Product Transform

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Consider a transform that acts on a function to extract the product series coefficient. For example if a function allows a series expansion

$$f(x) = \prod_{k=1}^{\infty}(1 + a_k x^k) = \sum_{k=0}^{\infty} b_k x^k \tag{1}$$

then we define the transform K of f to be

$$K_x[f](s) = a_s \tag{2}$$

likewise the inverse transform gives

$$K_s^{-1}[a_s](x) = f(x) \tag{3}$$

then by definition of the Pochhammer-q function we have

$$K_x[\frac{1}{2}(-1, x)_\infty] = 1 \tag{4}$$

$$K_x[(q, q)_\infty] = -1 \tag{5}$$

then certain functions with number theoretic potential can be defined by the inverse transform

$$K_s^{-1}[\lambda(s)] \tag{6}$$

$$K_s^{-1}[\mu(s)] \tag{7}$$

$$K_s^{-1}[\phi(s)] \tag{8}$$

$$K_s^{-1}[\Omega(s)] \tag{9}$$

$$\tag{10}$$

for functions $\lambda(s), \mu(s), ..$ the Liouville function, Moebius function, totient function etc. If we attempt to extract the coefficeints of a well known function, for example

$$\exp(x) = (1 + 1x)(1 + \frac{x^2}{2})(1 - \frac{x^3}{3})(1 + \frac{3x^4}{8})(1 - \frac{x^5}{5})(1 + \frac{13x^6}{72}) \cdots \tag{11}$$

we find that the coefficients of odd prime powers p are $-1/p$. We find that

$$\mathcal{K}_x\left[\frac{1}{1-x}\right](s) = \chi_{2^k}(s) \tag{12}$$

which is the characteristic function of non-zero powers of 2, which comes from the interpretation of that number of ways to partition each number into powers of 2 is 1.

We have

$$b_7 = (a_1a_2a_4 + a_3a_4 + a_2a_5 + a_1a_6 + a_7) \tag{13}$$

which is clearly a sum over the ways to make 7 from unique choices of k. Whereas terms like

$$a_5 = b_5 - b_2b_3 + b_1b_2b_2 - b_1b_4 + b_1b_1b_3 - b_1b_1b_1b_2 \tag{14}$$

where the signs seem to correspond to the number of terms and repeats are now allowed. This gives

$$b_n = \left(\sum_{k_1=1}^{n}[k_1 = n]a_{k_1}\right) + \left(\sum_{k_1=1}^{n}\sum_{k_2>k_1}^{n}[k_1 + k_2 = n]a_{k_1}a_{k_2}\right) + \cdots \tag{15}$$

and then

$$a_n = \left(\sum_{k_1=1}^{n}[k_1 = n]b_{k_1}\right) - \left(\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}[k_1 + k_2 = n]g_{k_1k_2}b_{k_1}b_{k_2}\right) + \cdots \tag{16}$$

where the g_{kl} correct for multiplicities.