

# The special least squares solutions of the complex matrix equation $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$

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## Abstract

In this paper, we study the problem of least square solutions of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ . First, by using the real representation method of the complex matrix, we present the real representation form of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ . In combination with the special structure of the real representation matrix of the complex matrix, the vec operator of the matrix, the Kronecker product of the matrix and the MP inverse property of the matrix, we obtain the Hermitian least squares solution of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ , and derive the Hermitian minimum norm least square solution, the real symmetric minimum norm least square solution and the real dissymmetric minimum norm least square solution of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ . At last, we give the expressions of three minimum norm least squares solutions and their corresponding algorithms, respectively.

The special least squares solutions of the complex matrix equation

$$(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$$

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**Abstract** In this paper, we study the problem of least square solutions of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ . First, by using the real representation method of the complex matrix, we present the real representation form of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ . In combination with the special structure of the real representation matrix of the complex matrix, the vec operator of the matrix, the Kronecker product of the matrix and the MP inverse property of the matrix, we obtain the Hermitian least squares solution of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ , and derive the Hermitian minimum norm least square solution, the real symmetric minimum norm least square solution and the real dissymmetric minimum norm least square solution of the complex matrix equation  $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$ . At last, we give the expressions of three minimum norm least squares solutions and their corresponding algorithms, respectively.

**Keywords** Matrix equation; Hermitian least squares solution; The real symmetric least square solution; The real opposition least square solution; Real representation matrix.

## 1. Introduction

Matrix equations have a wide range of applications, which the calculation of special least squares solution is always a research hotspot in the field of numerical algebra. The least squares solutions of complex matrix equations have been studied extensively<sup>[1]-[11]</sup>. For example, [1,2] by using the real representation method of the complex matrix, Zhang et al. study the Hermitian minimum norm least square solution of  $AXB + CXD = E$  and  $(AXB, CXD) = (E, F)$ , respectively; [3] by the Kronecker product of the Matrix-Vector and the MP inverse property of the matrix, Wang et al. proposed a direct method to solve the least squares solution of the complex matrix equation  $(AXB, CXD) = (E, F)$ ; [4] Yuan et al. proposed a new method to solve the Hermitian least square solution of the complex matrix equation, and gave the expression and algorithm of the least square solution of the Hermitian least square solution and the Hermitian minimum norm least square solution, et al.

This paper study the problem of least square solutions of complex matrix equations

$$(AXB, CXD) = (E, F). \quad (1)$$

In this paper, we use the following notations.

Let  $R^{m \times n}$ ,  $C^{m \times n}$ ,  $HC^{n \times n}$ ,  $SR^{n \times n}$ ,  $ASR^{n \times n}$  be the sets of  $m \times n$  real matrices, be the sets of  $m \times n$  real matrices, be the sets of  $m \times n$  Hermitian complex matrices, be the sets of  $m \times n$  real symmetric matrices, be the sets of  $m \times n$  real dissymmetric matrices.  $R^n$  be the sets of  $n$  column vector,

$A^T A^H$  denote the transpose and the conjugate transpose of matrix  $A$ , respectively.  $A^+$  denote the MP inverse of matrix  $A$ ,  $I_n = (e_1, e_2, \dots, e_n)$  denote the identity matrix of order  $n$ ,  $\|A\|$  denote Frobenius norm of complex matrices

For (1), we mainly discuss the following three issues.

**Problem 1** Let  $A, C \in C^{m \times n}$ ,  $B, D \in C^{n \times s}$ ,  $E, F \in C^{m \times s}$ , the Hermitian minimum norm least squares solution of (1) denote by  $C_H$ , that is

$$C_H = \{X | X \in HC^{n \times n}, \|AXB - E\| + \|CXD - F\| = \min\},$$

make sure  $C_H$ , and solve the Hermitian minimum norm least square solution  $\hat{X} \in C_S$  satisfying  $\|\hat{X}\| = \min_{X \in C_H} \|X\|$ .

