Mathematical aspects of fluid-multiferroic solid interaction problems

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Abstract

In the paper, we consider a three-dimensional model of fluid-solid interaction when a thermo-electro-magneto-elastic body occupying a bounded region Ω^+ is embedded in an inviscid fluid occupying an unbounded domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In this case, we have a six-dimensional thermo-electro-magneto-elastic field (the displacement vector with three components, electric potential, magnetic potential, and temperature distribution function) in the domain Ω^+, while we have a scalar acoustic pressure field in the unbounded domain Ω^-. The physical kinematic and dynamic relations are described mathematically by appropriate boundary and transmission conditions. With the help of the potential method and theory of pseudodifferential equations, we prove the uniqueness and existence theorems for the corresponding boundary-transmission problems in appropriate Sobolev-Slobodetskii and Hölder continuous function spaces.

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Introduction

The surge of interest in multiferroic materials over the past 15 years has been driven by their fascinating physical properties and huge potential for technological applications. Multiferroics belong to a newer class of thermo-electro-magneto-elastic materials in which ferromagnetic and ferroelectric properties occur simultaneously. Consequently, mathematical modeling related to multiferroic complex composite structures and the corresponding fluid-solid interaction problems became very important from the theoretical and practical points of view. Mathematically this type of interaction problems are described by non-standard boundary-transmission problems for different dimensional physical fields acting in adjacent domains. This type of interaction problems involving different dimensional physical fields appear in mathematical models of electro-magneto transducers, sensors, actuators, energy harvesters, servomechanisms, phased array microphones, ultrasound equipment, inkjet droplet actuators, sonar transducers, bioimaging, immunochemistry, and acousto-biotherapeutics (see, e.g., Neugschwandtner et al(*Journal of Applied Physics 89(8):4503–4511*, 2001), Safari et al(Safari A, 2008), Vopson(of Multiferroic Materials & *Critical Reviews in Solid State and Materials Sciences* 2015;40(4):223–250, 2015) and the references therein).

In this paper, we analyze a three-dimensional model of fluid-solid interaction, when a thermo-electro-magneto-elastic body occupying a bounded region Ω^+ is embedded in an inviscid fluid occupying an unbounded domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In the solid region we consider Green–Lindsay's generalized thermo-electro-magneto-elastic model. The essential feature of this model is that heat propagation has a finite speed in contrast to the classical model. In the case under consideration, we have a six-dimensional thermo-electro-magneto-elastic field in the solid region (the displacement vector with three components, electric potential, magnetic potential, and temperature distribution function), while in the fluid region we have a scalar acoustic pressure field. The physical kinematic and dynamic relations are described mathematically by appropriate boundary and transmission conditions. We consider the interaction problems for the so-called pseudo-oscillation equations, which are obtained form the corresponding dynamical equations by the Laplace transform. With the help of the potential method and theory of pseudodifferential equations, we prove the uniqueness and existence theorems for the corresponding boundary-transmission problems in appropriate Sobolev-Slobodetskii and Hölder continuous function spaces. We derive explicit representation formulas of solutions in the form of single layer potentials.

Many papers are devoted to the similar interaction problems, when in the solid region simpler mathematical models are considered. The Dirichlet type, Neumann type and mixed type steady state oscillation interaction problems of acoustic waves and piezoelectric structures are studied in the papers Chkadua(problems of interaction of different dimensional physical fields. *Journal of Physics: Conference Series* 2013;451:01, 2013), Chkadua et al(Chkadua G & piezoelectric structures. *Math. Meth. Appl. Sci.* 2015;38(11):2149-, 2015), Chkadua(Chkadua G. Solvability et al., 2017).

Similar interaction problems with the classical model of elasticity have been investigated by a number of authors. An exhaustive information concerning theoretical and numerical results, for the case when the both interacting media are isotropic, can be found in the references Bielak et al(Bielak J & boundary integral coupling methods for fluid-solid interaction. *Quart. Appl. Math.* 1991;49:107–119, 1991), Bielak et al(1991), Boström(of stationary acoustic waves by an elastic obstacle immersed, n.d.),(Boström A. Scattering of acoustic waves by a layered elastic obstacle in a fluid - an improved null field approach, 1984), Goswami et al(*J. Nondestruct. Eval. 9:101–112*, 1990), Hsiao et al(Hsiao GC & Hayes M. A. (eds): *Elastic Wave Propagation. IUTAM Symposium on Elastic Wave Propagation.* North-Holland, 1989), Hsiao(missing citation), Hsiao et al(Hsiao GC, 1917), Junger et al(1986), Kagawa et al(Kagawa Y, 1979), Luke et al(Luke CJ, 1995), Natroshvili et al(Natroshvili D & inverse fuid-structure interaction problems. *Rediconti di Matematica, Serie VII* 2000;20:57–92, 2000), Natroshvili et al(Natroshvili D & scalar fields. *Math. Meth. Appl. Sci.* 1996;19:1445–, 1996), Natroshvili et al(2005). Interaction problems of steady state oscillations for homogeneous and anisotropic elastic solids are analysed in the references Jentsch et al(1998; 1999), where the generalized Sommerfeld-Kupradze type radiation conditions for anisotropic solids are derived.

The present paper is organized as follows. In Section 2, we describe basic field equations in fluid and solid regions, introduce the partial differential operators of the generalized thermo-electro-magneto-elasticity theory (GTEME theory) in the solid region, formulate two type boundary-transmission problems, and prove the uniqueness theorems. It should be mentioned that in contrast to the classical case, the second order partial differential 6×6 matrix operators of the GTEME theory is neither positive definite nor formally self-adjoint. In Section 3, we introduce the scalar and vector layer potential operators associated with the corresponding differential operators in fluid and solid domains. We describe the jump properties of the layer potentials and introduce the corresponding boundary integral operators which play a crucial role in our further analysis. In Sections 4, we investigate the boundary-transmission problem (ID) formulated in Section 2 containing the Dirichlet type conditions on the interface surface for the electric potential, magnetic potential, and temperature function. Using the potential method, this problem is reduced to the equivalent system of pseudodifferential equations. It is shown that the corresponding pseudodifferential operator is strongly elliptic Fredholm operator with zero index and trivial null space. Therefore the pseudodifferential operator is invertible and the corresponding interaction problem (ID) is unconditionally solvable. In section 5, we investigate the boundary-transmission problem (IN) formulated in Section 2 containing the Neumann type conditions on the interface surface for the electric displacement vector, magnetic induction vector, and heat flux vector. In this case, the corresponding strongly elliptic pseudodifferential operator is again Fredholm operator with zero index and two dimensional null space. The explicit solutions of the adjoint pseudodifferential equation is found and the necessary and sufficient conditions for the problem (IN) to be solvable is written explicitly. Finally, in Appendix, for the readers convenience, we present some known results about the jump and mapping properties of scalar and vector potential operators employed in the main text of the paper.

Basic Equations and Operators, Statement of Problems, and Uniqueness Theorems

Generalized thermo-electro-magneto elastic field

The basic linear system of pseudo-oscillation equations for the thermo-electro-magneto-elasticity theory under Green–Lindsay's model is obtained from the corresponding dynamical equations by the Laplace transform and in matrix form reads as follows (see Straughan (missing citation), Aouadi (missing citation), Green et al (missing citation) and the references therein)

$$A(\partial,\tau)U(x,\tau)=\Phi(x,\tau),$$

where $U=(u,\varphi,\psi,\theta)^\top=(u_1,\cdots,u_6)^\top$, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $\varphi=u_4$ is the electric potential, $\psi=u_5$ is the magnetic potential, $\theta=u_6$ is the temperature distribution, $\Phi=(\Phi_1,\cdots,\Phi_6)^\top$ is a given vector-function, and $A(\partial,\tau)$ is the matrix differential operator

$$A(\partial_x,\tau)=[A_{pq}(\partial_x,\tau)]_{6\times 6}:=$$

$$\begin{bmatrix} [c_{rjkl}\partial_j\partial_l-\rho_1\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau)m_j\partial_j \\ [-\tau\lambda_{kl}\partial_l]_{1\times 3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l-\tau^2 h_0-\tau d_0 \end{bmatrix}_{6\times 6}.$$
(1)

where $\tau=\sigma+i\omega$ is a complex parameter with $\sigma>\sigma_0\geq 0$ and $\omega\in\mathbb{R}$), δ_{jk} is the Kronecker symbol, and summation over repeated indices is meant from 1 to 3, if not stated otherwise. Here and in what follows we employ the following notation for the material parameters: ρ_1 – the mass density, c_{rjkl} – the elastic constants, e_{jkl} – the piezoelectric constants, q_{jkl} – the piezomagnetic constants, $\varkappa_{jk}$ – the dielectric (permittivity) constants, μ_{jk} – the magnetic permeability constants, a_{jk} – the electromagnetic coupling coefficients, p_j, m_j, and λ_{rj} – coupling coefficients connecting dissimilar fields, η_{jk} – the heat conductivity coefficients, ν_0 and h_0 – two relaxation times, a_0 and d_0 – constitutive coefficients.
The constants involved in the above equations satisfy the symmetry conditions:

$$c_{rjkl}=c_{jrkl}=c_{klrj},\ \ e_{klj}=e_{kjl},\ \ q_{klj}=q_{kjl},$$
$$\varkappa_{kj}=\varkappa_{jk},\ \ \lambda_{kj}=\lambda_{jk},\ \ \mu_{kj}=\mu_{jk},\ \ a_{kj}=a_{jk},\ \ \eta_{kj}=\eta_{jk},\ \ r,j,k,l=1,2,3.$$
(2)

