t-s))\mathrm{d}s + r_2(t,h)\\
=& \frac{1}{\Gamma(\alpha)h}\int_0^{t} (t-s)^{\alpha-1}(F(s+h,x(s+h)) - F(s+h,x(s)))\mathrm{d}s\\
&+ \frac{1}{\Gamma(\alpha)h}\int_0^{t} (t-s)^{\alpha-1}(F(s+h,x(s)) - F(s,x(s)))\mathrm{d}s + r_2(t,h)\\
=& \frac{1}{\Gamma(\alpha)}\int_0^{t} (t-s)^{\alpha-1}[\frac{\partial F(s+h,x(s))}{\partial x} + r_3(s,h)]\frac{x(s+h)-x(s)}{h}\mathrm{d}s\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t} (t-s)^{\alpha-1}[\frac{\partial F(s,x(s))}{\partial t} + r_4(s,h)]\mathrm{d}s + r_2(t,h),
\end{aligned}
$$

where

$$
r_2(t,h) = \frac{1}{\Gamma(\alpha)h}\int_t^{t+h} s^{\alpha-1}F(t+h-s,x(t+h-s))\mathrm{d}s,\ \ h\neq 0,
$$

$$
r_3(s,h) = \begin{cases} \frac{F(s+h,x(s+h))-F(s+h,x(s))}{x(s+h)-x(s)} - \frac{\partial F(s+h,x(s))}{\partial x} & x(s+h)\neq x(s)\\ 0 & x(s+h)=x(s), \end{cases}
$$

$$
r_4(s,h) = \frac{F(s+h,x(s))-F(s,x(s))}{h} - \frac{\partial F(s,x(s))}{\partial t},\ \ h\neq 0.
$$

Then

$$
\lim_{h\to 0} r_3(s,h) = 0,\ \ \lim_{h\to 0} r_4(s,h) = 0,
$$

and

$$
\begin{aligned}
\lim_{h\to 0} r_2(t,h) =& \lim_{h\to 0}\frac{1}{\Gamma(\alpha)h}\int_t^{t+h} s^{\alpha-1}F(t+h-s,x(t+h-s))\mathrm{d}s\\
=& \lim_{h\to 0}\frac{1}{\Gamma(\alpha)h}\int_0^{h} (t+s)^{\alpha-1}F(h-s,x(h-s))\mathrm{d}s\\
=& \lim_{h\to 0}\frac{1}{\Gamma(\alpha)h}\int_0^{h} ((t+s)^{\alpha-1}F(h-s,x(h-s)) - F(0,x(0)))\mathrm{d}s\\
&+ \lim_{h\to 0}\frac{1}{\Gamma(\alpha)h}\int_0^{h} (t+s)^{\alpha-1}F(0,x(0))\mathrm{d}s.
\end{aligned}
$$

Moreover

$$
\lim_{h\to 0}\frac{1}{\Gamma(\alpha)h}\int_0^{h} (t+s)^{\alpha-1}F(0,x(0))\mathrm{d}s = \lim_{h\to 0}\frac{F(0,x(0))}{\Gamma(\alpha)h}(\frac{1}{\alpha}(t+h)^{\alpha} - \frac{1}{\alpha}t^{\alpha}) = \frac{t^{\alpha-1}F(0,x(0))}{\Gamma(\alpha)},
$$

and

$$\begin{aligned}&\left\|\frac{1}{\Gamma(\alpha)h}\int_0^h((t+s)^{\alpha-1}F(h-s,x(h-s))-F(0,x(0)))\mathrm{d}s\right\|\\ \le&\frac{1}{\Gamma(\alpha)h}\int_0^h(t+s)^{\alpha-1}\left\|F(h-s,x(h-s))-F(0,x(0))\right\|\mathrm{d}s\\ \le&\sup_{l\in[0,h]}\left\|F(l,x(l))-F(0,x(0))\right\|\frac{1}{\Gamma(\alpha)h}\int_0^h(t+s)^{\alpha-1}\mathrm{d}s.\end{aligned}$$

When $h\to 0$,

$$\frac{1}{\Gamma(\alpha)h}\int_0^h(t+s)^{\alpha-1}\mathrm{d}s\to\frac{t^{\alpha-1}}{\Gamma(\alpha)},\quad\sup_{l\in[0,h]}\left\|F(l,x(l))-F(0,x(0))\right\|\to 0.$$

Then

$$\lim_{h\to 0}\frac{1}{\Gamma(\alpha)h}\int_0^h((t+s)^{\alpha-1}F(h-s,x(h-s))-F(0,x(0)))\mathrm{d}s=0.$$

We get

$$\lim_{h\to 0}r_2(t,h)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}F(0,x_0).$$

Define

$$r_2(t,0)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}F(0,x_0),\ \ r_3(s,0)=r_4(s,0)=0.$$

Then $r_2(t,h)$, $r_3(s,h)$ and $r_4(s,h)$ are continuous with respect to h.

Thus

$$\begin{aligned}&\frac{x(t+h)-x(t)}{h}\\ =&r_1(t,h)+r_2(t,h)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}A\left(\frac{x(s+h)-x(s)}{h}\right)\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[\frac{\partial F(s+h,x(s))}{\partial x}+r_3(s,h)]\frac{x(s+h)-x(s)}{h}\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[\frac{\partial F(s,x(s))}{\partial x}+r_4(s,h)]\mathrm{d}s.\end{aligned}\tag{3.11}$$

Define

$$\psi(t,h)=\frac{x(t+h)-x(t)}{h}.$$

For fix $t_0\in[0,T]$, set

$$\widetilde{x_0}(t_0,h)=r_1(t_0,h)+r_2(t_0,h)+\frac{1}{\Gamma(\alpha)}\int_0^{t_0}(t_0-s)^{\alpha-1}\left[\frac{\partial F(s,x(s))}{\partial x}+r_4(s,h)\right]\mathrm{d}s\in X,$$

$$\widetilde{F}(t,h)=\Big(\frac{\partial F(t+h,x(t))}{\partial x}+r_3(s,h)\Big).$$

By (3.11), we get

$$\begin{aligned}\psi(t_0,h)=&\widetilde{x_0}(t_0,h)+\frac{1}{\Gamma(\alpha)}\int_0^{t_0}(t_0-s)^{\alpha-1}A\psi(s,h)\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{t_0}(t_0-s)^{\alpha-1}\widetilde{F}(s,h)\psi(s,h)\mathrm{d}s.\end{aligned}\tag{3.12}$$

By Lemma 3.6, there exists a unique solution $v(t,h,\widetilde{x}_0(t_0,h)) \in C([0,T],X)$ for integral equations

$$
\begin{aligned}
v(t) = & \widetilde{x_0}(t_0,h) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Av(s)\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{F}(s,h)v(s)\mathrm{d}s, \ 0 \le t \le T.
\end{aligned} \tag{3.13}
$$

Moreover, $v(t,h,\widetilde{x}_0(t_0,h))$ is continuous with respect to h and $\widetilde{x}_0 \in X$.

