Pullback D-attractors for doubly nonlinear parabolic equations

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Abstract

In this paper, we study a class of doubly nonlinear parabolic equations (1.1). The nonlinearity B(u) brings great difficulties to the study of the problem. First we show that the problem has a unique solution. Then we prove that the process corresponding to the problem is norm-to-weak continuous. After that, by using Legendre transform, we obtain uniform estimates and asymptotic compactness properties that allow us to ensure the existence of pullback D-attractors for the associated process to the problem

Pullback $\mathcal{D}$-attractors for doubly nonlinear parabolic equations*

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Abstract In this paper, we study a class of doubly nonlinear parabolic equations (1.1). The nonlinearity $\beta(u)$ brings great difficulties to the study of the problem. First we show that the problem has a unique solution. Then we prove that the process corresponding to the problem is norm-to-weak continuous. After that, by using Legendre transform, we obtain uniform estimates and asymptotic compactness properties that allow us to ensure the existence of pullback $\mathcal{D}$-attractors for the associated process to the problem.

Keywords Pullback $\mathcal{D}$-attractors; Parabolic equations; Norm-to-weak continuous process; Legendre transform

AMS Subject Classification: 35K57, 35B40, 35B41

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1 Introduction

We are interested in the long time behavior of doubly nonlinear parabolic equation of the form

$$\begin{cases} \dfrac{\partial \beta(u)}{\partial t} - \Delta u + f(u) = g(x,t), \quad x \in \Omega, t \in \mathbb{R} \\ u(x,t)|_{\partial\Omega} = 0, \\ u(x,\tau) = u_\tau(x). \end{cases} \tag{1.1}$$

in a bounded smooth domain Ω, $g(x,t) \in L^2(\tau, T; L^2(\Omega))$. Such equations appear, e.g., in the study of gas filtration(so called porous medium equation). The study of equation of the form (1.1) can be found in [3-5,10,13]). It has been extensively studied when $\beta(u) = u, g(x,t) = g(x)$ and the existence of attractors have been proved in([1,7,9, 14,15,17,18]). For more general equation (1.1) with $g(x,t) = g(x)$, the existence of

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attractors are constructed in ([2-4,8,10]), for the non-autonomous case, as far as I know, the existence of attractors has not been studied.

Our aim in this paper is to study the existence of pullback $\mathcal{D}$-attractors of (1.1), and extend the result of [7] to the non-autonomous case. We make the following assumptions:

$$\beta(s) \in \mathcal{C}^1(\mathbb{R}),\ \beta(0) = 0, \beta'(s) \geq \beta_0, \beta_0 > 0\ s \in \mathbb{R}; \tag{1.2}$$

$$\beta_1|s|^{r+2} - \beta_3 \leq \beta(s)s \leq \beta_2|s|^{r+2} + \beta_3,\ \beta_1, \beta_2 > 0, \beta_3 \geq 0, r \geq 0; \tag{1.3}$$

$$f(s) \in \mathcal{C}(\mathbb{R}), \gamma_1|s|^q - \gamma_3 \leq f(s)s \leq \gamma_2|s|^q + \gamma_3,\ s \in \mathbb{R}, \gamma_1, \gamma_2 > 0, \gamma_3 \geq 0, q \geq r+2; \tag{1.4}$$

There exists a constant $C_0 \geq 0$, such that

$$C_0\beta(s) + f(s) \text{ is increasing.} \tag{1.5}$$

By hypotheses (1.2)-(1.5), β and f are nonlinear functions with polynomial growth of arbitrary order. Here β is more general than in [2-4,10](where β is linear growth), which is an essential difficulty in proving the existence of attractor. To the problem (1.1), the key points are to obtain the norm-to-weak continuous and compactness of process generated by (1.1). By using Legendre transform and the asymptotic a priori estimate method introduced in [7], we show that the existence of pullback $\mathcal{D}$-attractor.

This article is organized as follows. In Section 2, we recall some basic concepts about the pullback $\mathcal{D}$-attractor. In Section 3, we show that the uniqueness of solution and norm-to-weak continuous of process generated by (1.1). In section 4, we verify the asymptotic compactness of the process $U(t,\tau)$ in $L^q(\Omega)$), and prove the existence of the $(L^{r+2}(\Omega), L^q(\Omega))$ pullback $\mathcal{D}$-attractor under the hypotheses (1.2)-(1.5).

Throughout this paper we use the following notation: $H = L^2(\Omega)$, and the norms in $H_0^1(\Omega)$ and $L^p(\Omega)(1 \leq p \leq \infty)$ are denoted by $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$ and $|u|_p^p = \int_\Omega |u|^p dx$, respectively; $\Omega(u \geq M) = \{x \in \Omega : u(x) \geq M\}$ and $\Omega(u \leq -M) = \{x \in \Omega : u(x) \leq -M\}$; $m(\Omega)$ or $|\Omega|$ denote Lebesgue measure of Ω; sometimes for special differentiation, we denote the different positive constants by $c, c_1, c_2, \cdots$.

2 Preliminaries

Let X be a complete metric space, and $\{U(t,\tau)\} = \{U(t,\tau) : t \geq \tau\}$ be a two-parameter family of mappings act on X : $U(t,\tau) : X \to X, t \geq \tau$.

Definition 2.1 ([2,9,17]) *A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a norm-to-weak continuous process in X if*

(1) $U(t,s)U(s,\tau)=U(t,\tau), \forall t\geq s\geq \tau$,

(2) $U(\tau,\tau)=Id$, *is the identity operator,* $\tau\in\mathbb{R}$,

(3)$U(t,\tau)x_n \rightharpoonup U(t,\tau)x$, *if* $x_n\to x$ *in* X.

Let $B(X)$ is the set of all bounded subsets of X, $\mathcal{D}$ is a nonempty class of parameterised sets $\widehat{\mathcal{D}}=\{D(t):t\in\mathbb{R}\}\subset B(X)$.