**Problem 2** Let  $A, C \in C^{m \times n}$ ,  $B, D \in C^{n \times s}$ ,  $E, F \in C^{m \times s}$ , the real symmetric minimum norm least squares solution of (1) denote by  $C_S$ , that is

$$C_S = \{X | X \in SR^{n \times n}, \|AXB - E\| + \|CXD - F\| = \min\},$$

make sure  $C_S$ , and solve the symmetric minimum norm least square solution  $\hat{X} \in C_S$  satisfying  $\|\hat{X}\| = \min_{X \in C_S} \|X\|$ .

**Problem 3** Let  $A, C \in C^{m \times n}$ ,  $B, D \in C^{n \times s}$ ,  $E, F \in C^{m \times s}$ , the real dissymmetric minimum norm least squares solution of (1) denote by  $C_A$ , that is

$$C_A = \{X | X \in ASR^{n \times n}, \|AXB - E\| + \|CXD - F\| = \min\},$$

make sure  $C_A$ , and solve the symmetric minimum norm least square solution  $\hat{X} \in C_A$  satisfying  $\|\hat{X}\| = \min_{X \in C_A} \|X\|$ .

## 2 Preliminaries

In this paper, we need the following concepts and conclusions:

**Definition 1** <sup>[12]</sup> For  $A = (a_{ij}) \in C^{m \times n}$ ,  $B = (b_{ij}) \in C^{p \times q}$ , called the following block matrix

$$A \otimes B = (a_{ij}B) \in C^{mp \times nq}$$

is the Kronecker product of matrix  $A$  and matrix  $B$ .

**Definition 2** <sup>[1]</sup> For  $A = A_1 + A_2i \in C^{m \times n}$ , which  $A_1, A_2 \in R^{m \times n}$ .

$$A^R =$$

$$\begin{pmatrix} A_1 & amp; -A_2 \\ A_2 & amp; A_1 \end{pmatrix} \in R^{2m \times 2n},$$

is called to the real representation matrix of  $A$ , Let  $A_c^R$  denote the first column block

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \text{ of } A^R.$$

$A^R$  and  $A_c^R$  have some properties as follows:

**Lemma 1** <sup>[1]</sup> For  $A, B \in C^{m \times n}$ ,  $k \in C$ . Then the following properties hold.

$$(1) A = B \Leftrightarrow A^R = B^R; (2) (A + B)^R A^R + B^R; (3) (kA)^R = kA^R.$$

**Lemma 2** <sup>[1]</sup> For  $A, B \in C^{m \times n}$ ,  $D \in C^{n \times s}$ ,  $k \in C$ . Then the following properties hold.

$$(1) A = B \Leftrightarrow A_c^R = B_c^R; (2) (kA)_c^R = kA_c^R, \\ (3) (AD)_c^R = A^R D_c^R; (4) (A + B)_c^R A_c^R + B_c^R.$$

For  $A \in C^{m \times n}$ , have  $\|A\| = \frac{1}{2} \|A^R\| = \|A_c^R\|$  and .

The vec operator of real matrix have some properties as follows:

**Lemma 3** <sup>[12]</sup> For  $A \in R^{m \times n}$ ,  $B \in R^{n \times p}$ ,  $C \in R^{p \times q}$ . Then the following properties hold.

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B).$$

For complex matrix  $X = X_0 + X_1i$ , the following relationship between  $X^R X_C^R X_0$  and  $X_1$

as follows:

**Lemma 4** <sup>[1]</sup> For  $X = X_0 + X_1i \in C^{m \times n}$ . Then the following properties hold. (1)  $\text{vec}(X^R) = H \text{vec}(X_C^R)$ ,

in which  $H =$

$$\begin{pmatrix} \text{diag}(I_{2n}, \dots, I_{2n}) \\ \text{diag}(Q_n, \dots, Q_n) \end{pmatrix} \in R^{4mn \times 2mn}, Q_n =$$

$$\begin{pmatrix} 0 & amp; -I_n \\ I_n & amp; 0 \end{pmatrix}.$$

$$(2) \text{vec}(X_c^R) = K \begin{pmatrix} \text{vec} X_0 \\ \text{vec} X_1 \end{pmatrix},$$

in which  $K =$

$$\begin{pmatrix} I_n & \text{amp}; 0 & & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\ 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; I_n & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\ 0 & \text{amp}; I_n & & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\ 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; I_n & \text{amp}; \cdots & \text{amp}; 0 \\ \cdots & \text{amp}; \cdots & & \cdots & \text{amp}; \cdots & \text{amp}; \cdots & \cdots & \text{amp}; \cdots & \text{amp}; \cdots \\ 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; I_n & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\ 0 & \text{amp}; 0 & & \cdots & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; I_n \end{pmatrix} \in C^{2n^2 \times 2n^2}.$$

(3) if  $X = X_0$ , have

$$\text{vec}(X_c^R) = \text{Vec}(X_0),$$

in which

$$V =$$

$$\begin{pmatrix} I_n & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\ 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\ 0 & I_n & \cdots & \text{amp}; 0 \\ 0 & 0 & \cdots & \text{amp}; 0 \\ \cdots & \text{amp}; \cdots & \text{amp}; \cdots & \cdots \\ 0 & 0 & \cdots & \text{amp}; I_n \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in C^{2n^2 \times n^2}.$$

By calculation, give definition and relationship between the independent elements of real (dis)symmetric matrix and vec of its independent elements:

**Definition 3** <sup>[1]</sup> For  $X = (x_{ij}) \in \text{SR}^{n \times n}$ , let  $\alpha_1 = (x_{11}, x_{21}, \dots, x_{n1})$ ,  $\alpha_2 = (x_{22}, x_{32},$

$\dots, x_{n2})$ ,  $\dots$ ,  $\alpha_{n-1} = (x_{(n-1)(n-1)}, x_{n(n-1)})$ ,  $\alpha_n = x_{nn}$ , the elements of  $\alpha_1, \alpha_2, \dots, \alpha_n$  are called as the independent elements of real symmetric matrix  $X = (x_{ij}) \in \text{SR}^{n \times n}$ , for short independent entry. Denoted by

$$\text{vec}_S(X) = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \in R^{\frac{n(n+1)}{2}}$$

is called a column straight of the independent elements of real symmetric matrix  $X = (x_{ij}) \in \text{SR}^{n \times n}$ .

**Definition 4** <sup>[1]</sup> For arbitrary real dissymmetric matrix  $X = (x_{ij}) \in \text{ASR}^{n \times n}$ , and let  $\beta_1 = (x_{21}, x_{31}, \dots, x_{n1})$ ,  $\beta_2 = (x_{32}, x_{42}, \dots, x_{n2})$ ,  $\dots$ ,  $\beta_{n-1} = (x_{(n-1)(n-2)}, x_{n(n-2)})$ ,  $\beta_n = x_{n(n-1)}$ , the elements of  $\beta_1, \beta_2, \dots, \beta_n$  are called as the independent elements of real dissymmetric matrix  $X = (x_{ij}) \in \text{ASR}^{n \times n}$ . Denoted by

$$\text{ves}_A(X) = (\beta_1, \beta_2, \dots, \beta_n)^T \in R^{\frac{n(n-1)}{2}}$$

is called a column straight of the independent elements of real dissymmetric matrix