Some authors require more extended symmetry conditions for piezoelectric and piezomagnetic constants: $e_{klj} = e_{kjl} = e_{ljk}$, $q_{klj} = q_{kjl} = q_{ljk}$ (see, e.g., Li (missing citation), (missing citation), Aouadi (missing citation), (missing citation)). However in our further analysis we will require only the symmetry properties described in (2). From physical considerations it follows that (see, e.g., Nowacki(missing citation), Li(missing citation), Aouadi (missing citation), Straughan (missing citation), Green et al(missing citation)):

$$c_{rjkl}\xi_{rj}\xi_{kl} \geq \delta_0 \xi_{kl}\,\xi_{kl}, \;\; \varkappa_{kj}\xi_k\xi_j \geq \delta_1|\xi|^2, \;\; \mu_{kj}\xi_k\xi_j \geq \delta_2|\xi|^2, \;\; \eta_{kj}\xi_k\xi_j \geq \delta_3|\xi|^2,$$
$$\text{for all } \; \xi_{kj} = \xi_{jk} \in \mathbb{R} \; \text{ and for all } \; \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,$$
$$\nu_0 > 0, \;\; h_0 > 0, \;\; d_0\nu_0 - h_0 > 0,$$

where δ_0, δ_1, δ_2, and δ_3 are positive constants depending on material parameters.

Due to the symmetry conditions (2), with the help of (**??**) we easily derive

$$c_{rjkl}\zeta_{rj}\overline{\zeta_{kl}} \geq \delta_0 \zeta_{kl}\overline{\zeta_{kl}}, \;\; \varkappa_{kj}\zeta_k\overline{\zeta_j} \geq \delta_1|\zeta|^2, \;\; \mu_{kj}\zeta_k\overline{\zeta_j} \geq \delta_2|\zeta|^2, \;\; \eta_{kj}\zeta_k\overline{\zeta_j} \geq \delta_3|\;\zeta|^2,$$
$$\text{for all } \; \zeta_{kj} = \zeta_{jk} \in \mathbb{C} \; \text{ and for all } \; \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3.$$
(3)

More careful analysis related to the positive definiteness of the potential energy and the thermodynamical laws insure that the following 8×8 matrix

$$M = [M_{kj}]_{8\times 8} :=$$

$$\begin{bmatrix} [\varkappa_{jl}]_{3\times3} & [a_{jl}]_{3\times3} & [p_j]_{3\times1} & [\nu_0 p_j]_{3\times1} \\ [a_{jl}]_{3\times3} & [\mu_{jl}]_{3\times3} & [m_j]_{3\times1} & [\nu_0 m_j]_{3\times1} \\ [p_j]_{1\times3} & [m_j]_{1\times3} & d_0 & h_0 \\ [\nu_0 p_j]_{1\times3} & [\nu_0 m_j]_{1\times3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8\times8}$$
(4)

is positive definite. Note that the positive definiteness of M remains valid if the parameters p_j and m_j in (4) are replaced by the opposite ones, $-p_j$ and $-m_j$. Moreover, it follows that the matrices

$$\Lambda^{(1)} :=$$

$$\begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} \end{bmatrix}_{6\times6}, \quad \Lambda^{(2)} :=$$

$$\begin{bmatrix} d_0 & h_0 \\ h_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2}$$
(5)

are positive definite as well, i.e.,

with some positive constants κ_1 and κ_2 depending on the material parameters involved in (5). The principal homogeneous symbol matrix of the operator $A(\partial_x, \tau)$ is

$$A^{(0)}(-i\xi) = -A^{(0)}(\xi) =$$

$$\begin{bmatrix} [-c_{rjkl}\xi_j\xi_l]_{3\times 3} & [-e_{lrj}\xi_j\xi_l]_{3\times 1} & [-q_{lrj}\xi_j\xi_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}\xi_j\xi_l]_{1\times 3} & -\varkappa_{jl}\xi_j\xi_l & -a_{jl}\xi_j\xi_l & 0 \\ [q_{jkl}\xi_j\xi_l]_{1\times 3} & -a_{jl}\xi_j\xi_l & -\mu_{jl}\xi_j\xi_l & 0 \\ [0]_{1\times 3} & 0 & 0 & -\eta_{jl}\xi_j\xi_l \end{bmatrix}_{6\times 6}.$$
(6)

From the symmetry conditions (2), inequalities (**??**), and positive definiteness of the matrix $\Lambda^{(1)}$ defined in (5) it follows that there is a positive constant C_0 depending only on the material parameters, such that

Therefore, $-A(\partial_x, \tau)$ is a non-selfadjoint strongly elliptic differential operator. The over bar denotes complex conjugation and the central dot denotes the scalar product in the respective complex-valued vector space. Further, let us introduce the generalized stress operator $\mathcal{T}(\partial_x, n, \tau)$ associated with the pseudo-oscillation operator $A(\partial_x, \tau)$

$$\mathcal{T} = \mathcal{T}(\partial_x, n, \tau) = \big[\ \mathcal{T}_{pq}(\partial_x, n, \tau)\ \big]_{6\times 6} :=$$

$$\begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -(1+\nu_0\tau)p_jn_j \\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -(1+\nu_0\tau)m_jn_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}.$$
(7)

Evidently, for a smooth six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial_x, n, \partial_t)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_jn_j, -B_jn_j, -T_0^{-1}q_jn_j)^\top. \tag{8}$$

Recall that $E = (E_1, E_2, E_3)^\top = -\varphi$ and $H = (H_1, H_2, H_3)^\top = -\psi$ are electric and magnetic fields, respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials and ϑ is the temperature change to a reference temperature T_0, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density, and the corresponding constitutive equations read as

$$\begin{aligned}
\sigma_{rj}(x,\tau) &= c_{rjkl}\varepsilon_{kl}(x,\tau) + e_{lrj}\partial_l\varphi(x,\tau) + q_{lrj}\partial_l\psi(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\vartheta(x,\tau),\\
D_j(x,\tau) &= e_{jkl}\varepsilon_{kl}(x,\tau) - \varkappa_{jl}\partial_l\varphi(x,\tau) - a_{jl}\partial_l\psi(x,\tau) + (1+\nu_0\tau)p_j\vartheta(x,\tau),\\
B_j(x,\tau) &= q_{jkl}\varepsilon_{kl}(x,\tau) - a_{jl}\partial_l\varphi(x,\tau) - \mu_{jl}\partial_l\psi(x,\tau) + (1+\nu_0\tau)m_j\vartheta(x,\tau),\\
q_j(x,\tau) &= -T_0\eta_{jl}\partial_l\vartheta(x,\tau).
\end{aligned}$$

The components of the vector $\mathcal{T}U$ given by (8) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the forth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector, respectively, with opposite sign, and finally the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector.

In Green's formulas there appears also the boundary operator $\mathcal{P}(\partial_x, n, \tau)$ associated with the adjoint differential operator $A^*(\partial_x, \tau) := [\overline{A(-\partial_x, \tau)}]^\top = A^\top(-\partial_x, \overline{\tau})$,

$$\mathcal{P} = \mathcal{P}(\partial_x, n, \tau) = \big[\, \mathcal{P}_{pq}(\partial_x, n, \tau) \,\big]_{6\times 6} =$$

$$\begin{bmatrix}
[c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} & [\overline{\tau}\lambda_{rj}n_j]_{3\times 1}\\
[e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -\overline{\tau}p_jn_j\\
[q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -\overline{\tau}m_jn_j\\
[0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l
\end{bmatrix}_{6\times 6}. \tag{9}$$

Scalar acoustic wave field

Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with C^∞-smooth boundary $S = \partial\Omega^+$ if not otherwise stated, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. We assume that the unbounded exterior domain Ω^- is filled by a homogeneous isotropic inviscid fluid medium with the constant density ρ_2. Further, let the propagation of acoustic pressure wave in Ω^- be described by a complex-valued scalar function (scalar field) w being a solution of the homogeneous Helmholtz equation (cf. Colton et al(missing citation))

$$\Delta \mathrm{w}(x,\tau) - \rho_2 \tau^2 \mathrm{w}(x,\tau) = 0,$$

(10)

satisfying the following asymptotic condition at infinity

$$\mathrm{w}(x) = O(|x|^m) \qquad \text{as} \qquad |x| \to \infty$$

(11)

with some natural number $m \in \mathbb{N}$. Note that, if $\mathrm{Re}\tau = \sigma > \sigma_0 \geq 0$, due to relation (11), it then follows that

$$D^\alpha \mathrm{w}(x) = O(|x|^{-n}) \qquad \text{as} \qquad |x| \to \infty$$

(12)

for all multi-indices α and $n \in \mathbb{N}$. Actually, w decays exponentially at infinity.