Definition 2.2([2,6,7,9,11,15]) *It is said that* $\hat{\mathcal{B}}\in\mathcal{D}$ *is pullback* $\mathcal{D}-$ *absorbing for the process* $\{U(t,\tau)\}$ *if for any* $t\in\mathbb{R}$ *and any* $\hat{\mathcal{D}}\in\mathcal{D}$, *there exists a* $\tau_0(t,\hat{\mathcal{D}})\leq t$ *such that* $U(t,\tau)D(\tau)\subset B(t)$ *for all* $\tau\leq\tau_0(t,\hat{\mathcal{D}})$.

Definition 2.3([2,6,7,9,11,15]) *The process* $\{U(t,\tau)\}$ *is said to be pullback* $\mathcal{D}-$*asymptotically compact if for any* $t\in R$, *any* $\hat{\mathcal{D}}\in\mathcal{D}$, *and any sequence* $\tau_n\to-\infty$, *any sequence* $x_n\in D(\tau_n)$, *the sequence* $\{U(t,\tau_n)x_n\}$ *is relatively compact in* X.

Definition 2.4([2,6,7,9,11,15]) *The family* $\hat{\mathcal{A}}=\{\mathcal{A}(t):t\in\mathbb{R}\}\subset B(X)$ *is said to be a pullback* $\mathcal{D}-$ *attractor for* $U(t,\tau)$ *if*

(1)$\mathcal{A}(t)$ *is compact for all* $t\in\mathbb{R}$,

(2)$\hat{\mathcal{A}}$ *is invariant, i.e.,* $U(t,\tau)\mathcal{A}(\tau)=\mathcal{A}(t)$ *for all* $t\geq\tau$,

(3)$\hat{\mathcal{A}}$ *is pullback* $\mathcal{D}-$*attracting, i.e.,*

$$\lim_{\tau\to-\infty} dist((U(t,\tau)D(\tau),\mathcal{A}(t))=0, \text{ for all } \hat{\mathcal{D}}\in\mathcal{D}, \text{ and all } t\in\mathbb{R},$$

(4)if $\{C(t)\}_{t\in R}$ *is another family of closed attracting sets, then* $\mathcal{A}(t)\subset C(t)$ *for all* $t\in\mathbb{R}$.

Let X be a complete metric space and B be a bounded subset of X. The Kuratowski measure of noncompactness $\alpha(B)$ of B is defined by

$$\alpha(B)=inf\{\delta>0\ |B \text{ has a finite open cover of sets of diameter} \leq\delta\}.$$

Definition 2.5([9]) *A process* $\{U(t,\tau)\}$ *is called pullback* ω*-D-limit compact if for any* $\varepsilon>0$ *and* $\hat{D}\in\mathcal{D}$, *there exists a* $\tau_0(\hat{D},t)\leq t$ *such that* $\alpha(\bigcup_{\tau\leq\tau 0}U(t,\tau)D(\tau))\leq\varepsilon$.

Lemma 2.1([9]) *Assume* $\{U(t,\tau)\}$ *is pullback* ω*-D-limit compact, then for any sequence* $\{\tau_n\}\subset R_t, \tau_n\to-\infty$ *as* $n\to\infty$, *and sequence* $x_n\in D(\tau_n)$ *there exists a convergent subsequence of* $\{U(t,\tau_n)x_n\}$ *whose limit lies in* $\omega(\hat{D},t)=\bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s}U(t,\tau)B(\tau)}$.

Theorem 2.1([9]) *Suppose that the process* $U(t,\tau)$ *is norm-to-weak continuous and pullback* ω*-D-limit compact,* $\hat{B}\in\mathcal{D}$ *is a family of pullback* $\mathcal{D}-$*absorbing sets for* $U(t,\tau)$.*Then the family* $\hat{\mathcal{A}}=\{\mathcal{A}(t):t\in R\}\subset B(X)$ *defined by*

$$\mathcal{A}(t)=\omega(\hat{B},t)=\bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s}U(t,\tau)B(\tau)},$$

is a pullback $\mathcal{D}$*-attractor for* $U(t,\tau)$.

Theorem 2.2([7]) *Let* $\{U(t,\tau)\}_{t\geq\tau}$ *is norm-to-weak continuous process in* $L^p(\Omega)$, $\hat{\mathcal{B}}=\{B(t):t\in\mathbb{R}\}\in\mathcal{D}$ *is pullback* $\mathcal{D}$*-absorbing sets in* $L^p(\Omega)$, *and* $U(t,\tau)$ *satisfy the* *following two assumptions:*

(1) $\{U(t,\tau)\}_{t\geq\tau}$ is pullback ω-D-limit compact in $L^q(\Omega)(1\leq q<p)$;

(2) for any $\varepsilon>0$, there exist $M(\varepsilon,\widehat{\mathcal{B}})$ and $\tau_0=\tau_0(\varepsilon,\widehat{\mathcal{B}})\leq t$ such that $\int_{\Omega(|U(t,\tau)u_\tau|\geq M)}|U(t,\tau)u_\tau|^p dx)^{\frac{1}{p}}<\varepsilon$ for any $u_\tau\in B(\tau)$, and $\tau\leq\tau_0$.

Then there exists a pullback $\mathcal{D}-$attractor $\hat{\mathcal{A}}=\{\mathcal{A}(t):t\in\mathbb{R}\}$ in $L^p(\Omega)$ and

$$\mathcal{A}(t)=\bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s}U(t,\tau)B(\tau)}^{L^p(\Omega)},$$

where $\overline{\bigcup_{\tau\leq s}U(t,\tau)B(\tau)}^{L^p(\Omega)}$ denote closure in $L^p(\Omega)$.

By Lemma 2.1, the process $\{U(t,\tau)\}$ is pullback ω-D-limit compact, then $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. In practice, as long as the process is pullback $\mathcal{D}$-asymptotically compact, then the Theorem 2.1 and the Theorem 2.2 are still hold.