Combining (3.12) and (3.13), we get

$$\psi(t_0,h) = v(t_0,h,\widetilde{x_0}(t_0,h)).$$

Then we get

$$x'(t_0) = \lim_{h\to 0} \psi(t_0,h) = \lim_{h\to 0} v(t_0,h,\widetilde{x_0}(t_0,h)) = v(t_0,0,\widetilde{x_0}(t_0,0)).$$

Since $A \in \mathbb{L}(X)$, $D(A) = X$. Combining Lemma 3.2, (3.4) exists a unique solution $x(t)$, moreover $x(t) \in C^1([0,T],X)$.

4 Continuity of solution on fractional order for linear equations

In this section, we consider the continuity of solutions on fractional order for linear Caputo fractional evolution equations, i.e., as $\alpha \to \beta$, the relationship of solutions between

$$
\begin{cases}
{}^C D_t^{\alpha} x(t) = Ax(t) + F(t), \ 0 < t \le T, \\
x(0) = x_0,
\end{cases} \tag{4.1}
$$

and

$$
\begin{cases}
{}^C D_t^{\beta} x(t) = Ax(t) + F(t), \ 0 < t \le T, \\
x(0) = x_0.
\end{cases} \tag{4.2}
$$

Before giving the main results of this section, we give and proof a lemma, which plays a key role in the proofs of our main results of this section and the next section.

Lemma 4.1. *Let $\alpha > 0$, $\beta > 0$ and $0 \le t \le T$ for some $T \in (0,+\infty)$. When $\alpha \to \beta$,*

$$\varepsilon(\alpha,\beta) := \int_0^t \left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} \right] ds \to 0.$$

Proof. As $\alpha > 0$, $\beta > 0$, then

$$
\begin{aligned}
\int_0^t \left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} \right] \mathrm{d}s = & \left(\frac{1}{\Gamma(\alpha)} \frac{1}{\alpha}(t-s)^{\alpha} - \frac{1}{\Gamma(\beta)} \frac{1}{\beta}(t-s)^{\beta} \right) \Big|_{s=0}^{s=t} \\
= & \frac{1}{\Gamma(\alpha+1)} t^{\alpha} - \frac{1}{\Gamma(\beta+1)} t^{\beta}.
\end{aligned}
$$

When $\alpha \to \beta$,

$$\frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \frac{1}{\Gamma(\beta+1)}t^{\beta} \to 0,$$

which, together with $0 \le t \le T$, implies

$$\int_0^t \left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right] \mathrm{d}s \to 0.$$

In the following, we state and proof our main results.

Theorem 4.1. *Let $A \in \mathbb{L}(X)$ be a closed operator, $F(t) \in C^1([0,T],X)$ satisfying*

$$\|F(t)\| \le C_2, \forall t \in [0,T].$$

Then the solution of (4.1) converges to the solution of (4.2) as $\alpha \to \beta$ for $0 < \alpha, \beta < 1$.

Proof. From Theorem 3.1, it follows that (4.1) as well as (4.2) has a unique solution, and the solutions have the forms

$$x_\alpha(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax_\alpha(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\mathrm{d}s, \ 0 \le t \le T,$$

and

$$x_\beta(t) = x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\alpha-1}Ax_\beta(s)\mathrm{d}s + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F(s)\mathrm{d}s, \ 0 \le t \le T.$$

As $A \in \mathbb{L}(X)$ being a closed operator, then there exists a constant C_1 such that

$$\|Ax\| \le C_1, \forall x \in C^1([0,T],X).$$

Thus

$$\begin{aligned}
&\|x_\alpha(t) - x_\beta(t)\| \\
\le & \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax_\alpha(s)\mathrm{d}s - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}Ax_\beta(s)\mathrm{d}s\right\| \\
&+ \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\mathrm{d}s - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F(s)\mathrm{d}s\right\| \\
\le & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Ax_\alpha(s) - Ax_\beta(s)\|\mathrm{d}s + \int_0^t \left\|\left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right]Ax_\beta(s)\right\|\mathrm{d}s \\
&+ \int_0^t \left\|\left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right]F(s)\right\|\mathrm{d}s \\
\le & \frac{\|A\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x_\alpha(s) - x_\beta(s)\|\mathrm{d}s + \int_0^t \left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right]\|Ax_\beta(s)\|\mathrm{d}s \\
&+ \int_0^t \left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right]\|F(s)\|\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
\leq&\frac{\|A\|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x_\alpha(s)-x_\beta(s)\|\mathrm{d}s+C_1\int_0^t\left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}-\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right]\mathrm{d}s\\
&+C_2\int_0^t\left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}-\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right]\mathrm{d}s\\
=&\frac{\|A\|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x_\alpha(s)-x_\beta(s)\|\mathrm{d}s+[C_1+C_2]\int_0^t\left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}-\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}\right]\mathrm{d}s\\
:=&\frac{\|A\|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x_\alpha(s)-x_\beta(s)\|\mathrm{d}s+\varepsilon_1(\alpha,\beta).
\end{aligned}
$$

From Lemma 4.1, it follows that

$$\lim_{\alpha\to\beta}\varepsilon_1(\alpha,\beta)=0.$$

Then, for any $\varepsilon>0$, there exists $\delta>0$ such that for $0<|\beta-\alpha|<\delta$, we have $0<\varepsilon_1(\alpha,\beta)<\varepsilon$, which imply

$$\|x_\alpha(t)-x_\beta(t)\|\leq\frac{\|A\|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x_\alpha(s)-x_\beta(s)\|\mathrm{d}s+\varepsilon,$$

as $0<|\beta-\alpha|<\delta$.

From Lemma 2.3, it follows that

$$\|x_\alpha(t)-x_\beta(t)\|\leq\varepsilon+\varepsilon\int_0^t\left[\sum_{n=1}^{\infty}\frac{\|A\|^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\right]\mathrm{d}s.$$

Noted that

$$
\begin{aligned}
\int_0^t\left[\sum_{n=1}^{\infty}\frac{\|A\|^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\right]\mathrm{d}s\leq\sum_{n=1}^{\infty}\frac{\|A\|^n}{\Gamma(n\alpha)}\int_0^t(t-s)^{n\alpha-1}\mathrm{d}s&=\sum_{n=1}^{\infty}\frac{\|A\|^n}{\Gamma(n\alpha)}\frac{t^{n\alpha}}{n\alpha}\\
&=\sum_{n=1}^{\infty}\frac{(\|A\|t^\alpha)^n}{\Gamma(n\alpha+1)}\leq E_\alpha(\|A\|t^\alpha)<\infty,
\end{aligned}
$$

which implies

$$\|x_\alpha(t)-x_\beta(t)\|\to 0,$$

as $\varepsilon\to 0$ $(\alpha\to\beta)$. This completes the proof.

Moreover, we find a relationship of solutions between the linear Caputo fractional evolution equations and the classic linear evolution equations, i.e., as $\alpha\to 1^-$, the relationship of solutions between

$$
\begin{cases}
{}^C D_t^\alpha x(t)=Ax(t)+F(t),\ 0<t\leq T,\\
x(0)=x_0,
\end{cases}
\tag{4.3}
$$

and

$$
\begin{cases}
x'(t)=Ax(t)+F(t),\ 0<t\leq T,\\
x(0)=x_0.
\end{cases}
\tag{4.4}
$$

Firstly, we give some Lemmas.

