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On establishing qualitative theory to nonlinear Boundary Value Problem of fractional Differential Equations

Amjad Ali[1], Nabeela Khan[2], and Seema Khan[2]

[1]University of Malakand
[2]Govt P.G Jahanzeb College

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Abstract

In concerned article, we investigate a class of boundary value problem of non-linear fractional differential equations. The aforesaid work is committed to the existence, uniqueness and stability analysis for boundary value problem of fractional differential equation. We used the tools of analysis and fixed point theory to establish the conditions for deserted results. At the end, we provided two examples to illustrate the concerned problem.

Introduction

The Fractional Calculus is known as the generalization of traditional calculus. In the last few decades, the aforesaid field attended more attention of researchers due to its verity of application in diverse field of social science and physical science, like physics, chemistry, economics and mechanics. One of the important aspects of aforementioned field that attended the attention of large number of researchers has existence of the solution for boundary value problems (BVPs) of fractional differential equations (FDEs). FDEs are widely applicable in image and signal processing, control theory, model identification, optimization theory, optics, fitting of experimental data for further detail, we refers **????????** to the readers. Furthermore, some other important applications of FDE are found in diverts fields of engineering, such as fluid dynamic like statistical, electromagnetic, statistical mechanics, fluid flow, polarization, colored noise, solid mechanics, traffic model, colored noise, processes, diffusion, economics and bioengineering see**??????????**, in references.
The researchers used various techniques and tools of analysis and fixed point theory to explored the concerned theory, for more detail we refer the readers **???**. However, the conditions for existence of solution of FDEs, in aforementioned articles needs the operator must be compactness, which restrict the concerned area of research to some specific limitions. Meanwhile, the researchers needs some weaker conditions for compactness of the operator. In order to resolve the aforesaid problem, Mawhin **?** used the tools of topological degree theory, to developed the essential condition for existence of solution for BVPs of FDEs and IEs. Furthermore, Isais **?**, used the degree theory to established some useful conditions for existence of solutions of FDEs. Recently, Wang at el **?**, used the techniques of topological degree theory to developed the conditions for existence of

the following non-local cauchy problem given by

$$\begin{aligned} &D^{\varsigma} v(t) = f(t, v(t)), \quad t \in [0, T], \\ &v(0) - v_0 = g(v), \end{aligned} \tag{1}$$

where D^{ς} represents the Caputo non integer order derivative, $v_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous function. Furthermore, Ali and Khan **?**, study the following BVPs of FDEs with non-local boundary conditions involving fractional integral is given by

where ${}^{c}D^{\varsigma}$ represents Caputo fractional derivatives and $g(v)$ is non-local function, $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous function.

One of the important feature of concerned field, due to which the researchers paid more attention, has the area devoted of stability analysis of BVPs of FDEs. There are various types stabilities present in the literature of fractional calculus. Ulam in (1940) initiated an important type of stability. Ulam cyan(S. M. Ulam & Sons, n.d.), proposed a question that "Under what conditions does there exists an additive mapping near an approximately additive mapping?". In response to has question Hyers cyan(the stability of the linear functional equatio Natl. Acad. Sci. U. S.A, n.d.; cyanthe stability of the linear transformation in Banach space J. Math. Soc, n.d.), replied that "additive mapping in complete norm spaces". Latter on it was tract out is a class of stability as so called Hyers-Ulam stability. The aforementioned stability was very much explore for conventional derivatives. However, for fractional differential equations the concerned stability was very rarely investigate and need further exploration.

There are various class of stabilities present in literature of concerned field for FDEs and IEs. Such as Lypunove stability cyan(2011), asymptotic stability cyan(2016), exponential stability cyan(R. Agarwal, n.d.; cyanT. M. Rassias "On the stability of the linear mapping in Banach space *Proc. Amer. Math. Soc.* 72(2), 1978) and many more. One of the interesting category of stability that was origination by Ulam and Hyres commonly known as Hyres-Ulam stability in (1940). Rassias [**?**,]]28, initiated a particular kind of stability is known as Generalized Hyers-Ulam-Rassias stability. Obloza cyan(J. Wang & data dependence for fractional differential, n.d.), was the author who investigate the concerned stability for DEs. Although the concerned stability was will studded for traditional differential equations. Furthermore, for FDEs the area concerning to the stability analysis was at its initial stages and a very few articles order had been published, we refers [**?**,]]30, in the references therein. This area of research need more attention of researchers to furnished the theory further. Motivated by the aforementioned importance of the concerned area, we consider the following non integer order derivatives with boundary conditions involving conventional order derivative is given by

(2)

where $3 < \varsigma \leq 4$ and $\forall\ s, t \in AC^4[0, 1]$, $f : J \times \mathbb{R} \to \mathbb{R}$ is continues and $h(1) = v$ is non-local function. In concerned work, we established necessary conditions for existence solutions and stability analysis for the proposed problem. We justified the developed conditions with help of some examples.

Axillary results and definitions

The concerned section, is devoted to some fundamental definitions and results of fractional calculus, which are necessary for further investigation. For further detailed study, we refer to readers cyan(L. Zhou et al., 2008; cyanM. Raja, 2011; cyanA. Mohebbi & Whitam-Brore-Kaup *Commun. Nonl. Sci. Numer. Simulat.* 17(12), 4610; cyanC. Goodrich "Existence of a positive solution to a class of fractional differential equations *J. Comp. Math. Appl.* 23(9), 1055; cyanMagin, n.d.; cyanS. F. Lacroix "Trait.6 du Calcul Differentiel et du *Calc. Integ. Paris.* 3, 1819).