3 Uniqueness of solution and norm-to-weak continuous of process

The existence of weak solution for (1.1) can be obtained by the standard Faedo-Galerkin approximation method(see[1,3,14]). Here we only state the result.

Lemma 3.1 *Assume that $g(x,t)\in L^2(\Omega)$, β and f satisfying (1.2)-(1.5), $u_\tau(x)\in L^{r+2}(\Omega)$. Then for any initial data $u_\tau(x)\in L^{r+2}(\Omega)$, there exists solution $u(x,t)$ for Eq.(1.1) which satisfies*

$$u(x,t)\in C(\tau,T;L^1(\Omega))\cap L^2(\tau,T;H_0^1(\Omega))\cap L^q(\tau,T;L^q(\Omega)).$$

We now show that the solution is uniqueness and continuous dependence on initial conditions.

Theorem 3.2 *Assume that $g(x,t)\in L^2(\Omega), u_\tau(x)\in L^{r+2}(\Omega)$, β and f satisfying (1.2)-(1.5). Then there exists a unique solution of Eq.(1.1)*

Proof Suppose that $u(t),v(t)$ be two solution of (1.1) with initial conditions $u_\tau(x)$, $v_\tau(x)$, then

$$\frac{\partial(\beta(u)-\beta(v))}{\partial t}-\triangle(u-v)+f(u)-f(v)=0,$$

i.e.,

$$\frac{\partial(\beta(u)-\beta(v))}{\partial t}-\triangle(u-v)+(C_0\beta(u)+f(u))-(C_0\beta(v)+f(v))=C_0(\beta(u)-\beta(v)).$$

We define the sign function by

$$sign(\tau) = \begin{cases} 1 & if\ \tau > 0, \\ 0 & if\ \tau = 0, \\ -1 & if\ \tau < 0. \end{cases}$$

Multiplying (3.1) by $sign(u-v)$ and integrating in Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\beta(u) - \beta(v)|dx - \int_\Omega \Delta(u-v)sign(u-v) \\ &+ \int_\Omega [(C_0\beta(u) + f(u)) - (C_0\beta(v) + f(v))]sign(\beta(u) - \beta(v))dx] \\ &= C_0 \int_\Omega |\beta(u) - \beta(v)|dx. \end{aligned}$$

Using (1.6), we get

$$\int_\Omega [(C_0\beta(u) + f(u)) - (C_0\beta(v) + f(v))]sign(\beta(u) - \beta(v))dx \geq 0.$$

Since $sign(u-v) = \lim_{\varepsilon \to 0^+} \dfrac{u-v}{\varepsilon + |u-v|}$, by dominated convergence theorem, we have

$$\begin{aligned} &-\int_\Omega \Delta(u-v)sign(u-v)dx = -\lim_{\varepsilon \to 0^+} \int_\Omega \Delta(u-v)\frac{u-v}{\varepsilon + |u-v|}dx \\ &= \lim_{\varepsilon \to 0^+} \int_\Omega \nabla(u-v)\nabla(\frac{u-v}{\varepsilon + |u-v|})dx = \lim_{\varepsilon \to 0^+} \int_\Omega \varepsilon \tfrac{|\nabla(u-v)|^2}{(\varepsilon + |u-v|)^2)}dx \geq 0. \end{aligned}$$

So

$$\frac{d}{dt}\int_\Omega |\beta(u) - \beta(v)|dx \leq C_0 \int_\Omega |\beta(u) - \beta(v)|dx.$$

By Gronwall inequality, we get

$$\int_\Omega |\beta(u(t) - \beta(v(t))|dx \leq e^{C_0(t-\tau)} \int_\Omega |\beta(u_\tau) - \beta(v_\tau)|dx,$$

From (1.2), we have

$$\int_\Omega |u(t) - v(t)|dx \leq \frac{1}{\beta_0} e^{C_0(t-\tau)} \int_\Omega |\beta(u_\tau) - \beta(v_\tau)|dx.$$

Which gives continuous dependence on initial conditions and uniqueness of solution in $L^1(\Omega)$.

By Theorem 3.2, we can define the process $\{U(t,\tau\}_{t\geq\tau}$ in $L^1(\Omega)$ as the following:

$$U(t,\tau)u_\tau : L^{r+2}(\Omega) \to L^1(\Omega),$$

which is continuous in $L^1(\Omega)$.

Since β be a continuous increasing function with $\beta(0)=0$. We define for $t\in\mathbb{R}$,

$$\psi(t)=\int_0^t \beta(\tau)d\tau.$$

Then the Legendre transform ψ^* is defined by

$$\psi^*(\tau)=\sup_{s\in\mathbb{R}}\{\tau s-\psi(s)\}.$$

Note that

$$\psi^*(\tau)\geq 0,\ \psi^*(\beta(\tau))+\psi(\tau)=\tau\beta(\tau),\psi^*(\beta(\tau))\leq \tau\beta(\tau). \tag{3.1}$$

Theorem 3.3 *Assume that the conditions (1.2)-(1.5) are satisfied, $g(x,t)\in L^2(\Omega)$. Then the process $U(t,\tau)$ is norm-to-weak continuous in $L^q(\Omega)$ and $H_0^1(\Omega)$.*

Proof Let $u_{m\tau}(x)\to u_\tau(x)$ in $L^{r+2}(\Omega)$, $u_m(t),u(t)$ are the solutions of Eq.(1.1) corresponding to initial date $u_{m\tau}(x),u_\tau(x)$. In (1.1), replace $u(t)$ by $u_m(t)$. Multiply (1.1) by $u_m(t)$ and integrating in Ω, we get

$$\frac{d}{dt}\int_\Omega \psi^*(\beta(u_m(t)))dx+|\nabla u_m|_2^2+(f(u_m),u_m)=(g,u_m).$$