Formulation of interaction problems

By $C^k(\overline{\Omega^\pm})$ we denote the subspace of functions from $C^k(\Omega^\pm)$ whose derivatives up to the order k are continuously extendable to S from $\Omega^\pm$. We denote by $H^s(\Omega^\pm)$ and $H^s(S)$ the well known L_2-based Sobolev-Slobodetski and Bessel potential spaces.
The symbols $\{\,\cdot\,\}^+_S$ and $\{\,\cdot\,\}^-_S$ denote one-sided limits (traces) on S from Ω^+ and Ω^-, respectively. We drop the subscript S if it does not lead to misunderstanding.

Assume that the domain $\overline{\Omega^+}$ is occupied by an anisotropic homogeneous material with the above described generalized thermo-electro-magneto-elastic properties and it is immersed in an inviscid fluid occupying the exterior domain Ω^-.

The fluid-solid interaction in the case under consideration is described by the boundary-transmission problems for the equations of the generalized thermo-electro-magneto-elasticity theory and the Helmholtz equation. Throughout the paper we assume that

$$\tau = \sigma + i\omega, \;\; \sigma > \sigma_0 \geqslant 0, \;\; \omega \in \mathbb{R},$$

(13)

if not otherwise stated.

Interaction Dirichlet Type Problem (ID): Find a vector-function $U = (u, u_4, u_5, u_6)^\top \in [H^1(\Omega^+)]^6$ and scalar function $\mathrm{w} \in H^1(\Omega^-)$ satisfying the differential equations

$$A(\partial_x, \tau)U = 0 \text{ in } \Omega^+, \tag{14}$$

$$\Delta \mathrm{w} - \rho_2 \tau^2 \mathrm{w} = 0 \text{ in } \Omega^-,$$

(15)

the transmission conditions

$$\{u \cdot n\}^+ = -(\rho_2 \tau^2)^{-1} \{\partial_n \mathrm{w}\}^- + g_0 \text{ on } S, \tag{16}$$

$$\{[\mathcal{T}U]_j\}^+ = -\{\mathrm{w}\}^- n_j + f_j \text{ on } S, \qquad j = 1, 2, 3,$$

(17)

and the Dirichlet type boundary conditions

$$\{u_r\}^+ = f_r^{(D)} \text{ on } S, \qquad r = 4, 5, 6,$$

(18)

where $g_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f_r^{(D)} \in H^{1/2}(S)$ $r = 4, 5, 6$.

Interaction Neumann Type Problem (IN): Find a vector-function $U = (u, u_4, u_5, u_6)^\top \in [H^1(\Omega^+)]^6$ and scalar function $\mathrm{w} \in H^1(\Omega^-)$ satisfying the differential equations (14) and (15) respectively, transmission conditions (16), (17), and the Neumann boundary condition

$$\{[\mathcal{T}U]_r\}^+ = f_r^{(N)} \text{ on } S \quad \text{with} \quad f_r^{(N)} \in H^{-1/2}(S) \quad r = 4, 5, 6.$$

(19)

In both boundary-transmission problems we require that the scalar pressure function w satisfies the decay condition (12).

Uniqueness theorems

Let $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. The homogeneous problem (ID) has only the trivial solution, while the general solution of the homogeneous problem (IN) is the vector $U = (0, 0, 0, b_1, b_2, 0)$ and $\mathrm{w} = 0$, where b_1 and b_2 are an arbitrary complex constants.

Let a pair (U, w) be a solution to the homogeneous problem (ID).
Let us write Green's second formula for the Helmholtz equation (15) in the domain Ω^-,

$$\int_{\Omega^-} [(\Delta - \rho_2\tau^2)\mathrm{w}\,\overline{\mathrm{w}} - \mathrm{w}(\Delta - \rho_2\tau^2)\overline{\mathrm{w}}]dx = -\langle\{\partial_n \mathrm{w}\}^-, \{\mathrm{w}\}^-\rangle_S + \langle\{\partial_n \overline{\mathrm{w}}\}^-, \{\overline{\mathrm{w}}\}^-\rangle_S. \tag{20}$$

Here and in what follows, the symbol $\langle\cdot,\cdot\rangle_S$ denotes the duality between the mutually adjoint function spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$, which extends the usual L_2 scalar product

$$\langle f, g\rangle_S = \int_S f\,\overline{g}\,dS \quad \text{for} \quad f, g \in L_2(S).$$

Therefore from (15) nd (20) we obtain

$$\mathrm{Im}\,\langle\{\partial_n \overline{\mathrm{w}}\}^-, \{\overline{\mathrm{w}}\}^-\rangle_S = 0. \tag{21}$$

Now, let us write Green's first formula for the Helmholtz equation (15) in the domain Ω^-,

$$\int_{\Omega^-} (\Delta - \rho_2\tau^2)\mathrm{w}\,\overline{\mathrm{w}}dx + \rho_2\tau^2 \int_{\Omega^-} |\mathrm{w}|^2 dx + \int_{\Omega^-} |\nabla \mathrm{w}|^2 dx = -\langle\{\partial_n \mathrm{w}\}^-, \{\mathrm{w}\}^-\rangle_S. \tag{22}$$

Take into account (15) and separate the real and imaginary parts of (22) to obtain

$$\begin{aligned} &2\sigma\omega \int_{\Omega^-} |\mathrm{w}|^2 dx = 0, \\ &(\sigma^2 - \omega^2) \int_{\Omega^-} |\mathrm{w}|^2 dx + \int_{\Omega^-} |\nabla \mathrm{w}|^2 dx = -\langle\{\partial_n \mathrm{w}\}^-, \{\mathrm{w}\}^-\rangle_S. \end{aligned} \tag{23}$$

(24)

If $\omega \neq 0$, from (23) we conclude that $\mathrm{w} = 0$ in Ω^-.
Now, let us write Green's formula for the operator $A(\partial_x, \tau)$ in the domain Ω^+ (see Subsection 2.7.3 in Buchukuri et al (missing citation)),

$$\int_{\Omega^+}\Big[[A(\partial_x,\tau)U]_j\,\overline{u}_j + [\overline{A(\partial_x,\tau)U}]_4\,u_4 + [\overline{A(\partial_x,\tau)U}]_5\,u_5 + \\ + \frac{1+\nu_0\tau}{\overline{\tau}}[\overline{A(\partial_x,\tau)U}]_6\,u_6 + \mathcal{E}(U,\overline{U})\Big]dx = \langle\{\mathcal{T}U\}_j^+,\{u_j\}^+\rangle_S + \\ +\langle\{\overline{\mathcal{T}U}\}_4^+,\{u_4\}^+\rangle_S + \langle\{\overline{\mathcal{T}U}\}_5^+,\{u_5\}^+\rangle_S + \frac{1+\nu_0\tau}{\overline{\tau}}\langle\{\overline{\mathcal{T}U}\}_6^+,\{u_6\}^+\rangle_S, \tag{25}$$

where

Since (U, w) is a solution of the homogeneous problem (ID), from (25) and we obtain

$$\int_{\Omega^+}\mathcal{E}(U,\overline{U})dx = \langle\{\mathcal{T}U\}_j^+,\{u_j\}^+\rangle_S = -\langle\{\mathrm{w}\}^-\,n_j\,,\,\{u_j\}^+\rangle_S = (\rho_2\,\overline{\tau}^{\,2})^{-1}\langle\{\mathrm{w}\}^-,\{\partial_n\mathrm{w}\}^-\rangle_S,$$

i.e.

$$\langle\{\mathrm{w}\}^-,\{\partial_n\mathrm{w}\}^-\rangle_S = \rho_2\,\overline{\tau}^{\,2}\int_{\Omega^+}\mathcal{E}(U,\overline{U})dx.$$

(26)

Substituting (26) into (24) for $\omega = 0$ we get

$$\sigma^2\int_{\Omega^-}|\mathrm{w}|^2dx + \int_{\Omega^-}|\nabla\mathrm{w}|^2dx + \rho_2\,\sigma^2\int_{\Omega^+}\mathcal{E}(U,\overline{U})dx = 0.$$

(27)

Keeping in mind that for $\omega = 0$, the following inequality

$$\int_{\Omega^+} \mathcal{E}(U, \overline{U}) dx \geq 0$$

holds (see the proof of Theorem 2.25 in Buchukuri et al (missing citation)), from (27) we deduce that $\mathrm{w} = 0$ in Ω^-. Thus, $\mathrm{w} = 0$ in Ω^- for arbitrary $\omega \in \mathbb{R}$. Therefore from (26) we get

$$\int_{\Omega^+} \mathcal{E}(U, \overline{U}) dx = 0$$

(28)

for $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. Due to the relations (3) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (5), we find that

$$c_{ijlk} \partial_i u_j \overline{\partial_l u_k} \geq 0, \quad \eta_{jl} \partial_l \vartheta \overline{\partial_j \vartheta} \geq 0,$$
$$\Big[\varkappa_{jl} \partial_l \varphi \overline{\partial_j \varphi} + a_{jl} (\partial_l \psi \overline{\partial_j \varphi} + \partial_j \varphi \overline{\partial_l \psi}) + \mu_{jl} \partial_l \psi \overline{\partial_j \psi}\Big] \geq \lambda_0 \big(|\nabla \varphi|^2 + |\nabla \psi|^2\big),$$

(29)

where λ_0 is a positive constant.
Using inequalities (29), positive definiteness of the matrix M defined by (4), and the inequality $\sigma(d_0 \nu_0 - h_0) > 0$ (see ()) we obtain that (see proof of Theorem 2.25 in Buchukuri et al (missing citation))

$$u = 0, \;\; u_4 = \varphi = b_1, \;\; u_5 = \psi = b_2, \;\; u_6 = \vartheta = 0 \;\text{ in }\; \Omega^+,$$

(30)

i.e.

$$U = (0, 0, 0, b_1, b_2, 0)^\top,$$

(31)

where b_1 and b_2 are arbitrary complex constants.
The homogeneous Dirichlet conditions on S then imply $b_1 = b_2 = 0$, i.e. $U = 0$ in Ω^-.