Consider the following first order differential equations

$$\begin{cases} x'(t) = F(t, x(t)), \ 0 < t \leq T, \\ x(0) = x_0. \end{cases} \tag{4.5}$$

Lemma 4.2. *([28]) $x(t) \in C^1([0,T], X)$ is a solution of (4.5) if and only if $x(t) \in C([0,T], X)$ is a solution of the following nonlinear Volterra integral equations*

$$x(t) = x_0 + \int_0^t F(s, x(s))\mathrm{d}s, \ 0 \leq t \leq T. \tag{4.6}$$

Lemma 4.3. *([28]) If $F(t,x)$ is continuous with respect to t on $[0,T]$ and Lipschitz continuous with respect to x, namely, there exists a positive constant L such that*

$$\|F(t,x) - F(t,y)\| \leq L\|x - y\|, \quad \forall t \in [0,T], \quad x, y \in X.$$

Then (4.6) has a unique solution $x(t)$, moreover $x(t) \in C([0,T], X)$.

Lemma 4.4. *Let $A \in \mathbb{L}(X)$ be a closed operator and $F(t) \in C([0,T], X)$. Then (4.4) is equivalent to the following integral equations*

$$x(t) = x_0 + \int_0^t Ax(s)\mathrm{d}s + \int_0^t F(s)\mathrm{d}s, \ 0 \leq t \leq T. \tag{4.7}$$

Moreover (4.4) exists a unique solution $x(t)$, moreover $x(t) \in C^1([0,T], X)$.

Proof. Let $\widetilde{F}(t,x) = Ax + f(t)$. Since $A \in \mathbb{L}(X)$, thus

$$\|\widetilde{F}(t,x) - \widetilde{F}(t,y)\| \leq \|A\|\|x - y\|, \quad \forall t \in [0,T], \quad x, y \in X.$$

Note that A is a closed operator and $F(t) \in C([0,T], X)$, then $\widetilde{F}(t,x)$ is continuous with respect to t on $[0,T]$. By lemma 4.3, (4.7) has a unique solution $x(t) \in C([0,T], X)$. Combining Lemma 4.2, (4.4) is equivalent to (4.7). Moreover (4.4) exists a unique solution $x(t)$, moreover $x(t) \in C^1([0,T], X)$.

We have the following result.

Theorem 4.2. *Let $A \in \mathbb{L}(X)$ be a closed operator, $F(t) \in C^1([0,T], X)$ satisfying*

$$\|F(t)\| \leq C_4, \forall t \in [0,T].$$

Then the solution of (4.3) converges to the solution of (4.4) as $\alpha \to 1^-$.

Proof. From Theorem 3.1 and Lemma 4.4, it follows that (4.3) as well as (4.4) has a unique solution, and the solutions have the forms

$$x_\alpha(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax_\alpha(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\mathrm{d}s, \ 0 \leq t \leq T,$$

and

$$x_1(t) = x_0 + \int_0^t Ax_1(s)\mathrm{d}s + \int_0^t F(s)\mathrm{d}s,\ 0 \le t \le T.$$

As $A \in \mathbb{L}(X)$ being a closed operator, then there exists a constant C_3 such that

$$\|Ax\| \le C_3, \forall x \in C^1([0,T], X).$$

Thus

$$\begin{aligned}
&\|x_\alpha(t) - x_1(t)\| \\
\le & \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax_\alpha(s)\mathrm{d}s - \int_0^t Ax_1(s)\mathrm{d}s \right\| \\
&+ \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s)\mathrm{d}s - \int_0^t F(s)\mathrm{d}s \right\| \\
\le & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|Ax_\alpha(s) - Ax_1(s)\|\mathrm{d}s + \int_0^t \left\| \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] Ax_1(s) \right\| \mathrm{d}s \\
&+ \int_0^t \left\| \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] F(s) \right\| \mathrm{d}s \\
\le & \frac{\|A\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\|\mathrm{d}s + \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \|Ax_1(s)\|\mathrm{d}s \\
&+ \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \|F(s)\|\mathrm{d}s \\
\le & \frac{\|A\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\|\mathrm{d}s + C_3 \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \mathrm{d}s \\
&+ C_4 \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \mathrm{d}s \\
= & \frac{\|A\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\|\mathrm{d}s + [C_3 + C_4] \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \mathrm{d}s \\
:= & \frac{\|A\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\|\mathrm{d}s + \varepsilon_2(\alpha).
\end{aligned}$$

Noted that

$$\varepsilon_2(\alpha) = [C_3+C_4] \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \mathrm{d}s = [C_3+C_4] \left[\frac{1}{\Gamma(\alpha)} \frac{t^a}{a} - t \right] = [C_3+C_4] \left[\frac{t^a}{\Gamma(\alpha+1)} - t \right].$$

So

$$\lim_{\alpha \to 1^-} \varepsilon_2(\alpha) = \lim_{\alpha \to 1^-} [C_3 + C_4] \left(\frac{t^a}{\Gamma(\alpha+1)} - t \right) = t - t = 0.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < 1 - \alpha < \delta$, we have $0 < \varepsilon_2(\alpha) < \varepsilon$, which imply

$$\|x_\alpha(t) - x_1(t)\| \le \frac{\|A\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\|\mathrm{d}s + \varepsilon,$$

as $0 < 1 - \alpha < \delta$.

From Lemma 2.3, it follows that

$$\|x_\alpha(t) - x_1(t)\| \le \varepsilon + \varepsilon \int_0^t \left[\sum_{n=1}^{\infty} \frac{\|A\|^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] \mathrm{d}s.$$

Noted that

$$\begin{aligned} \int_0^t \left[\sum_{n=1}^{\infty} \frac{\|A\|^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] \mathrm{d}s &\le \sum_{n=1}^{\infty} \frac{\|A\|^n}{\Gamma(n\alpha)} \int_0^t (t-s)^{n\alpha-1} \mathrm{d}s = \sum_{n=1}^{\infty} \frac{\|A\|^n}{\Gamma(n\alpha)} \frac{t^{n\alpha}}{n\alpha} \\ &= \sum_{n=1}^{\infty} \frac{(\|A\| t^\alpha)^n}{\Gamma(n\alpha+1)} \le E_\alpha(\|A\| t^\alpha) < \infty, \end{aligned}$$

which implies

$$\|x_\alpha(t) - x_1(t)\| \to 0,$$

as $\varepsilon \to 0$ $(\alpha \to 1^-)$. This completes the proof.

5 Continuity of solution on fractional order for semilinear equations

In this section, we consider the continuity of solution on fractional order for semilinear Caputo fractional evolution equations, i.e., as $\alpha \to \beta$, the relationship of solutions between

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + F(t, x(t)), \ 0 < t \le T, \\ x(0) = x_0, \end{cases} \tag{5.1}$$

and

$$\begin{cases} {}^C D_t^\beta x(t) = Ax(t) + F(t, x(t)), \ 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{5.2}$$

Throughout this section, we require the following assumptions.