Thanks to Poincaré inequality $\lambda|u|_2^2\leq|\nabla u|_2^2$, and Cauchy inequality, we have

$$|\int_\Omega g(x)u_m dx|\leq\frac{\lambda}{2}|u_m|_2^2+\frac{1}{2\lambda}|g(x,t)|_2^2\leq\frac{1}{2}|\nabla u_m|_2^2+\frac{1}{2\lambda}|g(x,t)|_2^2.$$

So

$$\frac{d}{dt}\int_\Omega \psi^*(\beta(u_m(t)))dx+\frac{1}{2}|\nabla u_m|_2^2+\gamma_1|u_m|_q^q\leq\gamma_3|\Omega|+\frac{1}{2\lambda}|g(x,t)|_2^2.$$

Integrating from τ to T, we obtain

$$\begin{aligned}&\textstyle\int_\Omega \psi^*(\beta(u_m(T)))dx+\frac{1}{2}\int_\tau^T|\nabla u_m|_2^2dt+\gamma_1\int_\tau^T|u_m|_q^q dt\\ &\leq\textstyle\int_\Omega\psi^*(\beta(u_{m\tau}))dx+\gamma_3|\Omega|(T-\tau)+\frac{1}{2\lambda}\int_\tau^T|g(x,s)|_2^2ds.\\ &\leq\textstyle\int_\Omega u_{m\tau}\beta(u_{m\tau})dx+\gamma_3|\Omega|(T-\tau)+\frac{1}{2\lambda}\int_\tau^T|g(x,s)|_2^2ds\\ &\leq\textstyle\beta_2|u_{m\tau}|_{r+2}^{r+2}+\beta_3|\Omega|+\gamma_3|\Omega|(T-\tau)+\frac{1}{2\lambda}\int_\tau^T|g(x,s)|_2^2ds.\end{aligned}$$

$u_{m\tau}\to u_\tau$ in $L^{r+2}(\Omega)$, so there exists $M>0$, such that $|u_{m\tau}|_{r+2}^{r+2}\leq M$. We get $u_m(t)$ are bounded in $L^2(\tau,T;H_0^1(\Omega))$ and $L^q(\tau,T;L^q(\Omega))$, there exists weak convergence subsequence $u_{m_k}(t)$ convergence to $v(t)$ in $L^2(\tau,T;H_0^1(\Omega))$ and $L^q(\tau,T;L^q(\Omega))$, obviously, $v(t)$ be a solution of (1.1) satisfies initial value $v(x,\tau)=u_\tau(x)$. By the unique of solution for (1.1), we have $u(t)=v(t)$, i.e., $u_{m_k}\rightharpoonup u(t)$ in $L^2(\tau,T;H_0^1(\Omega))$ and $L^q(\tau,T;L^q(\Omega))$. By Definition 2.1, Theorem 3.3 holds.

Remark 3.4 *The proess $\{U(t,\tau\}_{t\geq\tau}$ is norm-to-weak continuous in $L^2(\Omega)$.*

4 Pullbck $\mathcal{D}$-attractor in $L^q(\Omega)$

By theorem 3.3, we can define process $\{U(t,\tau)\}_{t\geq\tau}$ as the following:

$$U(t,\tau): L^{r+2}(\Omega) \to L^q(\Omega). \tag{4.1}$$

Moreover, we suppose for any $t \in \mathbb{R}$, we have

$$\int_{-\infty}^{t} e^{\delta s}|g(x,s)|_2^2 ds < \infty, \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\delta r}|g(x,r)|_2^2 drds < \infty, \tag{4.2}$$

here $\delta = \frac{\gamma_1 q}{\beta_2(r+2)}$.

Lemma 4.1 *Assume that the conditions (1.2)-(1.5) are satisfied and $g(x,t)$ satisfies (4.2), $u(t)$ be a weak solution of (1.1). Then there exists $T>0$, for any $t-\tau\geq T$, we have the following inequality:*

$$\begin{aligned} &|\nabla u(t)|_2^2 + |u(t)|_q^q \leq c((t-\tau)e^{-\delta(t-\tau)}|u_\tau|_{r+2}^{r+2} + 1 \\ &+ \textstyle\int_{-\infty}^{t} e^{\delta(s-t)}|g(x,s)|_2^2 ds + \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\delta(r-t)}|g(x,r)|_2^2 drds). \end{aligned} \tag{4.3}$$

Proof Multiplying (1.1) by $u(t)$ and integrating over Ω, we obtain

$$\frac{d}{dt}\int_\Omega \psi^*(\beta(u))dx + |\nabla u|_2^2 + (f(u),u) = (g,u).$$

By (1.4), we have

$$\frac{d}{dt}\int_\Omega \psi^*(\beta(u))dx + |\nabla u|_2^2 + \gamma_1|u|_q^q - \gamma_3|\Omega| \leq (g,u). \tag{4.4}$$

Thanks to Poincaré inequality and Young inequality, one gets

$$|(g,u)| \leq \frac{\lambda}{2}|u|_2^2 + \frac{1}{2\lambda}|g(x,t)|_2^2 \leq \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2\lambda}|g(x,t)|_2^2,$$

By (4.4), we have

$$\frac{d}{dt}\int_\Omega \psi^*(\beta(u))dx + \frac{1}{2}|\nabla u|_2^2 + \gamma_1|u|_q^q \leq \gamma_3|\Omega| + +\frac{1}{2\lambda}|g(x,t)|_2^2. \tag{4.5}$$

Using Young inequality, we obtain

$$|u|_{r+2}^{r+2} = \int_\Omega |u|^{r+2}dx \leq \frac{r+2}{q}|u|_q^q + \frac{q-r-2}{q}|\Omega|.$$

We find that

$$\frac{\gamma_1 q}{r+2}(|u|_{r+2}^{r+2} - \frac{q-r-2}{q}|\Omega|) \leq \gamma_1|u|_q^q. \tag{4.6}$$