It is evident that b_1 and b_2 in (31) remain arbitrary complex constants in the case of the homogeneous problem (IN), which completes the proof.

Let a pair $(V, \mathrm{w}) \in [H^1(\Omega^+)]^6 \times H^1(\Omega^-)$ be a solution of the homogeneous boundary-transmission problem associated with the adjoint differential operators:

$$\begin{aligned}
A^*(\partial_x, \tau)V &= 0 && \text{in } \Omega^+, \\
(\Delta - \rho_2 \overline{\tau}^2)\mathrm{w} &= 0 && \text{in } \Omega^-, \\
\{v \cdot n\}^+ + (\rho_2 \overline{\tau}^2)^{-1} \{\partial_n \mathrm{w}\}^- &= 0 && \text{on } S, \\
\{[\mathcal{P}V]_j\}^+ + \{\mathrm{w}\}^- n_j &= 0 && \text{on } S, \qquad j = 1, 2, 3, \\
\{[\mathcal{P}V]_k\}^+ &= 0 && \text{on } S, \qquad k = 4, 5, 6,
\end{aligned}$$

where $V = (v, v_4, v_5, v_6)^\top$ with $v = (v_1, v_2, v_3)^\top$ and $\mathcal{P}$ is defined in (9). By the similar arguments applied in the proof of Theorem , one can prove that $\mathrm{w} = 0$ in Ω^- and $V = (0, 0, 0, b_1, b_2, 0)^\top$ in Ω^+, where b_1 and b_2 are arbitrary complex constants.

Layer potentials

Potentials associated with the Helmholtz equation

Let us introduce the single and double layer potentials

$$V_\tau(g)(x) := \int_S \gamma(x - y, \tau) g(y) d_y S, \tag{32}$$

$$W_\tau(f)(x) := \int_S \partial_{n(y)} \gamma(x - y, \tau) f(y) d_y S,$$

(33)

where

$$\gamma(x, \tau) := -\frac{\exp\big(-\sqrt{\rho_2}\,\tau |x|\big)}{4\pi |x|}, \qquad \mathrm{Re}\tau > 0,$$

is the fundamental solution of the Helmholtz equation (10). These potentials satisfy the decay condition (12) at infinity.
For these potentials the following theorems are valid (see Colton et al (missing citation), McLean (missing citation)).

Let $g \in H^{-1/2}(S)$, $f \in H^{1/2}(S)$. Then the following jump relations hold on the manifold S

$$\{V_\tau(g)\}^\pm = \mathcal{H}_\tau(g), \qquad \{\partial_n V_\tau(g)\}^\pm = \mp 2^{-1} g + \mathcal{K}_\tau(g),$$
$$\{W_\tau(f)\}^\pm = \pm 2^{-1} f + \mathcal{N}_\tau(f), \qquad \{\partial_n W_\tau(f)\}^+ = \{\partial_n W_\tau(f)\}^- =: \mathcal{L}_\tau(f),$$

(34)

where $\mathcal{H}_\tau$, $\mathcal{N}_\tau$, and $\mathcal{K}_\tau$ are integral operators with weakly singular kernels

$$\mathcal{H}_\tau(g)(z) := \int_S \gamma(z-y,\tau)g(y)d_yS, \qquad z\in S, \tag{35}$$

$$\mathcal{N}_\tau(f)(z) := \int_S \partial_{n(y)}\gamma(z-y,\tau)f(y)d_yS, \qquad z\in S, \tag{36}$$

$$\mathcal{K}_\tau(g)(z) := \int_S \partial_{n(z)}\gamma(z-y,\tau)g(y)d_yS, \qquad z\in S,$$

(37)

while $\mathcal{L}_\tau$ is a singular integro-differential operator (a pseudodifferential operator of order 1).

Mapping properties of the above potentials and the boundary integral operators are described in Appendix (see Theorems -).

Fundamental solution and potentials associated with the pseudo-oscillation operator of generalized thermo-electro-magneto-elasticity theory

The full symbol of the pseudo-oscillation operator $A(\partial_x,\tau)$ is elliptic provided $\mathrm{Re}\tau\neq 0$, i.e. (see Ch.3 in Buchukuri et al(missing citation)),

$$\text{A (-i}\xi,\tau) = -\begin{bmatrix} [c_{rjkl}\xi_j\xi_l+\rho_1\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\xi_j\xi_l]_{3\times 1} & [q_{lrj}\xi_j\xi_l]_{3\times 1} & [-i(1+\nu_0\tau)\lambda_{rj}\xi_j]_{3\times 1} \\ [-e_{jkl}\xi_j\xi_l]_{1\times 3} & \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & -i(1+\nu_0\tau)p_j\xi_j \\ [-q_{jkl}\xi_j\xi_l]_{1\times 3} & a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & -i(1+\nu_0\tau)m_j\xi_j \\ [-i\tau\lambda_{kl}\xi_l]_{1\times 3} & i\tau p_l\xi_l & i\tau m_l\xi_l & \eta_{jl}\xi_j\xi_l+\tau^2h_0+\tau d_0 \end{bmatrix}_{6\times 6}.$$

$$\det A(-i\xi,\tau)\neq 0, \qquad \forall\xi\in\mathbb{R}^3\backslash\{0\},$$

where

Moreover, the entries of the inverse matrix $A^{-1}(-i\xi,\tau)$ are locally integrable functions decaying at infinity as $O(|\xi|^{-2})$. Therefore, one can construct the fundamental matrix $\Gamma(x,\tau)=[\Gamma_{rk}(x,\tau)]_{6\times 6}$ of the operator $A(\partial_x,\tau)$ by the distributional Fourier transform technique,

$$\Gamma(x,\tau)=F^{-1}_{\xi\to x}[A^{-1}(-i\xi,\tau)].$$

(38)

The properties of the fundamental matrix $\Gamma(x,\tau)$ in a neighbourhood of the pole and at infinity are studied in the reference Buchukuri et al(missing citation) (Ch. 3).

Let us introduce the single and double layer vector potentials associated with the pseudo-oscillation operator $A(\partial,\tau)$:

$$\begin{aligned} \mathbf{V}_\tau(h) &= \int_S \Gamma(x-y,\tau)\, h(y)\, d_yS, \\ \mathbf{W}_\tau(h) &= \int_S \left[\mathcal{P}(\partial_y, n(y), \overline{\tau})\Gamma^\top(x-y,\tau)\right]^\top h(y)\, d_yS, \end{aligned} \tag{39}$$

(40)

where $h=(h_1,h_2,h_3,h_4,h_5,h_6)^\top$ is a density vector-function and $\mathcal{P}$ is defined in (9).
These pseudo-oscillation potentials have the following jump properties (see Theorem 4.4 in Buchukuri et al (missing citation)).

Let $h^{(1)}\in[H^{-1+s}(S)]^6$, $h^{(2)}\in[H^s(S)]^6$, $s>0$. Then the following jump relations hold on S

$$\begin{aligned} \{\mathbf{V}_\tau(h^{(1)})(z)\}^\pm &= \int_S \Gamma(z-y,\tau)\, h^{(1)}(y)\, d_yS, \quad z\in S, \\ \{\mathbf{W}_\tau(h^{(2)})(z)\}^\pm &= \pm 2^{-1}h^{(2)}(z) + \int_S \left[\mathcal{P}(\partial_y, n(y), \overline{\tau})\Gamma^\top(z-y,\tau)\right]^\top h^{(2)}(y)\, d_yS, \quad z\in S, \\ \{\mathcal{T}\mathbf{V}_\tau(h^{(1)})(z)\}^\pm &= \mp 2^{-1}h^{(1)}(z) + \int_S \mathcal{T}(\partial_z, n(z),\tau)\Gamma(z-y,\tau)\, h^{(1)}(y)\, d_yS, \quad z\in S, \\ \{\mathcal{T}\mathbf{W}_\tau(h^{(2)})(z)\}^+ &= \{\mathcal{T}\mathbf{W}_\tau(h^{(2)})(z)\}^-, \quad z\in S. \end{aligned}$$

Further we introduce the boundary pseudodifferential operators

$$\begin{aligned} \mathbf{H}_\tau h(z) &= \int_S \Gamma(z-y,\tau)\, h(y)\, d_yS, \quad z\in S, \\ \mathbf{K}_\tau h(z) &= \int_S \mathcal{T}(\partial_z, n(z),\tau)\Gamma(z-y,\tau)\, h(y)\, d_yS, \quad z\in S, \\ \mathbf{N}_\tau h(z) &= \int_S \left[\mathcal{P}(\partial_y, n(y), \overline{\tau})\Gamma^\top(z-y,\tau)\right]^\top h(y)\, d_yS, \quad z\in S, \\ \mathbf{L}_\tau h(z) &= \{\mathcal{T}\mathbf{W}_\tau h(z)\}^+ = \{\mathcal{T}\mathbf{W}_\tau(h)(z)\}^-, \quad z\in S. \end{aligned}$$

Note that $\mathbf{H}_\tau$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathbf{K}_\tau$ and $\mathbf{N}_\tau$ are singular integral operators (pseudodifferential operator of order 0), and $\mathbf{L}_\tau$ is a singular integro-differential operator (pseudodifferential operator of order 1). Mapping properties of these potentials and the corresponding boundary operators are described in Appendix (see Theorems -).