**Lemma 5** <sup>[1]</sup> For  $X = (x_{ij}) \in \text{SR}^{n \times n}$ . Then

$$X \in \text{SR}^{n \times n} \Leftrightarrow \text{ves}(X) = K_n \text{ves}_S(X), K_n \in R^{n^2 \times \frac{n(n+1)}{2}},$$

in which  $K_n =$

$$\begin{pmatrix} e_1 & \text{amp}; e_2 & \text{amp}; e_3 & \text{amp}; \cdots & \text{amp}; e_{n-1} \\ 0 & \text{amp}; e_1 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\ 0 & 0 & e_1 & \text{amp}; \cdots & \text{amp}; 0 \\ \cdots & \text{amp}; \cdots & \text{amp}; \cdots & \text{amp}; \cdots & \text{amp}; \cdots \\ 0 & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; e_1 \\ 0 & 0 & 0 & \text{amp}; \cdots & \text{amp}; 0 \end{pmatrix}$$

$$\begin{array}{cccccc}
 e_n & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\
 0 & \text{amp}; e_2 & \text{amp}; e_3 & \text{amp}; \cdots & \text{amp}; e_{n-1} \\
 0 & 0 & 0 & \cdots & 0 \\
 \cdots & \cdots & \cdots & \text{amp}; & \text{amp}; \\
 0 & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; e_2 \\
 e_1 & 0 & 0 & \text{amp}; \cdots & \text{amp}; 0 \\
 \\ 
 0 & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; 0 \\
 e_n & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; 0 \\
 0 & \cdots & 0 & \text{amp}; 0 & \text{amp}; 0 \\
 \vdots & \cdots & \cdots & \text{amp}; \cdots & \text{amp}; \cdots \\
 0 & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; e_n & \text{amp}; 0 \\
 e_2 & \cdots & 0 & \text{amp}; e_{n-1} & \text{amp}; e_n
 \end{array}$$

**Lemma 6** <sup>[1]</sup> For  $X = (x_{ij}) \in R^{n \times n}$ . Then

$$X \in \text{ASR}^{n \times n} \Leftrightarrow \text{ves}(X) = L_n \text{ves}_A(X), L_n \in R^{n^2 \times \frac{n(n+1)}{2}},$$

in which  $L_n =$  (

$$\begin{array}{cccccc}
 e_2 & \text{amp}; e_3 & \text{amp}; \cdots & \text{amp}; e_{n-1} & \text{amp}; e_n \\
 -e_1 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; 0 \\
 0 & -e_1 & \cdots & \text{amp}; 0 & \text{amp}; 0 \\
 \cdots & \cdots & \cdots & \text{amp}; & \text{amp}; \\
 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; -e_1 & \text{amp}; 0 \\
 0 & 0 & \cdots & \text{amp}; 0 & \text{amp}; -e_1 \\
 \\ 
 0 & \text{amp}; \cdots & \text{amp}; 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\
 e_3 & \text{amp}; \cdots & \text{amp}; e_{n-1} & \text{amp}; e_n & \text{amp}; \cdots & \text{amp}; 0 \\
 -e_2 & 0 & 0 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; 0 \\
 \cdots & \cdots & \cdots & \text{amp}; & \text{amp}; & \text{amp}; \\
 0 & \text{amp}; 0 & \text{amp}; -e_2 & \text{amp}; 0 & \text{amp}; \cdots & \text{amp}; e_n \\
 0 & 0 & 0 & \text{amp}; -e_2 & \text{amp}; & \text{amp}; -e_{n-1}
 \end{array}$$

**Lemma 7** <sup>[1]</sup> Let  $A \in R^{m \times n}$ ,  $b \in R^n$ . Then the least square solutions of the incompatible linear equations  $Ax = b$  is  $x = A^+b + (I - AA^+)y$ , where  $y \in R^n$  is arbitrary vector.

## 2 The minimum norm least square solution of complex matrix equations $(AXB, CXD) = (E, F)$

In this paper, the special structure of the real representation matrix of the complex matrix and the MP inverse property of the matrix, we study Hermitian least square solutions of complex matrix equations(1), and give the expressions of Hermitian minimum norm least square solution, real symmetric minimum norm least square solution and real dissymmetric minimum norm least square solution.