Using (1.3) and (3.1), we get

$$0 \le \int_\Omega \psi^*(\beta(u))dx \le \int_\Omega u\beta(u)dx \le \beta_2|u|_{r+2}^{r+2} + \beta_3|\Omega|.$$

Hence

$$\gamma_1|u|_q^q \ge \frac{\gamma_1 q}{\beta_2(r+2)}\int_\Omega \psi^*(\beta(u))dx - \frac{\gamma_1 q\beta_3}{\beta_2(r+2)}|\Omega| - \frac{\gamma_1 q(q-r-2)}{q(r+2)}|\Omega|.$$

Let $\delta = \frac{\gamma_1 q}{\beta_2(r+2)}, c_1 = (\frac{\gamma_1 q\beta_3}{\beta_2(r+2)} + \frac{\gamma_1 q(q-r-2)}{q(r+2)} + \gamma_3)|\Omega|$, by (4.5), we obtain

$$\frac{d}{dt}\int_\Omega \psi^*(\beta(u))dx + \delta\int_\Omega \psi^*(\beta(u))dx \le c_1 + \frac{1}{2\lambda}|g(x,t)|_2^2. \tag{4.7}$$

By the Gronwall lemma, for all $t \ge \tau$, one deduces

$$\int_\Omega \psi^*(\beta(u(t)))dx \le e^{-\delta(t-\tau)}\int_\Omega \psi^*(\beta(u(\tau)))dx + \frac{c_1}{\delta} + \frac{1}{2\lambda}e^{-\delta t}\int_{-\infty}^t e^{-\delta s}|g(x,s)|_2^2 ds. \tag{4.8}$$

Multiplying (1.1) by u_t and integrating in Ω, we get

$$\int_\Omega \beta'(u)u_t^2 dx + \frac{d}{dt}(|\nabla u|_2^2 + \int_\Omega F(u)dx) = (g(x,t), u_t),$$

where $F(u) = \int_0^u f(s)ds$. By (1.2), we have

$$|(g(x,t), u_t)| \le \int_\Omega |g(x,t)u_t|dx \le \frac{1}{2}\int_\Omega \beta'(u)u_t^2 dx + \frac{1}{2\beta_0}|g(x,t)|_2^2.$$

Therefore, one has

$$\frac{d}{dt}(|\nabla u|_2^2 + \int_\Omega F(u)dx) \le \frac{1}{2\beta_0}|g(x,t)|_2^2. \tag{4.9}$$

It follows from (1.4) that there exist $\gamma_1', \gamma_2' > 0, \gamma_3' \ge 0$ such that

$$\gamma_1'|s|^q - \gamma_3' \le F(s) \le \gamma_2'|s|^q + \gamma_3'. \tag{4.10}$$

Let $\rho = \min\{\frac{1}{2}, \frac{\gamma_1}{r_2'}\}$. Using (4.5), one has

$$\frac{d}{dt}\int_\Omega \psi^*(\beta(u))dx + \rho(|\nabla u|_2^2 + \int_\Omega F(u)dx) \le c_2(1 + |g(x,t)|_2^2). \tag{4.11}$$

So, by (4.9) and (4.11)

$$\frac{d}{dt}(e^{\delta t}\int_\Omega \psi^*(\beta(u))dx) \le \delta e^{\delta t}\int_\Omega \psi^*(\beta(u))dx + e^{\delta t}(-\rho(|\nabla u|_2^2 + \int_\Omega F(u)dx) + c_2(1+|g(x,t)|_2^2)).$$

Using (4.8), one has

$$\begin{aligned}
&\int_\tau^t e^{\delta s}(|\nabla u|_2^2+\int_\Omega F(u)dx)ds\\
&\le c_3(e^{\delta\tau}\int_\Omega \psi^*(\beta(u_\tau))dx+\int_\tau^t e^{\delta s}\int_\Omega \psi^*(\beta(u(s)))dxds+e^{\delta t}+\int_\tau^t e^{\delta s}|g(x,s)|_2^2 ds)\\
&\le c_4((1+t-\tau)e^{\delta\tau}\int_\Omega \psi^*(\beta(u_\tau))dx+e^{\delta t}+\int_\tau^t e^{\delta s}|g(x,s)|_2^2 ds+\int_\tau^t\int_\tau^s e^{\delta r}|g(x,r)|_2^2 dr).
\end{aligned}\tag{4.12}$$

In fact, by (4.9), we obtain

$$\begin{aligned}
&\tfrac{d}{dt}((t-\tau)e^{\delta t}(|\nabla u|_2^2+\int_\Omega F(u)dx))\\
&=(1+\delta(t-\tau))e^{\delta t}(|\nabla u|_2^2+\int_\Omega F(u)dx)+(t-\tau)e^{\delta t}\tfrac{d}{dt}(|\nabla u|_2^2+\int_\Omega F(u)dx) \qquad (4.13)\\
&\le c_5((1+t-\tau)e^{\delta t}(|\nabla u|_2^2+\int_\Omega F(u)dx)+(t-\tau)e^{\delta t}|g(x,t)|_2^2).
\end{aligned}$$