Existence of solutions to the interaction problem (ID)

By Theorems and (see Appendix) the operators

$$\mathbf{H}_\tau : [H^s(S)]^6 \to [H^{s+1}(S)]^6, \qquad \mathcal{H}_\tau : H^s(S) \to H^{s+1}(S),$$

are invertible for all $s \in \mathbb{R}$ and we can look for a solution pair (U, w) of the problem (ID) in the form of single layer potentials:

$$U = \mathbf{V}_\tau\big(\mathbf{H}_\tau^{-1} g\big) \ \text{ in } \ \Omega^+, \qquad \mathrm{w} = V_\tau\big(\mathcal{H}_\tau^{-1} h\big) \ \text{ in } \ \Omega^-,$$

(41)

where $g = (\widetilde{g}, g_4, g_5, g_6)^\top \in [H^{1/2}(S)]^6$, $\widetilde{g} = (g_1, g_2, g_3)^\top$, $h \in H^{1/2}(S)$ are unknown densities. Theorems , , and imply the inclusion $U \in [H^1(\Omega^+)]^6$ and $\mathrm{w} \in H^1(\Omega^-)$.
Transmission conditions (16), (17), and the Dirichlet type conditions (18) lead to the following system of pseudodifferential equations with respect to the unknowns $\widetilde{g}, g_4, g_5, g_6$, and h:

$$\widetilde{g} \cdot n + (\rho_2 \tau^2)^{-1}\big(2^{-1} I_1 + \mathcal{K}_\tau\big)\mathcal{H}_\tau^{-1} h = g_0 \quad \text{on } S, \tag{42}$$
$$\big[\big(-2^{-1} I_6 + \mathbf{K}_\tau\big)\mathbf{H}_\tau^{-1} g\big]_j + n_j h = f_j \quad \text{on } S, \quad j = 1,2,3, \tag{43}$$
$$g_r = f_r^{(D)} \quad \text{on } S, \quad r = 4,5,6.$$

(44)

Here and in what follows I_m stands for the $m \times m$ unit matrix.
The matrix operator generated by the left hand side expressions in the system (42)-(44) reads as

$$\mathbf{Q}_{\tau,D} = \big[\mathbf{Q}_{\tau,D}^{lm}\big]_{7\times 7} := [$$

$$\begin{matrix} [n]_{1\times 3} & [0]_{1\times 3} & (\rho_2\tau^2)^{-1}\mathcal{A}_\tau \\ [\mathbf{A}_\tau^{jk}]_{3\times 3} & [\mathbf{A}_\tau^{j,k+3}]_{3\times 3} & [n]_{3\times 1} \\ [0]_{3\times 3} & I_3 & [0]_{3\times 1} \end{matrix} \quad {}_{7\times 7}, \qquad j,k = 1,2,3,$$

where

$$\mathbf{A}_\tau = \big[\mathbf{A}_\tau^{jk}\big]_{6\times 6} := \big(-2^{-1}I_6 + \mathbf{K}_\tau\big)\mathbf{H}_\tau^{-1}, \tag{45}$$
$$\mathcal{A}_\tau := \big(2^{-1}I_1 + \mathcal{K}_\tau\big)\mathcal{H}_\tau^{-1}$$

(46)

are the Steklov-Poincaré type operators on S associated with the operators $A(\partial_x,\tau)$ and the Helmholtz operator respectively. These operators are strongly elliptic pseudodifferential operators of order 1 (for details see Buchukuri et al (missing citation),(missing citation)).
System (42)-(44) can be rewritten in matrix form

$$\mathbf{Q}_{\tau,D}\Phi = F, \qquad \Phi = \big(g,h\big)^\top, \qquad F = \big(g_0, f_1, f_2, f_3, f_4^{(D)}, f_5^{(D)}, f_6^{(D)}\big)^\top$$

(47)

By Theorems and (see Appendix), the operator $\mathbf{Q}_{\tau,D}$ possesses the following mapping property

$$\mathbf{Q}_{\tau,D} : [H^{1/2}(S)]^7 \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^3.$$

(48)

In view of (44) equations (42) and (43) can be rewritten in the following equivalent form as a system with respect to $\widetilde{g} = (g_1, g_2, g_3)^\top$ and h:

$$\begin{array}{rcll} \widetilde{g}\cdot n + (\rho_2\tau^2)^{-1}\mathcal{A}_\tau h & = & g_0 & \text{on } \ S, \\ \big[\mathbf{A}_\tau(\widetilde{g},0,0,0)^\top\big]_j + n_j h & = & \widetilde{F}_j & \text{on } \ S, \quad j=1,2,3, \end{array} \tag{49}$$

(50)

where $\widetilde{F}_j := f_j - [\mathbf{A}_\tau(0,0,0,f_4^{(D)}, f_5^{(D)}, f_6^{(D)})^\top]_j, \quad j = 1,2,3.$
Denote by $\mathbf{R}_{\tau,D}$ the operator generated by the left hand side expression of system (49)-(50),

$$\mathbf{R}_{\tau,D} = \Big[\mathbf{R}_{\tau,D}^{jk}\Big]_{4\times 4} := [$$

$$[n]_{1\times 3} \qquad (\rho_2\tau^2)^{-1}\mathcal{A}_\tau$$
$$\widetilde{\mathbf{A}}_\tau \qquad [n]_{3\times 1} \qquad {}_{4\times 4},$$

where $\widetilde{\mathbf{A}}_\tau := [\mathbf{A}_\tau^{jk}]_{3\times3},\ j,k = 1,2,3.$
Evidently the operator

$$\mathbf{R}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4$$

(51)

is bounded. Let

$$\mathbf{R}^{(0)}_{\tau,D} := \begin{bmatrix} [0]_{1\times3} & (\rho_2\tau^2)^{-1}\mathcal{A}_\tau \\ \widetilde{\mathbf{A}}_\tau & [0]_{3\times1} \end{bmatrix}_{4\times4}.$$

It can be easily verified that the operator

$$\mathbf{R}_{\tau,D} - \mathbf{R}^{(0)}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4$$

is compact.
The strong ellipticity property of the operators (45) and (46) implies that the operators

$$\widetilde{\mathbf{A}}_\tau : [H^{1/2}(S)]^3 \to [H^{-1/2}(S)]^3, \qquad \mathcal{A}_\tau : H^{1/2}(S) \to H^{-1/2}(S)$$

are Fredholm operators with zero index (see Hörmander (missing citation), Hsiao et al (missing citation), McLean (missing citation), Buchukuri et al(missing citation)).
Therefore operator (51) and consequently operator (48) are Fredholm with index zero.
Now, we show that the null space of the operator $\mathbf{R}_{\tau,D}$ is trivial. Let $(\widetilde{g}, h)^\top$ with $\widetilde{g} \in [H^{1/2}(S)]^3$ and $h \in H^{1/2}(S)$ be a solution of the homogeneous system

$$\mathbf{R}_{\tau,D}(\widetilde{g}, h)^\top = 0,$$

(52)

and set

$$\widetilde{U} = (\widetilde{u}, \widetilde{u}_4, \widetilde{u}_5, \widetilde{u}_6)^\top = \mathbf{V}_\tau\big(\mathbf{H}_\tau^{-1}(\widetilde{g}, 0, 0, 0)\big), \qquad \widetilde{\mathrm{w}} = V_\tau\big(\mathcal{H}_\tau^{-1} h\big).$$

With the help of equation (52) it can be easily checked that $\widetilde{U}$ and $\widetilde{\mathrm{w}}$ solve the homogeneous problem (ID). Therefore by the uniqueness theorem for the problem (ID) (see Theorem), we deduce $\widetilde{U} = 0$ in Ω^+ and $\widetilde{\mathrm{w}} = 0$ in Ω^-. Then $\{\widetilde{U}\}^+ = (\widetilde{g}, 0, 0, 0)^\top = 0$ and $\{\widetilde{\mathrm{w}}\}^- = h = 0$ on S. Consequently, the operators

$$\begin{aligned} \mathbf{R}_{\tau,D} \quad &: \quad [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4, \\ \mathbf{Q}_{\tau,D} \quad &: \quad [H^{1/2}(S)]^7 \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^3 \end{aligned}$$

are invertible.
Therefore system (42)-(44) is uniquely solvable and the following assertion holds.