(H_1) There exists a constants C_6 such that

$$\|F(t, x)\| \le C_6, \text{ for all } t \in [0, T] \ x \in X.$$

(H_2) $F(t, x)$ is continuous with respect to t on $[0, T]$ and Lipschitz continuous with respect to x, namely, there exists a positive constant L such that

$$\|F(t, x) - F(t, y)\| \le L\|x - y\|, \quad \forall t \in [0, T], \quad x, y \in X.$$

We have the following result.

Theorem 5.1. *Let the assumptions of Theorem 3.2 be satisfied and the conditions H_1 and H_2 hold. Then the solution of (5.1) converges to the solution of (5.2) as $\alpha \to \beta$ for $0 < \alpha, \beta < 1$.*

Proof. From Theorem 3.2, it follows that (5.1) as well as (5.2) has a unique solution, and the solutions have the forms

$$x_\alpha(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A x_\alpha(s) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x_\alpha(s)) \mathrm{d}s, \ 0 \le t \le T,$$

and

$$x_\beta(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} A x_\beta(s) \mathrm{d}s + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F(s, x_\beta(s)) \mathrm{d}s, \ 0 \le t \le T.$$

As $A \in \mathbb{L}(X)$ being a closed operator, then there exists a constant C_5 such that

$$\|Ax\| \le C_5, \forall x \in C^1([0,T], X).$$

Thus

$$\begin{aligned}
&\|x_\alpha(t) - x_\beta(t)\| \\
\le & \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A x_\alpha(s) \mathrm{d}s - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A x_\beta(s) \mathrm{d}s \right\| \\
&+ \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x_\alpha(s)) \mathrm{d}s - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F(s, x_\beta(s)) \mathrm{d}s \right\| \\
\le & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|A x_\alpha(s) - A x_\beta(s)\| \mathrm{d}s + \int_0^t \left\| \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \right] A x_\beta(s) \right\| \mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|F(s, x_\alpha(s)) - F(s, x_\beta(s))\| \mathrm{d}s \\
&+ \int_0^t \left\| \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \right] F(s, x_\beta(s)) \right\| \mathrm{d}s \\
\le & \frac{\|A\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_\beta(s)\| \mathrm{d}s + \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \right] \|A x_\beta(s)\| \mathrm{d}s \\
&+ \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_\beta(s)\| \mathrm{d}s + \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \right] \|F(s, x_\beta(s))\| \mathrm{d}s \\
\le & \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_\beta(s)\| \mathrm{d}s + C_5 \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \right] \mathrm{d}s \\
&+ C_6 \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \right] \mathrm{d}s \\
= & \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_\beta(s)\| \mathrm{d}s + [C_5 + C_6] \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \right] \mathrm{d}s \\
:= & \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_\beta(s)\| \mathrm{d}s + \varepsilon_3(\alpha, \beta).
\end{aligned}$$

From Lemma 2.4, it follows that

$$\lim_{\alpha \to \beta} \varepsilon_3(\alpha, \beta) = 0.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < |\beta - \alpha| < \delta$, we have $0 < \varepsilon_3(\alpha, \beta) < \varepsilon$, which imply

$$\|x_\alpha(t) - x_\beta(t)\| \le \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_\beta(s)\| \mathrm{d}s + \varepsilon,$$

as $0 < |\beta - \alpha| < \delta$.

From Lemma 2.3, it follows that

$$\|x_\alpha(t) - x_\beta(t)\| \le \varepsilon + \varepsilon \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha - 1} \right] \mathrm{d}s.$$

Noted that

$$\begin{aligned} \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha - 1} \right] \mathrm{d}s &\le \sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} \int_0^t (t-s)^{n\alpha-1} \mathrm{d}s = \sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} \frac{t^{n\alpha}}{n\alpha} \\ &= \sum_{n=1}^{\infty} \frac{((\|A\| + L)t^\alpha)^n}{\Gamma(n\alpha + 1)} \le E_\alpha(\|A\| + L)t^\alpha) < \infty, \end{aligned}$$

which implies

$$\|x_\alpha(t) - x_\beta(t)\| \to 0,$$

as $\varepsilon \to 0$ ($\alpha \to \beta$). This completes the proof.

Moreover, we find a relationship of solutions between the semilinear Caputo fractional evolution equations and the classic semilinear evolution equations, i.e., as $\alpha \to 1^-$, the relationship of solutions between

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + F(t, x(t)), \ 0 < t \le T, \\ x(0) = x_0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} x'(t) = Ax(t) + F(t, x(t)), \ 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{5.4}$$

Firstly, we give a Lemma.

Lemma 5.1. *Let $A \in \mathbb{L}(X)$ be a closed operator. If $F(t,x)$ is continuous with respect to t on $[0,T]$ and Lipschitz continuous with respect to x, namely, there exists a positive constant L such that*

$$\|F(t,x) - F(t,y)\| \le L\|x - y\|, \quad \forall t \in [0,T], \quad x, y \in X.$$

Then (5.4) is equivalent to the following integral equations

$$x(t) = x_0 + \int_0^t Ax(s)\mathrm{d}s + \int_0^t F(s, x(s))\mathrm{d}s, \ 0 \le t \le T. \tag{5.5}$$

Moreover (5.4) exists a unique solution $x(t)$ and $x(t) \in C^1([0,T], X)$.

Proof. Let $\widetilde{F}(t,x) = Ax + f(t,x)$. Since $A \in \mathbb{L}(X)$,

$$\|\widetilde{F}(t,x) - \widetilde{F}(t,y)\| \le (\|A\| + L)\|x - y\|, \quad \forall t \in [0,T], \quad x, y \in X.$$

Note that A is a closed operator and $F(t,x) \in C([0,T],X)$, then $\widetilde{F}(t,x)$ is continuous with respect to t on $[0,T]$. By Lemma 4.3, (5.5) has a unique solution $x(t) \in C([0,T],X)$. Combining Lemma 4.2, (5.4) is equivalent to (5.5). Moreover (5.4) exists a unique solution $x(t)$ and $x(t) \in C^1([0,T],X)$.

We have the following result.