For any $t-\tau\ge 1$, integrating from τ to t, we have

$$\begin{aligned}
&|\nabla u(t)|_2^2+\int_\Omega F(u(t))dx\\
&\le c_5((1+\tfrac{1}{t-\tau})e^{-\delta t}\int_\tau^t e^{\delta s}(|\nabla u(s)|_2^2+\int_\Omega F(u(s))dx)ds+e^{-\delta t}\int_\tau^t e^{\delta s}|g(x,s)|_2^2 ds)\\
&\le c_6((t-\tau)e^{-\delta(t-\tau)}\int_\Omega \psi^*(\beta(u_\tau))dx+1\\
&+e^{-\delta t}(\int_\tau^t e^{\delta s}|g(x,s)|_2^2 ds+\int_\tau^t\int_\tau^s e^{\delta r}|g(x,r)|_2^2 drds))\\
&\le c_7((t-\tau)e^{-\delta(t-\tau)}|u_\tau|_{r+2}^{r+2}+(t-\tau)e^{-\delta(t-\tau)}+1\\
&+e^{-\delta t}(\int_\tau^t e^{\delta s}|g(x,s)|_2^2 ds+\int_\tau^t\int_\tau^s e^{\delta r}|g(x,r)|_2^2 drds))\\
&\le c_7((t-\tau)e^{-\delta(t-\tau)}|u_\tau|_{r+2}^{r+2}+(t-\tau)e^{-\delta(t-\tau)}+1\\
&+e^{-\delta t}(\int_{-\infty}^t e^{\delta s}|g(x,s)|_2^2 ds+\int_{-\infty}^t\int_{-\infty}^s e^{\delta r}|g(x,r)|_2^2 drds)).
\end{aligned}\tag{4.14}$$

We find that exists $T>0$, for any $t-\tau>\max\{1,T\}$,

$$\begin{aligned}
&|\nabla u(t)|_2^2+\int_\Omega F(u(t))dx\le c((t-\tau)e^{-\delta(t-\tau)}|u_\tau|_{r+2}^{r+2}+1\\
&+\int_{-\infty}^t e^{\delta(s-t)}|g(x,s)|_2^2 ds+\int_{-\infty}^t\int_{-\infty}^s e^{\delta(r-t)}|g(x,r)|_2^2 drds).
\end{aligned}\tag{4.15}$$

By (4.10), we obtain Lemma 4.1.

Let $\mathcal{R}$ be the set of all functions $\rho:\mathbb{R}\to(0,+\infty)$ such that $\lim\limits_{t\to+\infty} te^{\delta t}\rho^{r+2}(t)=0$, denote by $\mathcal{D}$ the class of all families $\widehat{\mathcal{D}}=\{D(t):t\in\mathbb{R}\}$ such that $D(t)\subset\overline{B}(\rho(t))$ for some $\rho(t)\in\mathcal{R}$, $\overline{B}(\rho(t))$ the closed ball in $L^{r+2}(\Omega)$ with radius $\rho(t)$). Let

$$\rho_0(t)=[2c(1+\int_{-\infty}^t e^{\delta(s-t)}|g(x,s)|_2^2 ds+\int_{-\infty}^t\int_{-\infty}^s e^{\delta(r-t)}|g(x,r)|_2^2 drds)]^{\frac{1}{q}}.\tag{4.16}$$

$\overline{B}_q(\rho_0(t))$ denote close ball in $L^q(\Omega)$ with radius $\rho_0(t)$. Obviousy $\overline{B}_q(\rho_0(t))$ be a family of bounded pullback $\mathcal{D}$-absorbing sets for the process $\{U(t,\tau\}_{t\geq\tau}$ generated by (1.1) in $L^q(\Omega)$.

From (4.3), we also get that there exists a family of bounded pullback $\mathcal{D}$-absorbing sets in $L^2(\Omega)$ and $H_0^1(\Omega)$, therefore, the process generated by (1.1) is pullback ω-D-limit compact in $L^2(\Omega)$. By theorem 2.1, we have the following theorem.

Theorem 4.2 *Assume that the conditions (1.2)-(1.5) are satisfied, $g(x,t)$ satisfies (4.2). Then the process $U(t,\tau)$ generated by (1.1) exists a pullback $\mathcal{D}$-attractor in $L^2(\Omega)$.*

In the following, we will give the asymptotic a priori estimate of $\{U(t,\tau\}_{t\geq\tau}$ with respect to $L^q(\Omega)$ norm, which play a crucial role in the proof of the pullback $\mathcal{D}$-attractor in $L^q(\Omega)$

Theorem 4.3 *Assume that the conditions (1.2)-(1.5) are satisfied, $g(x,t)$ satisfies (4.2), $q > r+2$. Then the process $U(t,\tau)$ generated by (1.1) exists a pullback $\mathcal{D}$-attractor in $L^q(\Omega)$.*

Proof We know from Theorem 4.2 that the process $\{U(t,\tau\}_{t\geq\tau}$ is pullback ω-D-limit compact in $L^2(\Omega)$. Next we will prove that the process satisfies (2) of Theorem 2.2.

By (1.3) and (1.5), we find that there exists $M_1 > 0, \forall |u| \geq M_1$ such that

$$f(u)u \geq \frac{\gamma_1}{2}|u|^q, \quad \frac{\beta_1}{2}|u|^{r+1} \leq |\beta(u)| \leq 2\beta_2|u|^{r+1}. \tag{4.17}$$

Let $M_2 = \max\{1, \frac{\beta_1}{2}|M_1|^{r+1}\}$, $|u| \geq M_1$, then $|\beta(u)| \geq M_2$. Multiply (1.1) with $|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-2}(\beta(u)-M_2)_+$, we get

$$\begin{aligned}&\frac{r+1}{q}\frac{d}{dt}\textstyle\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx + \int_\Omega \nabla u\nabla(|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-2}(\beta(u)-M_2)_+)dx\\&+\textstyle\int_\Omega f(u)|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-2}(\beta(u)-M_2)_+dx\\&=\textstyle\int_\Omega g(x,t)|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-2}(\beta(u)-M_2)_+dx.\end{aligned} \tag{4.18}$$

Where $(\beta(u)-M_2)_+$ denote the positive part of $(\beta(u)-M_2)$, that is

$$(\beta(u)-M_2)_+ = \begin{cases} \beta(u)-M_2, & \beta(u) \geq M_2,\\ 0, & \beta(u) < M_2.\end{cases}$$