**Theorem 1** Let  $A, C \in C^{m \times n}$ ,  $B, D \in C^{n \times s}$ ,  $E, F \in C^{m \times s}$ ,  $X = X_0 + X_1i$ , where  $X_0 \in \text{SR}^{n \times n}$ ,  $X_1 \in \text{ASR}^{n \times n}$ . For complex matrix equations (1), the Hermitian minimum norm least square solution  $X$  satisfies

$$\left( \frac{\text{ves}_S(X_0)}{\text{ves}_A(X_1)} \right) = \left[ \left( \frac{(B^R)^T \otimes A^R}{(D^R)^T \otimes C^R} \right) \text{HVK}_n \right]^+ \left( \frac{E^R}{F^R} \right).$$

Proof. By lemma 1-6, we get

$$\|AXB - E\| + \|CXD - F\|$$

$$\begin{aligned}
&= \|A^R X^R B_c^R - E_c^R\| + \|C^R X^R D_c^R - F_c^R\| \\
&= \left\| \left( (B_c^R)^T \otimes A^R \right) \text{vec} (X^R) - \text{vec} (E_c^R) \right\| + \left\| \left( (D_c^R)^T \otimes C^R \right) \text{vec} (X^R) - \text{vec} (F_c^R) \right\| \\
&= \left\| \left( \frac{\left( (B_c^R)^T \otimes A^R \right) \text{vec} (X^R) - \text{vec} (E_c^R)}{\left( (D_c^R)^T \otimes C^R \right) \text{vec} (X^R) - \text{vec} (F_c^R)} \right) \right\| \\
&= \left\| \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \text{vec} (X^R) - \left( \frac{E_c^R}{F_c^R} \right) \right\| \\
&= \left\| \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \text{HK} \begin{pmatrix} \text{vec} X_0 \\ \text{vec} X_1 \end{pmatrix} - \left( \frac{E_c^R}{F_c^R} \right) \right\| \\
&= \left\| \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \text{HK} \begin{pmatrix} K_n \text{ves}_S (X_0) \\ L_n \text{ves}_A (X_1) \end{pmatrix} - \left( \frac{E_c^R}{F_c^R} \right) \right\|
\end{aligned}$$

Where  $H, K, K_n, L_n$  are defined in the form of lemma 4-6. Let  $G = \text{diag} (K_n, L_n)$ , then

$$\|AXB - E\| + \|CXD - F\| = \left\| \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \right) \text{HKG} \right\|$$

$$\begin{pmatrix} \text{ves}_S (X_0) \\ \text{ves}_A (X_1) \end{pmatrix} - \left( \frac{E_c^R}{F_c^R} \right),$$

So calculating the least squares solution of the complex matrix equation (1) is equivalent to calculating the least squares solution of the following real linear system

$$\begin{aligned}
&\left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \right) \text{HKG} \\
&\begin{pmatrix} \text{ves}_S (X_0) \\ \text{ves}_A (X_1) \end{pmatrix} = \left( \frac{E_c^R}{F_c^R} \right).
\end{aligned}$$

By lemma 7, the set of Hermitian least square solution of (1) is denoted by

$$C_H = \{X | \begin{pmatrix} \text{ves}_S (X_0) \\ \text{ves}_A (X_1) \end{pmatrix} = \left[ \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \right) \text{HKG} \right]^+ + \left[ I - \left[ \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \right) \text{HKG} \right]^+ \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \right) \text{HKG} \right] y\},$$

Where  $y$  is arbitrary vector. In particular, the Hermitian minimum norm least square solution of (1) is can be expressed as  $X = X_0 + X_1 i \in C^{n \times n}$ , in which  $X_0, X_1$  satisfies

$$\begin{pmatrix} \text{ves}_S (X_0) \\ \text{ves}_A (X_1) \end{pmatrix} = \left[ \left( \frac{\left( (B_c^R)^T \otimes A^R \right)}{\left( (D_c^R)^T \otimes C^R \right)} \right) \text{HKG} \right]^+ \left( \frac{E_c^R}{F_c^R} \right) \text{证毕.}$$

**Corollary 1** Let  $A, C \in C^{m \times n}, B, D \in C^{n \times s}, E, F \in C^{m \times s}, X = X_0 + X_1 i$ , in which  $X_1 = 0, X_0 \in \text{SR}^{n \times n}$ . For complex matrix equations (1), the real symmetric minimum norm least square solution  $X = X_0$  satisfies

$$\text{ves}_S(X_0) = \left[ \left( \frac{(B_c^R)^T \otimes A^R}{(D_c^R)^T \otimes C^R} \right) \text{HVK}_n \right]^+ \left( \frac{E_c^R}{F_c^R} \right),$$

Where  $H, V, K_n$  are defined in the form of lemma 4-6.