Theorem 5.2. *Let the assumptions of Theorem 3.2 be satisfied and the conditions H_1 and H_2 hold. Then the solution of (5.3) converges to the solution of (5.4) as $\alpha \to 1^-$.*

Proof. From Theorem 3.2 and Lemma 5.1, it follows that (5.3) as well as (5.4) has a unique solution, and the solutions have the forms

$$x_\alpha(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A x_\alpha(s) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x_\alpha(s)) \mathrm{d}s, \ 0 \le t \le T,$$

and

$$x_1(t) = x_0 + \int_0^t A x_1(s) \mathrm{d}s + \int_0^t F(s, x_1(s)) \mathrm{d}s, \ 0 \le t \le T.$$

As $A \in \mathbb{L}(X)$ being a closed operator, then there exists a constant C_7 such that

$$\|Ax\| \le C_7, \forall x \in C^1([0,T],X).$$

Thus

$$\begin{aligned}
&\|x_\alpha(t) - x_1(t)\| \\
\le& \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A x_\alpha(s) \mathrm{d}s - \int_0^t A x_1(s) \mathrm{d}s \right\| \\
&+ \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x_\alpha(s)) \mathrm{d}s - \int_0^t F(s, x_1(s)) \mathrm{d}s \right\| \\
\le& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|A x_\alpha(s) - A x_1(s)\| \mathrm{d}s + \int_0^t \left\| \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] A x_1(s) \right\| \mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|F(s, x_\alpha(s)) - F(s, x_1(s))\| \mathrm{d}s + \int_0^t \left\| \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] F(s, x_1(s)) \right\| \mathrm{d}s \\
\le& \frac{\|A\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\| \mathrm{d}s + \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \|A x_1(s)\| \mathrm{d}s \\
&+ \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\| \mathrm{d}s + \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \|F(s, x_1(s))\| \mathrm{d}s \\
\le& \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\| \mathrm{d}s + C_7 \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \mathrm{d}s \\
&+ C_6 \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \mathrm{d}s \\
=& \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\| \mathrm{d}s + [C_7 + C_6] \int_0^t \left[\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - 1 \right] \mathrm{d}s \\
:=& \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\| \mathrm{d}s + \varepsilon_4(\alpha).
\end{aligned}$$

Noted that

$$\varepsilon_4(\alpha) = [C_7{+}C_6] \int_0^t \left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - 1\right] \mathrm{d}s = [C_7{+}C_6] \left[\frac{1}{\Gamma(\alpha)} \frac{t^a}{a} - t\right] = [C_7{+}C_6] \left[\frac{t^a}{\Gamma(\alpha+1)} - t\right].$$

So

$$\lim_{\alpha \to 1^-} \varepsilon_4(\alpha) = \lim_{\alpha \to 1^-} [C_7 + C_6] \left(\frac{t^a}{\Gamma(\alpha+1)} - t\right) = t - t = 0.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < 1 - \alpha < \delta$, we have $0 < \varepsilon_4(\alpha) < \varepsilon$, which imply

$$\|x_\alpha(t) - x_1(t)\| \leq \frac{\|A\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_\alpha(s) - x_1(s)\| \mathrm{d}s + \varepsilon,$$

as $0 < 1 - \alpha < \delta$.

From Lemma 2.3, it follows that

$$\|x_\alpha(t) - x_1(t)\| \leq \varepsilon + \varepsilon \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1}\right] \mathrm{d}s.$$

Noted that

$$\begin{aligned}\int_0^t \left[\sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1}\right] \mathrm{d}s &\leq \sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} \int_0^t (t-s)^{n\alpha-1} \mathrm{d}s = \sum_{n=1}^{\infty} \frac{(\|A\| + L)^n}{\Gamma(n\alpha)} \frac{t^{n\alpha}}{n\alpha} \\ &= \sum_{n=1}^{\infty} \frac{((\|A\| + L)t^\alpha)^n}{\Gamma(n\alpha+1)} \leq E_\alpha(\|A\| + L)t^\alpha) < \infty,\end{aligned}$$

which implies

$$\|x_\alpha(t) - x_1(t)\| \to 0,$$

as $\varepsilon \to 0$ $(\alpha \to 1^-)$. This completes the proof.

6 Examples and numerical investigations

In this section, the numerical studies are performed to explore the continuity of solutions on fractional order for some fractional systems, which show reasonable agreement with the theoretic results.

6.1 Examples and numerical investigations for linear equations

Case $\alpha \to \beta$

Setting $A = -1$, $x_0 = 10$, $F(t) = \sin(t)$ and $t \in [0, 5]$ in (3.1), we have

$$\begin{cases} {}_cD_t^\alpha x(t) = -x(t) + \sin(t), \ t \in (0, 5], \\ x(0) = 10. \end{cases} \tag{6.1}$$

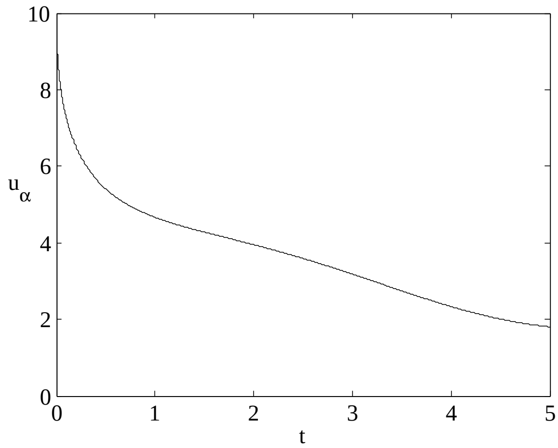
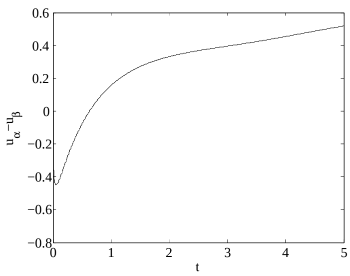
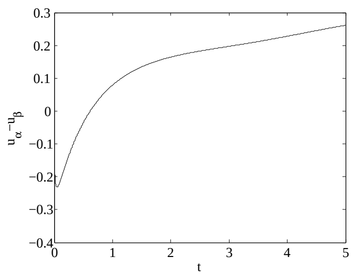


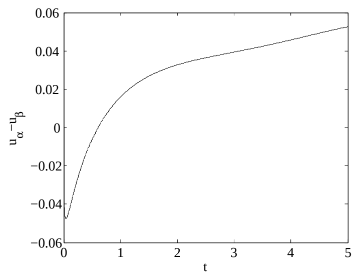
Figure 1: Numerical solution of (6.1) when $\alpha = 0.5$.



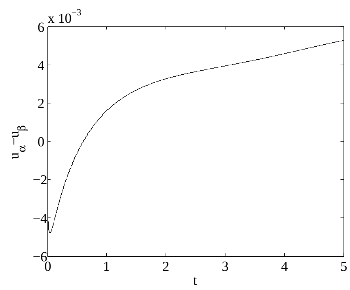
(a) $\alpha = 0.5,\ \beta = 0.6$



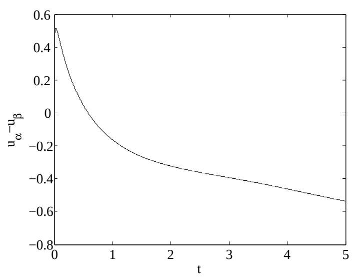
(b) $\alpha = 0.5,\ \beta = 0.55$



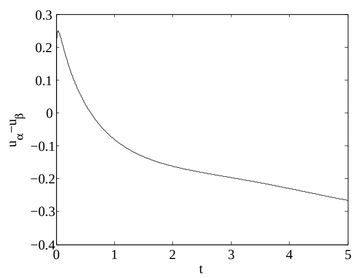
(c) $\alpha = 0.5,\ \beta = 0.51$



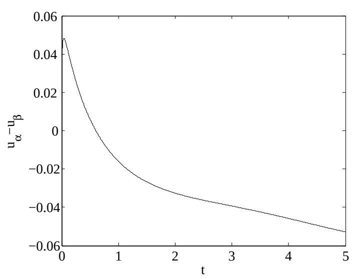
(d) $\alpha = 0.5,\ \beta = 0.501$



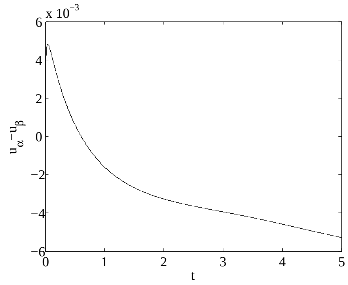
(e) $\alpha = 0.5,\ \beta = 0.4$



(f) $\alpha = 0.5,\ \beta = 0.45$



(g) $\alpha = 0.5,\ \beta = 0.49$



(h) $\alpha = 0.5,\ \beta = 0.499$

Figure 2: Difference of numerical solutions for (6.1).