Thus we have

$$\begin{aligned}
&\textstyle\int_\Omega \nabla u \nabla(|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-2}(\beta(u)-M_2)_+)dx\\
&= \textstyle\int_{\Omega(\beta(u)\geq M)} \nabla u \nabla(|(\beta(u)-M_2)|^{\frac{q}{r+1}-1})dx\\
&= (\tfrac{q}{r+1}-1)\textstyle\int_{\Omega(\beta(u)\geq M)} \beta'(u)|(\beta(u)-M_2)|^{\frac{q}{r+1}-2}|\nabla u|^2dx\\
&\geq 0,
\end{aligned}$$

$$\begin{aligned}
&\textstyle\int_\Omega f(u)|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-2}(\beta(u)-M_2)_+dx\\
&\geq \frac{\gamma_1}{2}\textstyle\int_\Omega |u|^{q-1}|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-1}dx\\
&\geq c_8\textstyle\int_\Omega |\beta(u)|^{\frac{q-1}{r+1}}|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-1}dx\\
&\geq \frac{c_8}{2}\textstyle\int_\Omega |\beta(u)|^{\frac{q-1}{r+1}}|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-1}dx+\frac{c_8}{2}\int_\Omega |\beta(u)|^{\frac{r}{r+1}}|\beta(u)|^{\frac{q-r-1}{r+1}}|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-1}dx\\
&\geq \frac{c_8}{2}M_2^{\frac{q-r-2}{r+1}}\textstyle\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx+\frac{c_8}{2}\int_\Omega |(\beta(u)-M_2)_+|^{\frac{2(q-r-1)}{r+1}}dx
\end{aligned}$$

and

$$\begin{aligned}
&|\textstyle\int_\Omega g(x,t)|(\beta(u)-M_2)_+|^{\frac{q}{r+1}-2}(\beta(u)-M_2)_+dx|\\
&\leq \textstyle\int_\Omega |g(x,t)||(\beta(u)-M_2)_+|^{\frac{q-r-1}{r+1}}dx\\
&\leq \frac{c_8}{2}\textstyle\int_\Omega |(\beta(u)-M_2)_+|^{\frac{2(q-r-1)}{r+1}}dx+\frac{1}{2c_8}\int_{\Omega(\beta(u)\geq M_2)}|g(x,t)|^2dx.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
&\frac{d}{dt}\textstyle\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx+c_9\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx\\
&\leq c_{10}\textstyle\int_{\Omega(\beta(u)\geq M_2)}|g(x,t)|^2dx.
\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}
&\frac{d}{dt}[(t-\tau)e^{c_9M_2^{\frac{q-r-2}{r+1}}t}\textstyle\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx]\\
&\leq e^{c_9M_2^{\frac{q-r-2}{r+1}}t}\textstyle\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx+c_{10}(t-\tau)e^{c_9M_2^{\frac{q-r-2}{r+1}}t}\int_{\Omega(\beta(u)\geq M_2)}|g(x,t)|^2dx.
\end{aligned}\tag{4.20}$$

Integrating from τ to t, we have

$$\begin{aligned}
&\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx\\
&\le \tfrac{1}{t-\tau}e^{-c_9M_2^{\frac{q-r-2}{r+1}}t}(\int_\tau^t e^{c_9M_2^{\frac{q-r-2}{r+1}}s}\int_\Omega |(\beta(u(s))-M_2)_+|^{\frac{q}{r+1}}dxds\\
&+c_{10}\int_\tau^t(s-\tau)e^{c_9M_2^{\frac{q-r-2}{r+1}}s}\int_{\Omega(\beta(u(s))\ge M_2)}|g(x,t)|^2dxds)\\
&\le c_{11}(\tfrac{1}{t-\tau}\int_\tau^t e^{-c_9M_2^{\frac{q-r-2}{r+1}}(t-s)}|u(s)|_q^q ds+\int_\tau^t e^{-c_9M_2^{\frac{q-r-2}{r+1}}(t-s)}\int_{\Omega(\beta(u(s))\ge M_2)}|g(x,t)|^2dxds)
\end{aligned} \tag{4.21}$$

Using (4.3), we obtain

$$\begin{aligned}
&\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx\\
&\le c_{12}[\frac{e^{-\delta(t-\tau)}}{(c_9M_2^{\frac{q-r-2}{r+1}}-\delta)(t-\tau)}|u_\tau|_{r+2}^{r+2}+\frac{1}{c_9M_2^{\frac{q-r-2}{r+1}}(t-\tau)}+\frac{e^{-\delta t}}{c_9M_2^{\frac{q-r-2}{r+1}}}\int_{-\infty}^t e^{\delta s}|g(x,s)|_2^2ds\\
&+\frac{e^{-\delta t}}{c_9M_2^{\frac{q-r-2}{r+1}}}\int_{-\infty}^t\int_{-\infty}^s e^{\delta s}|g(x,r)|_2^2drds].
\end{aligned} \tag{4.22}$$

Obviously, for any $\varepsilon>0$, $p>r+2$, there exist $M>0$, $\tau_0<t$, for any $M_2>M$, $\tau<\tau_0$, we have

$$\int_\Omega |(\beta(u)-M_2)_+|^{\frac{q}{r+1}}dx<\varepsilon. \tag{4.23}$$

Hence

$$\int_{\Omega(\beta(u)\ge M)}((|\beta(u|)-M)_+)^{\frac{q}{r+1}}dx<\varepsilon. \tag{4.24}$$

Repeat the same step above, mulitplying (1.1) with $|(\beta(u)+M_2)_-|^{\frac{q}{r+1}-2}(\beta(u)+M_2)_-$, we get

$$\int_{\Omega(\beta(u)\le -M)}(|\beta(u(t))|-M)^{\frac{q}{r+1}}dx<\varepsilon, \tag{4.25}$$

where $(\beta(u)+M_2)_-=\begin{cases}\beta(u)+M_2, & \beta(u)\le -M_2,\\ 0, & \beta(u)>-M_2.\end{cases}$