Let $S \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 \geqslant 0$, $\omega \in \mathbb{R}$, and

$$g_0 \in H^{-1/2}(S), \quad f_j \in H^{-1/2}(S), \quad j = 1, 2, 3, \quad f_r^{(D)} \in H^{1/2}(S), \quad r = 4, 5, 6.$$

Then the interaction Dirichlet type problem (ID) has a unique solution $(U, \mathrm{w}) \in [H^1(\Omega^+)]^6 \times H^1(\Omega^-)$, which can be represented by the single layer potentials

$$U = \mathbf{V}_\tau\big(\mathbf{H}_\tau^{-1} g\big) \quad \text{in} \ \ \Omega^+, \qquad \mathrm{w} = V_\tau\big(\mathcal{H}_\tau^{-1} h\big) \quad \text{in} \ \ \Omega^-,$$

(53)

where the densities $g \in [H^{1/2}(S)]^6$ and $h \in H^{1/2}(S)$ are defined from the uniquely solvable system (42)-(44).

If the boundary-transmission data of the problem are smooth functions, then the solution pair (U, w) is smooth as well and the following regularity result holds.

Let $S \in C^{m,\alpha}$, $0 < \beta < \alpha \le 1$, $m \ge 2$ $\ m \in \mathbb{N}$, and

$$g_0 \in C^{k-1,\beta}(S), \quad f_j \in C^{k-1,\beta}(S), \quad j = 1, 2, 3, \quad f_r^{(D)} \in C^{k,\beta}(S), \quad r = 4, 5, 6, \quad 1 \le k \le m-1, \quad k \in \mathbb{N}.$$

Then the problem (ID) has a unique solution $(U, \mathrm{w}) \in [C^{k,\beta}(\overline{\Omega^+})]^6 \times C^{k,\beta}(\overline{\Omega^-})$, which can be represented by the single layer potentials (53), where the densities $g \in [C^{k,\beta}(S)]^6$ and $h \in C^{k,\beta}(S)$ are defined from the uniquely solvable system (42)-(44).

From Theorem it follows that the strongly elliptic pseudodifferential operators

$$\mathcal{A}_\tau : C^{k,\beta}(S) \to C^{k-1,\beta}(S), \qquad \mathbf{A}_\tau : [C^{k,\beta}(S)]^6 \to [C^{k,\beta}(S)]^6$$

are Fredholm with zero index. Using the same arguments as above, with the help of the uniqueness theorem for the problem (ID) it can be shown that the operator

$$\mathbf{Q}_{\tau,D} \quad : \quad [C^{k,\beta}(S)]^7 \to [C^{k-1,\beta}(S)]^4 \times [C^{k,\beta}(S)]^3$$

has the trivial null space and consequently it is invertible. Therefore system (42)-(44) is uniquely solvable in the space $[C^{k,\beta}(S)]^7$, i.e. $g \in [C^{k,\beta}(S)]^6$, $h \in C^{k,\beta}(S)$. The regularity result then follows from the representation (53) and from Theorems and .

Existence of solutions to the interaction problem (IN)

As in the previous section, we can look for a solution of the problem (IN) in the form of single layer potentials

$$U = \mathbf{V}_\tau\big(\mathbf{H}_\tau^{-1} g\big) \ \text{ in } \ \Omega^+, \qquad \mathrm{w} = V_\tau\big(\mathcal{H}_\tau^{-1} h\big) \ \text{ in } \ \Omega^-,$$

(54)

where $g = (\widetilde{g}, g_4, g_5, g_6)^\top \in [H^{1/2}(S)]^6$ with $\widetilde{g} = (g_1, g_2, g_3)^\top$ and $h \in H^{1/2}(S)$ are unknown densities. From Theorems , , and it follows that $U \in [H^1(\Omega^+)]^6$ and $\mathrm{w} \in H^1(\Omega^-)$.

Transmission conditions (16), (17), and the Neumann type condition (19) lead to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$\begin{aligned} \widetilde{g} \cdot n + (\rho_2 \tau^2)^{-1} \mathcal{A}_\tau h &= g_0 \quad \text{on } S, && (55)\\ [\mathbf{A}_\tau g]_j + n_j h &= f_j \quad \text{on } S, \quad j = 1,2,3, && (56)\\ [\mathbf{A}_\tau g]_r &= f_r^{(N)} \quad \text{on } S, \quad r = 4,5,6, \end{aligned}$$

(57)

where $\mathbf{A}_\tau$ and $\mathcal{A}_\tau$ are Steklov-Poincaré type operators defined in (45) and (46).
Denote by $\mathbf{Q}_{\tau,N}$ the operator generated by the left hand side expressions of system (55)-(57):

$$\mathbf{Q}_{\tau,N} := \begin{bmatrix} [(n,0,0,0)]_{1\times 6} & (\rho_2\tau^2)^{-1}\mathcal{A}_\tau \\ [\mathbf{A}_\tau]_{6\times 6} & [n,0,0,0]_{6\times 1} \end{bmatrix}_{7\times 7}, \qquad j=1,2,3, \quad k=\overline{1,6}.$$

We can rewrite system (55)-(57) in matrix form

$$\mathbf{Q}_{\tau,N}\Phi = F, \quad \Phi = (g,h)^\top, \quad F = \big(g_0, f_1, f_2, f_3, f_4^{(N)}, f_5^{(N)}, f_6^{(N)}\big)^\top. \tag{58}$$

The operator $\mathbf{Q}_{\tau,N}$ possesses the following mapping property

$$\mathbf{Q}_{\tau,N} : [H^{1/2}(S)]^7 \to [H^{-1/2}(S)]^7.$$

Since the Steklov-Poincaré type operators $\mathcal{A}_\tau$ and $\mathbf{A}_\tau$ are strongly elliptic pseudodifferential operators of order 1, it follows that the operators $\mathcal{A}_\tau : H^{1/2}(S) \to H^{-1/2}(S)$ and $\mathbf{A}_\tau : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6$ are Fredholm operators with index zero. Hence, the operator

$$\mathbf{Q}_{\tau,N} : [H^{1/2}(S)]^7 \to [H^{-1/2}(S)]^7$$

is Fredholm with index zero.
Now, let us investigate the null spaces of the operator $\mathbf{Q}_{\tau,N}$ and its adjoint one. Let $g \in [H^{1/2}(S)]^6$ and $h \in H^{1/2}(S)$ be solutions of the homogeneous system (55)-(57)

$$\mathbf{Q}_{\tau,N}(g,h)^\top = 0, \tag{59}$$

and construct the potentials

$$\widetilde{U} = (\widetilde{u}, \widetilde{u}_4, \widetilde{u}_5, \widetilde{u}_6)^\top = \mathbf{V}_\tau\big(\mathbf{H}_\tau^{-1} g\big), \qquad \widetilde{\mathrm{w}} = V_\tau\big(\mathcal{H}_\tau^{-1} h\big).$$

(60)

Evidently, $\widetilde{U}$ and $\widetilde{\mathrm{w}}$ solve the homogeneous problem (IN) in view of representation (60) and equation (59). From the structure of a solution to the homogeneous problem (IN) presented in Theorem , we have

$$\widetilde{U} = (0,0,0,b_1,b_2,0)^\top \quad \text{in} \;\; \Omega^+, \qquad \widetilde{\mathrm{w}} = 0 \quad \text{in} \;\; \Omega^-,$$

where b_1 and b_2 are arbitrary complex constants. These relations imply $\{\widetilde{U}\}^+ = (0,0,0,b_1,b_2,0)^\top = g$ on S, i.e. $g_1 = g_2 = g_3 = g_6 = 0$, $g_4 = b_1$, $g_5 = b_2$ and $\{\widetilde{\mathrm{w}}\}^- = h = 0$ on S. Therefore the dimension of the null space of the operator $\mathbf{Q}_{\tau,N}$ equals to 2, $\dim \operatorname{Ker} \mathbf{Q}_{\tau,N} = 2$. Therefore $\dim \operatorname{Ker} \mathbf{Q}^*_{\tau,N} = 2$, where $\mathbf{Q}^*_{\tau,N} : [H^{1/2}(S)]^7 \to [H^{-1/2}(S)]^7$ is the operator adjoint to $\mathbf{Q}_{\tau,N} : [H^{1/2}(S)]^7 \to [H^{-1/2}(S)]^7$. Further, we will describe the null space of the adjoint operator $\mathbf{Q}^*_{\tau,N}$ to formulate explicitly the necessary and sufficient conditions for the problem (IN) to be solvable.

One can easily find that the operator adjoint to $\mathbf{Q}_{\tau,N}$ has the following form

$$\mathbf{Q}^*_{\tau,N} := [$$

$$\begin{matrix} [n,0,0,0]_{6\times 1} & [\mathbf{A}^*_\tau]_{6\times 6} \\ (\rho_2 \overline{\tau}^{\,2})^{-1} \mathcal{A}^*_\tau & [n,0,0,0]_{1\times 6} \end{matrix} \quad {}_{7\times 7},$$

where

It is evident that $\mathbf{A}^*_\tau$, $\mathbf{H}^*_\tau$, $\mathbf{K}^*_\tau$, $\mathcal{A}^*_\tau$, $\mathcal{H}^*_\tau$, and $\mathcal{K}^*_\tau$ are the adjoint operators respectively to the operators $\mathbf{A}_\tau$, $\mathbf{H}_\tau$, $\mathbf{K}_\tau$, $\mathcal{A}_\tau$, $\mathcal{H}_\tau$, and $\mathcal{K}_\tau$ with respect to the corresponding duality relations.