**Corollary 2** Let  $A, C \in C^{m \times n}, B, D \in C^{n \times s}, E, F \in C^{m \times s}, X = X_0 + X_1 i$ , in which  $X_1 = 0, X_0 \in \text{ASR}^{n \times n}$ . For complex matrix equations (1), the real dissymmetric minimum norm least square solution  $X = X_0$  satisfies

$$\text{ves}_A(X_0) = \left[ \left( \frac{(B_c^R)^T \otimes A^R}{(D_c^R)^T \otimes C^R} \right) \text{HVL}_n \right]^+ \left( \frac{E_c^R}{F_c^R} \right),$$

Where  $H, V, L_n$  are defined in the form of lemma 4-6.

### 3 Algorithm

For complex matrix equations (1), in which  $A, C \in C^{m \times n}, B, D \in C^{n \times s}, E, F \in C^{m \times s}$ . the following Theorem 1, Corollary 1 and Corollary 2, we propose the following algorithms for solving the Hermitian minimum norm least square solution, the real symmetric minimum norm least square solution and the real dissymmetric minimum norm least square solution of (1).

**Algorithm 1** (solving the Hermitian minimum norm least square solution of (1))

- (1) Input  $A, B, C, D, E, F, H, K, G$ ,
- (2) Form  $A^R, C^R, B_c^R, D_c^R, E_c^R, F_c^R$ ,
- (3) Calculat

$$\begin{pmatrix} \text{ves}_S(X_0) \\ \text{ves}_A(X_1) \end{pmatrix} = \left[ \left( \frac{(B_c^R)^T \otimes A^R}{(D_c^R)^T \otimes C^R} \right) H K G \right]^+ \begin{pmatrix} E_c^R \\ F_c^R \end{pmatrix}.$$

**Algorithm 2** (solving the real symmetric minimum norm least square solution of (1))

- (1) Input  $A, B, C, D, E, F, H, V, K_n$ ,
- (2) Form  $A^R, C^R, B_c^R, D_c^R, E_c^R, F_c^R$ ,
- (3) Calculat  $\text{ves}_S(X_0) = \left[ \left( \frac{(B_c^R)^T \otimes A^R}{(D_c^R)^T \otimes C^R} \right) \text{HVK}_n \right]^+ \begin{pmatrix} E_c^R \\ F_c^R \end{pmatrix}.$

**Algorithm 3** (solving the real dissymmetric minimum norm least square solution of (1))

- (1) Input  $A, B, C, D, E, F, H, V, L_n$ ,
- (2) Form  $A^R, C^R, B_c^R, D_c^R, E_c^R, F_c^R$ ,
- (3) Calculat  $\text{ves}_A(X_0) = \left[ \left( \frac{(B_c^R)^T \otimes A^R}{(D_c^R)^T \otimes C^R} \right) \text{HVL}_n \right]^+ \begin{pmatrix} E_c^R \\ F_c^R \end{pmatrix}.$

### 4 Conclusion

In this paper, by using the special structures of the real representation of complex matrix, the vec operator of the matrix, the Kronecker product of matrices and the properties of MP inverse of matrix, we transform the calculation problem of the special least squares solutions of complex matrix equations into the least squares problem of corresponding linear systems, and propose illustrate the effectiveness based on real matrix with the Hermitian minimum norm least square solution, the real symmetric minimum norm least square solution and the real dissymmetric minimum norm least square solution of the complex matrix equation  $(AXB, CXD) = (E, F)$ .

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