Combining (4.24) and (4.25), we have

$$\begin{aligned}
&\int_{\Omega(|\beta(u)|\ge 2M)}|\beta(u(t))|^{\frac{q}{r+1}}dx\\
&=\int_{\Omega(|\beta(u)|\ge 2M)}(|\beta(u(t))|-M+M)^{\frac{q}{r+1}}dx\\
&\le c_{13}(\int_{\Omega(|\beta(u)|\ge 2M)}(|\beta(u(t))|-M)^{\frac{q}{r+1}}dx+\int_{\Omega(|\beta(u)|\ge 2M)}M^{\frac{q}{r+1}}dx)\\
&\le c_{13}(\int_{\Omega(|\beta(u)|\ge M)}(|\beta(u(t))|-M)^{\frac{q}{r+1}}dx+\int_{\Omega(|\beta(u)|\ge M)}(|\beta(u(t)|-M)^{\frac{q}{r+1}}dx)\\
&\le c_{14}\varepsilon.
\end{aligned}$$

Thanks to (4.17), we conclude

$$\int_{\Omega(|u(t)|\geq M)} |u(t)|^q dx < c_{15}\varepsilon.$$

Hence, the (2) of Theorem 2.2 is satisfied, which say that the process is pullback ω-D-limit compact in $L^q(\Omega)$.

For $p = r+2$, the constant $M_2^{\frac{q-r-2}{r+1}} = 1$, the above prove can not obtain the right hand side of (4.22) tend to 0. For our purpose, we add another condition,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall e \subset \Omega, me < \delta, \forall t \in \mathbb{R}, \int_e |g(x,t)|^2 dx < \varepsilon. \tag{4.26}$$

Using (4.13) and (4.17), we get

$$\frac{\beta_1}{2} M_2^{\frac{q}{r+1}} m(\Omega(|\beta(u)| \geq M_2)) \leq \int_{\Omega(|\beta(u)|\geq M_2)} |\beta(t)|^{\frac{q}{r+1}} dx < +\infty.$$

We find that there exists $M > 0$, for any $M_2 > M$, $m(\Omega(|\beta(u)| \geq M_2)) < \varepsilon$, using (4.19) and the previous proof method, we also get

$$\int_{\Omega(|u(t)|\geq M)} |u(t)|^q dx < c_{16}\varepsilon,$$

which say that the process is pullback ω-D-limit compact in $L^q(\Omega)$, so we have the following theorem.

Theorem 4.4 *Assume that the conditions (1.2)-(1.5) are satisfied, $g(x,t)$ satisfy (4.2) and (4.26), $q = r+2$. Then the process $U(t,\tau)$ generated by (1.1) exists a pullback $\mathcal{D}$-attractor in $L^q(\Omega)$.*

References

[1] V.Chepyzhov, M.vishik, Attractors for Equations of Mathematics physics, Vol.49 of American Mathematical Society Colloquium Publications, AMS, Providence, RI, 2002.

[2] X.Dong, Y.Qin, Strong pullback attractors for a nonclassical diffusion equation, Discrete and Continuous Dynamical Systems-B, 2022, doi: 10.3934/dcdsb.2021313.

[3] A.Eden, B.Michaux, J.M.Rakotoson, Doubly Nonlinear Parabolic-Type Equations as Dynamical Systems, J. Diff. Eqns. 1991, 3(1), 87-131.

[4] A.Eden J.M.Rakotoson, Exponential attractors for a doubly nonlinear equation, J. Math. Anal. Appl. 1994, 185(2), 321-339.

[5] A.Friedman, S.Kamin, The asymptotic behavior of a gas in n-dimensional porous medium, Trans. AM.Math.Soc. 1980, 262,551-563.

[6] D.Li, Q.Chang and C.Sun, Pullback attractors for a critical degenerate wave equation with time-dependent damping, Nonl. Anal.: Real World Applications, 2022, 63, 103421.

[7] Y.Li, S.Wang and H.Wu, Pullback attractors for non-autonomous reaction-diffusion equation in L^p, Appl. Math. Comp., 2009, 207, 373-379.

[8] Y.Li, S.Wang and Y.Zhang, Global attractor for doubly nonlinear parabolic equation, ISRM AppL. Math. 2012, 956291.

[9] Y.Li, C.Zhong, Pullback attractors for the norm-to-weak continuous process and application to the non-autonomous reaction-diffusion equations, Appl. Math. Comp., 2007, 190, 1020-1029.

[10] A.Miranville, Finite dimensional global attractor for a class of doubly nonlinear parabolic equation, Cent. Euro. J. Math. 2006, 4, 163-182.

[11] Y.Qin, B.Yang, Existence and regularity of time-dependent pullback attractors for the non-autonomous nonclassical diffusion equations, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 2021:1-18.

[12] J.C.Robinson, Infinite-dimensional dynamical systems as introduction to dissipative parabolic PDEs and the Theory of Global Attractors, Cambridge University Press,2001.

[13] K.Shirakawa, Large time behavior for doubly nonlinear systems generated by subdifferentials, Adv. Math. Sci. Appl., 2000, 10, 77-92.

[14] R.Teman, Infinite-dimensional Dynamical Systems in Mechanics and Physics, Springer, New York, 1997.

[15] H.L.Van, Pullback attractors for a class of non-autonomous semilinear parabolic equations with infinite delay, Acta Math. Univ. Comenianae, 2021, 3, 289-308.

[16] J.Wei, Y.Li, Pullback exponential attractors for non-autonomous abstract retarded evolution equations, J. Appl. Anal. Comp., 2022, 12(4), 1595-1612.

[17] C.Zhong, M.Yang, C.Sun, The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations, J. Diff. Eqns. 2006, 223, 367-399.

[18] K.Zhu, Y.Xie, F.Zhou and Q.Zhou, Pullback attractors for p-Laplacian equations with delays, J. Math. Phys., 2021, 62(2), 022702.