t	α=0.5	α=0.6	α=0.55	α=0.51	α=0.501	α=0.4	α=0.45	α=0.49	α=0.499
0	10	10	10	10	10	10	10	10	10
0.5	5.417313	5.494595	5.451875	5.423592	5.417927	5.371333	5.390542	5.411345	5.416702
1	4.671882	4.51666	4.593493	4.656078	4.670299	4.834008	4.751937	4.687752	4.673466
1.5	4.280918	4.010343	4.146668	4.254204	4.278249	4.546053	4.413766	4.307575	4.283586
2	3.954464	3.62294	3.790219	3.921817	3.951203	4.277471	4.116548	3.987023	3.957723
2.5	3.5957	3.227966	3.413084	3.559339	3.592067	3.956876	3.776645	3.631994	3.599333
3	3.185281	2.78988	2.988217	3.14594	3.181348	3.578451	3.381749	3.224599	3.189214
3.5	2.749259	2.326192	2.537626	2.706896	2.745022	3.175245	2.961574	2.791649	2.753497
4	2.338456	1.884632	2.110709	2.292762	2.333884	2.800365	2.56817	2.384229	2.34303
4.5	2.009223	1.522525	1.764344	1.96	2.004295	2.508894	2.257285	2.058573	2.014152
5	1.805929	1.287856	1.544702	1.753339	1.800662	2.341628	2.071499	1.858693	1.811198

Table 1: Numerical solutions of (6.1) for different fractional order α.

Applying the fractional Adams-Bashforth-Moulton scheme in [29], the numerical solution of (6.1) is obtained. We can compare the numerical solutions when $\alpha \to \beta$ as bellow.

Table 1 and Figure 2 show that when α getting closer to 0.5, the numerical solution of (6.1) is getting closer to the numerical solution of (6.1) whose fractional order is 0.5 .

Case $\alpha \to 1^-$

Setting $A = -1$, $x_0 = 10$, $F(t) = \sin(t)$ and $t \in [0, 5]$ in (4.4), we have

$$\begin{cases} x'(t) = -x(t) + \sin(t), \ t \in (0, 5], \\ x(0) = 10. \end{cases} \tag{6.2}$$

Applying the Runge-Kutta method, the numerical solution of (6.2) is obtained. We can compare the numerical solutions of (4.3) and (4.4) when $\alpha \to 1^-$ as bellow.

t	u_1	α=0.9	α=0.95	α=0.99	α=0.999
0	10	10	10	10	10
0.5	6.169363	5.943919	6.05221	6.145155	6.166798
1	4.013279	4.109099	4.057428	4.021494	4.014145
1.5	2.806249	3.127074	2.968055	2.838886	2.809622
2	2.083763	2.512635	2.302783	2.128405	2.088336
2.5	1.561724	2.021764	1.796678	1.609617	1.566592
3	1.088342	1.54196	1.318514	1.134995	1.093052
3.5	0.609925	1.046372	0.829126	0.653957	0.614348
4	0.140743	0.563295	0.350579	0.182476	0.144926
4.5	-0.266721	0.149002	-0.062356	-0.226435	-0.262681
5	-0.55054	-0.137653	-0.349197	-0.511127	-0.546586

Table 2: Numerical solutions of (6.2) and (6.1) for different fractional order α.

Table 2 and Figure 4 show that when α getting closer to 1, the numerical solution of (6.1) is getting closer to the numerical solution of (6.2).

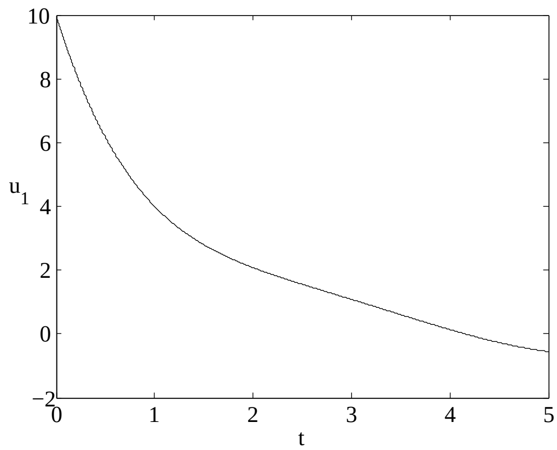
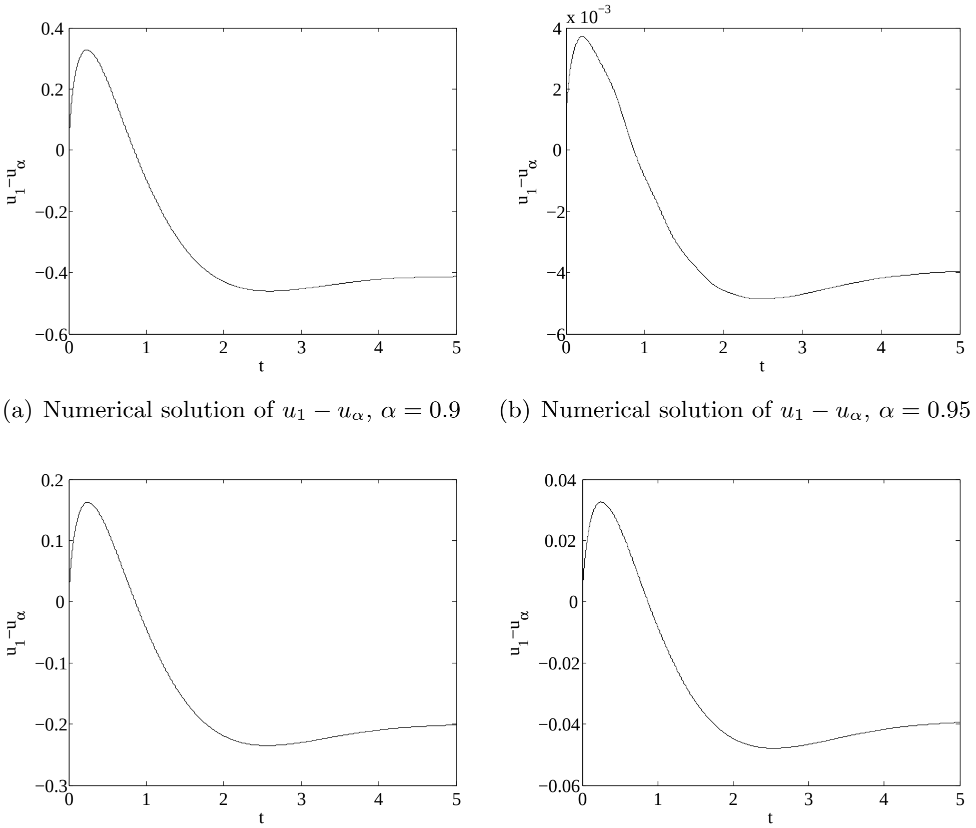


Figure 3: Numerical solution of (6.2).