Note that the fundamental matrix $\Gamma^*(x,\tau)$ of the adjoint operator $A^*(\partial,\tau)$ reads as $\Gamma^*(x,\tau) = \overline{\Gamma^\top(-x,\tau)} = \Gamma^\top(-x,\overline{\tau})$, while the fundamental solution $\gamma^*(x,\tau)$ of the adjoint Helmholtz operator $(\Delta + \varrho_2\, \overline{\tau}^{\,2})$ reads as $\gamma^*(x,\tau) = \overline{\gamma(-x,\tau)} = \gamma(-x,\overline{\tau}) = \gamma(x,\overline{\tau})$.

Therefore operators (**??**)-(**??**) can be rewritten as

It is evident that operators (**??**)-(**??**) are generated by the direct values on S of the single and double layer potentials constructed by the fundamental matrix $\Gamma^*(x,\tau)$ and fundamental solution $\gamma^*(x,\tau)$ (cf. Buchukuri et al(missing citation)):

It is easy to see that the potentials (**??**)-(**??**) have the same mapping properties as the potentials (32), (33), (39), (40).

To find the basis of the null space of the operator $\mathbf{Q}^*_{\tau,N}$ we proceed as follows. Let $\Psi := (\psi_1, \cdots, \psi_7)^\top \in [H^{1/2}(S)]^7$ be a solution of the homogeneous adjoint system

$$\begin{bmatrix} \mathbf{H}^*_\tau & [0]_{6\times 1} \\ [0]_{1\times 6} & -\rho_2\overline{\tau}^{\,2}\mathcal{H}^*_\tau \end{bmatrix}_{7\times 7}$$

$$\mathbf{Q}^*_{\tau,N}\Psi = 0.$$

(61)

By applying the injective matrix operator

to equation (61), we obtain the following equivalent equation

$$\widetilde{\mathbf{Q}}_{\tau,N}\Psi = 0,$$

(62)

where

$$\begin{matrix} \mathbf{H}^*_\tau & [0]_{6\times 1} \\ [0]_{1\times 6} & -\rho_2\overline{\tau}^{\,2}\mathcal{H}^*_\tau \end{matrix}\ {}_{7\times 7}\ [$$

$$\begin{matrix} [n,0,0,0]_{6\times 1} & [\mathbf{A}^*_\tau]_{6\times 6} \\ (\rho_2\overline{\tau}^{\,2})^{-1}\mathcal{A}^*_\tau & [n,0,0,0]_{1\times 6} \end{matrix}\ {}_{7\times 7}$$

$= [$

$$\begin{matrix} [(\mathbf{H}_\tau^*)^{kl} n_l]_{6\times 1} & \big(-2^{-1} I_6 + \mathbf{K}_\tau^*\big) \\ -\big(2^{-1} I_1 + \mathcal{K}_\tau^*\big) & -\rho_2 \overline{\tau}^{\,2}\, \mathcal{H}_\tau^* [n,0,0,0]_{1\times 6} \end{matrix}_{7\times 7}.$$

Construct the following potentials

$$\begin{aligned} \widetilde{U} &= \mathbf{V}_\tau^*(\Psi^{(1)}) + \mathbf{W}_\tau^*(\Psi^{(2)}) \quad \text{in } \Omega^-, \\ \widetilde{\mathrm{w}} &= -W_\tau^*(\psi_1) - \rho_2 \overline{\tau}^{\,2} V_\tau^*(\Psi' \cdot n) \quad \text{in } \Omega^+, \end{aligned} \tag{63}$$

(64)

where

$$\Psi^{(1)} := (n\psi_1, 0, 0, 0)^\top, \quad \Psi^{(2)} := (\Psi', \psi_5, \psi_6, \psi_7)^\top, \quad \Psi' = (\psi_2, \psi_3, \psi_4)^\top.$$

From (62)-(64) we easily deduce

$$\begin{aligned} &\{\widetilde{U}\}^- = [(\mathbf{H}_\tau^*)^{kl} n_l]_{6\times 1} \psi_1 + \big(-2^{-1} I_6 + \mathbf{K}_\tau^*\big) \Psi^{(2)} = 0 \quad \text{on} \quad S, \\ &\{\widetilde{\mathrm{w}}\}^+ = -\big(2^{-1} I_1 + \mathcal{K}_\tau^*\big)\psi_1 - \rho_2 \overline{\tau}^{\,2} \mathcal{H}_\tau^* [\Psi' \cdot n] = 0 \quad \text{on} \quad S. \end{aligned}$$

Therefore the vector $\widetilde{U} \in [H^1(\Omega^-)]^6$ solves the exterior homogeneous Dirichlet problem

$$\begin{aligned} &A^*(\partial, \tau)\widetilde{U} = 0 \quad \text{in} \quad \Omega^-, \\ &\{\widetilde{U}\}^- = 0 \quad \text{on} \quad S, \end{aligned}$$

and from the corresponding uniqueness result it follows that $\widetilde{U} = 0$ in Ω^- (see Theorem 2.30 in Buchukuri et al(missing citation)).
On the other hand, the function $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$ solves the interior homogeneous Dirichlet problem:

$$\begin{aligned} &\big(\Delta - \rho_2 \overline{\tau}^{\,2}\big)\widetilde{\mathrm{w}} = 0 \quad \text{in} \quad \Omega^+, \\ &\{\widetilde{\mathrm{w}}\}^+ = 0 \quad \text{on} \quad S. \end{aligned}$$

It can easily be shown that this problem possesses only the trivial solution, i.e. $\widetilde{\mathrm{w}} = 0$ in Ω^+ (see Colton et al(missing citation)).
Using the jump formulae for potentials (63) and (64) (see Theorems and) we derive that on the surface S the following relations hold:

$$\begin{aligned}
\{\widetilde{\mathrm{w}}\}^- &= \psi_1, && (65)\\
\{\partial_n \widetilde{\mathrm{w}}\}^- &= -\rho_2 \overline{\tau}^{\,2}\, \Psi' \cdot n, && (66)\\
\{[\mathcal{P}\widetilde{U}]_j\}^+ &= -n_j \psi_1, \qquad j = 1,2,3, && (67)\\
\{[\mathcal{P}\widetilde{U}]_k\}^+ &= 0, \qquad\qquad k = 4,5,6, && (68)\\
\{\widetilde{U}\}^+ &= \Psi^{(2)},
\end{aligned}$$

(69)

Hence we deduce that $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3, \widetilde{U}_4, \widetilde{U}_5, \widetilde{U}_6)^\top = (\widetilde{U}', \widetilde{U}_4, \widetilde{U}_5, \widetilde{U}_6)^\top$ with $\widetilde{U}' = (\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3,)^\top$ and $\widetilde{\mathrm{w}}$ solve the following homogeneous transmission problem

$$\begin{aligned}
A^*(\partial_x, \tau)\widetilde{U} &= 0 && \text{in } \Omega^+,\\
\big(\Delta - \rho_2 \overline{\tau}^{\,2}\big)\widetilde{\mathrm{w}} &= 0 && \text{in } \Omega^-,\\
\{\widetilde{U}' \cdot n\}^+ + \big(\rho_2 \overline{\tau}^{\,2}\big)^{-1}\{\partial_n \widetilde{\mathrm{w}}\}^- &= 0 && \text{on } S,\\
\{[\mathcal{P}\widetilde{U}]_j\}^+ + \{\widetilde{\mathrm{w}}\}^- n_j &= 0 && \text{on } S, \qquad j = 1,2,3,\\
\{[\mathcal{P}\widetilde{U}]_k\}^+ &= 0 && \text{on } S, \qquad k = 4,5,6.
\end{aligned}$$

From the uniqueness result (see Remark) it follows that $\widetilde{\mathrm{w}} = 0$ in Ω^- and $\widetilde{U} = (0,0,0,b_1,b_2,0)^\top$ in Ω^+ with arbitrary complex constants b_1 and b_2. Then from (65)-(69) we obtain

$$\psi_j = 0, \quad j = \overline{1,4}, \quad \psi_5 = b_1, \quad \psi_6 = b_2, \quad \psi_7 = 0, \quad \text{i.e.,} \quad \Psi = (0,0,0,0,b_1,b_2,0)^\top.$$

(70)

Since the operator $\mathbf{Q}_{\tau,N} : [H^{1/2}(S)]^7 \to [H^{-1/2}(S)]^7$ is Fredholm with zero index, from (70) we obtain that the following orthogonality condition

$$\langle F,\, \Psi \rangle_S = 0$$

(71)

is necessary and sufficient for matrix pseudodifferential equation (58) to be solvable. Therefore the boundary-transmission problem(IN) is solvable if and only if

$$\langle f_4^{(N)}, 1\rangle_S = 0, \qquad \langle f_5^{(N)}, 1\rangle_S = 0.$$

(72)

Now we can formulate the following existence theorem.