(a) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.9$

(b) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.95$

(c) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.99$

(d) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.999$

Figure 4: Difference for numerical solutions of (6.2) and (6.1) for different α.

6.2 Examples and numerical investigations of semilinear equations

Case $\alpha \to \beta$

Setting $A = -1$, $x_0 = 10$, $F(t, x(t)) = \sin(t)\sin(x(t))$ and $t \in [0, 5]$ in (3.2), we have

$$\begin{cases} {}_cD_t^\alpha x(t) = -x(t) + \sin(t)\sin(x(t)),\ t \in (0, 5], \\ x(0) = 10. \end{cases} \tag{6.3}$$

Applying the fractional Adams-Bashforth-Moulton scheme in [29], the numerical solution of (6.3) is obtained. We can compare the numerical solutions when $\alpha \to \beta$ as bellow.

t	α=0.5	α=0.6	α=0.55	α=0.51	α=0.501	α=0.4	α=0.45	α=0.49	α=0.499
0	10	10	10	10	10	10	10	10	10
0.5	5.124018	5.254179	5.184282	5.135298	5.125129	5.032457	5.073413	5.113125	5.122911
1	3.973182	3.851906	3.911538	3.960688	3.971929	4.103556	4.037021	3.985763	3.974437
1.5	3.478554	3.266652	3.373309	3.457573	3.476457	3.688947	3.58337	3.499517	3.480651
2	3.221847	2.967334	3.09521	3.196563	3.219319	3.47635	3.348438	3.247128	3.224375
2.5	3.039825	2.740786	2.890318	3.009867	3.036827	3.34395	3.19056	3.069831	3.042823
3	2.857141	2.491824	2.67325	2.820112	2.853432	3.236993	3.044633	2.894313	2.860852
3.5	2.621829	2.165152	2.388655	2.574382	2.617066	3.119342	2.865349	2.669692	2.626597
4	2.308861	1.773257	2.027849	2.250356	2.302957	2.959345	2.619555	2.368565	2.314777
4.5	1.980443	1.431716	1.685294	1.91744	1.974044	2.749393	2.331331	2.045676	1.986864
5	1.760844	1.232035	1.474215	1.699121	1.75456	2.554616	2.112185	1.825126	1.767154

Table 3: Numerical solutions of (6.3) for different fractional order α.

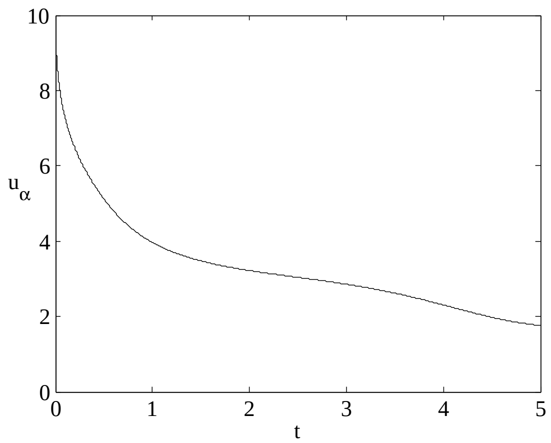


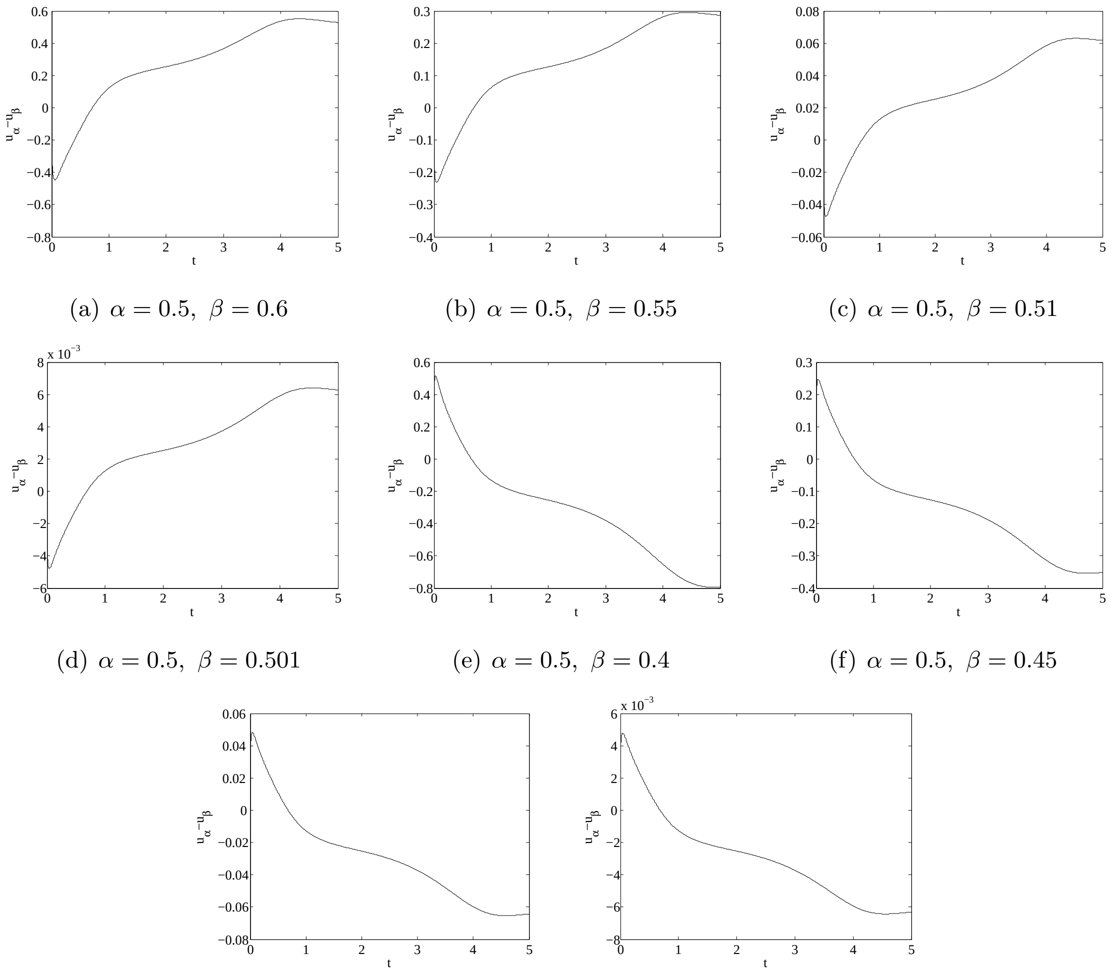
Figure 5: Numerical solution of (6.3) when $\alpha = 0.5$.