Let $S \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 \geqslant 0$, $\omega \in \mathbb{R}$, and

$$g_0 \in H^{-1/2}(S), \quad f_j \in H^{-1/2}(S), \quad j = 1,2,3, \quad f_r^{(N)} \in H^{-1/2}(S), \quad r = 4,5,6.$$

Then the interaction Neumann type problem (IN) is solvable in the space $(U, \mathrm{w}) \in [H^1(\Omega^+)]^6 \times H^1(\Omega^-)$, if and only if the condition (72) is fulfilled. The solutions of the problem (IN) are represented by potentials

$$U = \mathbf{V}_\tau\big(\mathbf{H}_\tau^{-1} g\big) \quad \text{in} \ \ \Omega^+, \qquad \mathrm{w} = V_\tau\big(\mathcal{H}_\tau^{-1} h\big) \quad \text{in} \ \ \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^6$ and $h \in H^{1/2}(S)$ are defined from system (55)-(57), and they are defined modulo the addend vector $(0,0,0,b_1,b_2,0)^\top$ with arbitrary comlex constants b_1 and b_2.

The following regularity result holds.

Let $S \in C^{m,\alpha}$, $0 < \beta < \alpha \le 1$, $m \ge 2$ $m \in \mathbb{N}$, and

$$g_0 \in C^{k-1,\beta}(S), \quad f_j \in C^{k-1,\beta}(S), \quad j = 1,2,3, \quad f_r^{(N)} \in C^{k-1,\beta}(S), \quad r = 4,5,6, \quad 1 \le k \le m-1, \quad k \in \mathbb{N}.$$

Then the problem (IN) is solvable in the space $[C^{k,\beta}(\overline{\Omega^+})]^6 \times C^{k,\beta}(\overline{\Omega^-})$, if and only if the conditions

$$\int_S f_4^{(N)} dS = 0, \qquad \int_S f_5^{(N)} dS = 0$$

are fulfilled. The solutions of the problem (IN) are represented by potentials (54) and they are defined modulo the complex constant addend vector $(0,0,0,b_1,b_2,0)^\top$.

Proof of this Theorem is similar to the proof of Theorem .

Appendix: Mapping Properties of Potentials

For the readers convenience here we collect some results describing properties of the layer potentials. Here we preserve the notation from the main text of the paper.

For the potentials associated with the Helmholtz equation the following theorems hold (see McLean (missing citation), Colton et al(missing citation)).

Let $s \in \mathbb{R}$, $S \in C^{\infty}$. Then the single and double layer scalar potentials can be extended to the following continuous operators

$$\begin{aligned} V_\tau : H^s(S) \to H^{s+3/2}(\Omega^+), \qquad & V_\tau : H^s(S) \to H^{s+3/2}(\Omega^-), \\ W_\tau : H^s(S) \to H^{s+1/2}(\Omega^+), \qquad & W_\tau : H^s(S) \to H^{s+1/2}(\Omega^-). \end{aligned}$$

Let $s \in \mathbb{R}$, $S \in C^{\infty}$. Then the operators (see (34)-(37))

$$\begin{aligned} \mathcal{H}_\tau : H^s(S) \to H^{s+1}(S), \qquad & \pm 2^{-1} I_1 + \mathcal{N}_\tau : H^s(S) \to H^s(S), \\ \mathcal{L}_\tau : H^{s+1}(S) \to H^s(S), \qquad & \pm 2^{-1} I_1 + \mathcal{K}_\tau : H^s(S) \to H^s(S), \end{aligned}$$

are continuous and invertible for $\mathrm{Re}\tau > 0$.

Let $S \in C^{m,\alpha}$, $0 < \beta < \alpha \leq 1$, and let $k \leq m-1$, $m \geq 2$ be nonnegative integers. Then the scalar potential operators

$$V_\tau \; : \; C^{k,\beta}(S) \to C^{k+1,\beta}(\overline{\Omega^{\pm}}), \qquad W_\tau \; : \; C^{k,\beta}(S) \to C^{k,\beta}(\overline{\Omega^{\pm}}),$$

are continuous, while the scalar boundary operators

$$\begin{aligned} \mathcal{H}_\tau : C^{k,\beta}(S) \to C^{k+1,\beta}(S), \qquad & \pm 2^{-1} I_1 + \mathcal{N}_\tau, : C^{k,\beta}(S) \to C^{k,\beta}(S), \\ \mathcal{L}_\tau : C^{k+1,\beta}(S) \to C^{k,\beta}(S), \qquad & \pm 2^{-1} I_1 + \mathcal{K}_\tau : C^{k,\beta}(S) \to C^{k,\beta}(S), \end{aligned}$$

are invertible.

For the vector potentials associated with the pseudo-oscillation operators $A(\partial, \tau)$ and $A^*(\partial, \tau)$ the following theorems hold (see Buchukuri et al(missing citation), (missing citation)).

Let $s \in \mathbb{R},\ \ S \in C^\infty$. Then the vector potentials $\mathbf{V}_\tau$ and $\mathbf{W}_\tau$ are continuous in the following spaces

$$\mathbf{V}_\tau : [H^s(S)]^6 \to [H^{s+3/2}(\Omega^+)]^6, \qquad \mathbf{V}_\tau : [H^s(S)]^6 \to [H^{s+3/2}_{loc}(\Omega^-)]^6,$$
$$\mathbf{W}_\tau : [H^s(S)]^6 \to [H^{s+1/2}(\Omega^+)]^6, \qquad \mathbf{W}_\tau : [H^s(S)]^6 \to [H^{s+1/2}_{loc}(\Omega^-)]^6.$$

Let $s \in \mathbb{R},\ \ S \in C^\infty$. Then the operators

$$\mathbf{H}_\tau : [H^s(S)]^6 \to [H^{s+1}(S)]^6, \qquad \mathbf{K}_\tau, \mathbf{N}_\tau : [H^s(S)]^6 \to [H^s(S)]^6, \qquad \mathbf{L}_\tau : [H^s(S)]^6 \to [H^{s-1}(S)]^6$$

are bounded. The operators $\mathbf{H}_\tau$ and $\mathbf{L}_\tau$ are strongly elliptic pseudodifferential operators of order -1 and 1 respectively, while the operators $\pm 2^{-1} I_6 + \mathbf{K}_\tau$ and $\pm 2^{-1} I_6 + \mathbf{N}_\tau$ are elliptic pseudodifferential operators of order 0.
Moreover, the operators $\mathbf{H}_\tau$, $2^{-1}I_6 + \mathbf{K}_\tau$, and $2^{-1}I_6 + \mathbf{N}_\tau$ are invertible, whereas the operators $\mathbf{L}_\tau$, $-2^{-1}I_6 + \mathbf{N}_\tau$, and $-2^{-1}I_6 + \mathbf{K}_\tau$ are Fredholm operators with zero index.

Let $h^{(1)} \in [H^{-1+s}(S)]^6$, $h^{(2)} \in [H^s(S)]^6$, $s > 0$. Then the following jump relations hold on S (see (**??**)-(**??**))

$$\begin{aligned}
\{\mathbf{V}^*_\tau(h^{(1)})(z)\}^\pm &= \int_S \Gamma^*(z-y,\tau)\, h^{(1)}(y)\, d_yS, \\
\{\mathbf{W}^*_\tau(h^{(2)})(z)\}^\pm &= \pm 2^{-1} h^{(2)}(z) + \int_S \left[\mathcal{T}(\partial_y, n(y), \overline{\tau})[\Gamma^*(z-y,\tau)]^\top\right]^\top h^{(2)}(y)\, d_yS, \\
\{\mathcal{P}\mathbf{V}^*_\tau(h^{(1)})(z)\}^\pm &= \mp 2^{-1} h^{(1)}(z) + \int_S \mathcal{P}(\partial_z, n(z), \tau)[\Gamma^*(z-y,\tau)]\, h^{(1)}(y)\, d_yS, \\
\{\mathcal{P}\mathbf{W}^*_\tau(h^{(2)})(z)\}^+ &= \{\mathcal{T}\mathbf{W}^*_\tau(h^{(2)})(z)\}^-.
\end{aligned}$$

Following theorems hold in the space of Hölder continuous functions (cf. Buchukuri et al(missing citation)).

Let $S \in C^{m,\alpha}$, $\ 0 < \beta < \alpha \le 1$, and let $k \le m-1$, $m \ge 2$ be nonnegative integers. Then the single and double layer vector potential operators

$$\mathbf{V}_\tau : [C^{k,\beta}(S)]^6 \to [C^{k+1,\beta}(\overline{\Omega^\pm})]^6, \qquad \mathbf{W}_\tau : [C^{k,\beta}(S)]^6 \to [C^{k,\beta}(\overline{\Omega^\pm})]^6$$

are continuous.

Let $S \in C^{m,\alpha}$, $0 < \beta < \alpha \leq 1$, and let $k \leq m-1$, $m \geq 2$ be nonnegative integers. Then the following boundary integral operators

are invertible, while the operators

are Fredholm with zero index.

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