Table 3 and Figure 6 show that when α getting closer to 0.5, the numerical solution of (6.3) is getting closer to the numerical solution of (6.3) whose fractional order is 0.5.

Case $\alpha \to 1^-$

Setting $A = -1$, $x_0 = 10$, $F(t, x(t)) = \sin(t)\sin(x(t))$ and $t \in [0, 5]$ in (5.4), we have

$$\begin{cases} x'(t) = -x(t) + \sin(t)\sin(x(t)),\ t \in (0, 5], \\ x(0) = 10. \end{cases} \tag{6.4}$$



(a) $\alpha = 0.5,\ \beta = 0.6$ (b) $\alpha = 0.5,\ \beta = 0.55$ (c) $\alpha = 0.5,\ \beta = 0.51$

(d) $\alpha = 0.5,\ \beta = 0.501$ (e) $\alpha = 0.5,\ \beta = 0.4$ (f) $\alpha = 0.5,\ \beta = 0.45$

(g) $\alpha = 0.5,\ \beta = 0.49$ (h) $\alpha = 0.5,\ \beta = 0.499$

Figure 6: Difference of numerical solutions for (6.3).

Applying the Runge-Kutta method, the numerical solution of (6.4) is obtained. We can compare the numerical solutions of (5.3) and (5.4) when $\alpha \to 1^-$ as bellow.

t	u_1	α=0.9	α=0.95	α=0.99	α=0.999
0	10	10	10	10	10
0.5	6.115991	5.859458	5.985998	6.091125	6.115209
1	3.510434	3.559828	3.53078	3.513968	3.511052
1.5	2.272599	2.552636	2.415196	2.300793	2.27452
2	1.729996	2.101008	1.922883	1.768311	1.731841
2.5	1.345234	1.762551	1.5641	1.38994	1.348437
3	0.945308	1.381591	1.17239	0.991852	0.949135
3.5	0.545073	0.952198	0.752635	0.587118	0.548797
4	0.250764	0.58589	0.417477	0.283997	0.253925
4.5	0.097841	0.361633	0.225853	0.122852	0.100311
5	0.036261	0.252826	0.138817	0.055805	0.038145

Table 4: Numerical solutions of (6.4) and (6.3) for different fractional order α.

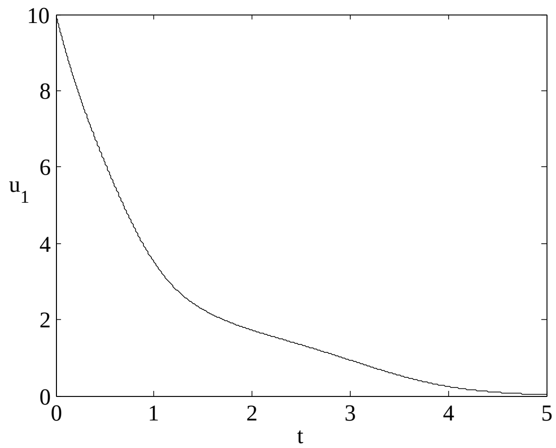
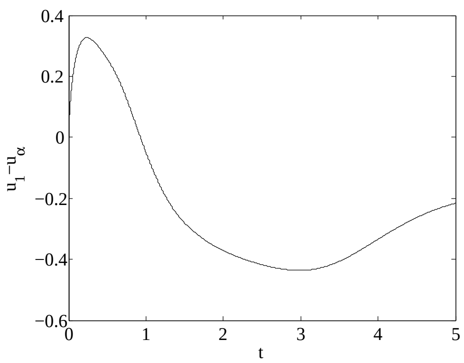


Figure 7: Numerical solution of (6.4).

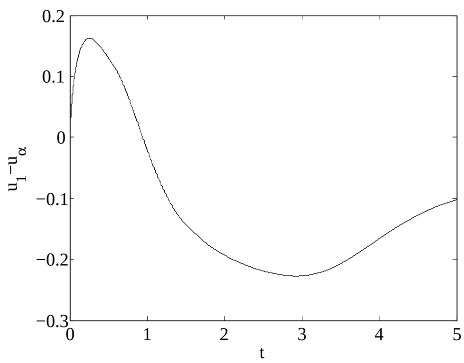
Table 4 and Figure 8 show that when α getting closer to 1, the numerical solution of (6.3) is getting closer to the numerical solution of (6.4).

7 Conclusion

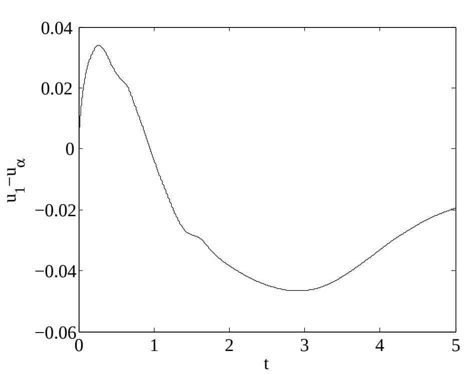
In this study, we firstly established the well-posedness for a type of Caputo fractional evolution equations. Then, we considered the continuity of solutions with respect to fractional order of those equations. Particularly, if the fractional order α converges to 1, then the solution of Caputo fractional evolution equations becomes the solution of classic evolution equation. Numerical studies are performed to explore the continuity on fractional order for some fractional



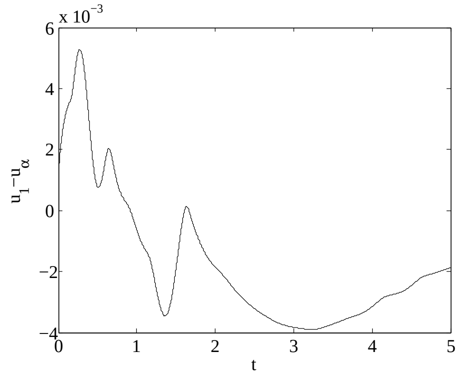
(a) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.9$



(b) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.95$



(c) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.99$



(d) Numerical solution of $u_1 - u_\alpha$, $\alpha = 0.999$

Figure 8: Difference for numerical solutions of (6.4) and (6.3) for different α.

systems, which show reasonable agreement with the theoretic results. To the best of our knowledge, only a few researchers pay attention to the continuity of solutions on fractional order of Caputo fractional evolution equations, particularly, few ones pay attention to the relationship between the solutions of integer order differential equations and the fractional ones. Future work may include exploring the well-posedness and long time behavior of Caputo fractional evolution equations for the order in (1, 2). Especially, the continuity of solution with respect to fractional order of Caputo fractional evolution equations for the order in (1, 2), and the relationship between the the solution of Caputo fractional evolution equations for the order in (1, 2) and the solutions of classic evolution equations of first order and second order.

Acknowledgements

This research was supported by the Basic and Applied Basic Research Foundation of Guangdong Province (No.2021A1515010055), the Key Construction Discipline Scientific Research Ability Promotion Project of Guangdong Province (No.2021ZDJS055), the NNSF of China (No.11871225), the Young Creative Talents Program of Guangdong Province (2019KQNCX096).

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