Definition 0.1. *For all $\varsigma > 0$, Gamma function is usually represented by $\Gamma(\varsigma)$ and given by*

$$\Gamma(\varsigma) = \int\limits_0^1 e^{-t}(t)^{\varsigma-1}dt.$$

Definition 0.2. *The fractional order ($\gamma > 0$) integral of a function $u(t) : J \to \mathbb{R}$ is given by*

$$I^{\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int\limits_0^t (t-\sigma)^{\gamma-1}u(\sigma)d\sigma,$$

provided that integral at the right is defined on $(0, \infty)$ point wise.

Definition 0.3. *The famous non integer order Caputo's function $u(t)$ on any closed interval $[a, b]$ is given by*

$$^{c}D^{\gamma}_{0+}u(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-\sigma)^{n-\gamma-1}u^{(n)}(s)\, d\sigma,$$

where $n = [\gamma] + 1$, where $[\gamma]$ is greatest integer function, but not greater then γ.

Lemma 0.3.1. *The solution of non-integer order differential equation*

$$^{c}D^{\gamma}u(t) = 0, \qquad \gamma \in (0, \infty], \tag{1}$$

is given by

$$u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in (-\infty, \infty)$, *where* $i = 0, 1, 2, ..., n$.
Lemma 0.3.2. *For FDEs, the following result holds*

$$I^{\gamma} D^{\gamma} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in (-\infty, \infty)$, *where* $i = 0, 1, 2, ..., n$.
Definition 0.4. *Let us defined*

$$\mathbb{X} = \{u(t) \in CJ : \ \parallel u \parallel = \max_{t \in [0,1]} |u(t)|\},$$

then $(\mathbb{X}, \|u\|)$ *is a Banach Space.*
Definition 0.5. *Let* $T : V \to U$ *be a mapping, which is bounded and continuous. Then* T *is* ς*-Lipschitz, if* $\exists\ K \geq 0$*, such that*

$$\varsigma((B)) \leq K\varsigma(B), \quad \forall B \subset V \ \ bounded.$$

We also recall that $T : V \to U$ *is Lipschitz, if* $\exists\ K > 0$ *such that*

$$\|Fx - Fy\| \leq K\|x - y\|, \quad \forall \ \ x, y \in V,$$

and T *is strict contraction, if* $K < 1$.
Proposition 0.1. *If* $T, \mathbb{G} : V \to U$ *are both* ς*-Lipschitz mapings with constant* K *and* $K^{'}$*, then* $T + G : V \to U$ *is also* ς*-Lipschitz with* $K + K^{'}$ *constant.*
Proposition 0.2. *The mapping* T *is* ς*-Lipschitz, if* $T : V \to \varsigma$ *is Lipschitz with constant* K.
Proposition 0.3. *If* $T : V \to U$ *is compact, then* T *is* ς*-Lipschitz with zero constant.*
Theorem 0.6. *Let* E *be a measurable set and Let* $\{f_n\}$ *be a sequence of measurable function such that*

$$\lim_{n \to \infty} f_n(v) = f(v) \in E, and \ \ for \ \ every \ \ n \in N, \quad |f_n(v)| \leq h(v) \in E,$$

where g *is integrable on* E*, then*

$$\lim_{n \to \infty} \int\limits_{E} f_n(v) dv = \int\limits_{E} f(v) dv$$

Definition 0.7. *The Banach space* $\mathbb{X}$ *is compact, if every sequence* $\mathbb{S}_{\Bbbk}$ *contained a convergent sub-sequence in* $\mathbb{X}$.

Definition 0.8. *A space $\mathbb{X}$, where every Cauchy sequence of elements of $\mathbb{X}$ converges to an element of $\mathbb{X}$ is called a complete space.*
Definition 0.9. *If $A \subseteq X$ is relatively compact, if every sequence of A contain a sub-sequence is convergent in it.*
Definition 0.10. *The linear operator $T : V \to U$ continuous at v_0, if for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that*

$$|Tv - Tv_0| < \varepsilon, \quad \forall \quad |v - v_0| < \delta.$$

OR T is continuous, if $v_n \to v$, then

$$Fv_n \to Fv.$$

OR

$$\lim_{n\to\infty} |v_n - v| \to 0,$$

$$\Rightarrow \lim_{n\to\infty} |Fv_n - Fv| \to 0.$$

Definition 0.11. *The linear operator $T : V \to U$ is said to be uniformly continuous, if for $\varepsilon > 0$, $\exists\ \delta > 0$ such that*

$$|Tv - Tv_0| < \varepsilon, \quad \forall \quad |v - v_0| < \delta.$$

OR T is continuous, if $v_n \to v$, then

$$Fv_n \to Fv.$$

OR

$$\lim_{n\to\infty} |v_n - v| \to 0,$$

$$\Rightarrow \lim_{n\to\infty} |Fv_n - Fv| \to 0.$$

Definition 0.12. *The linear operator $T : V \to U$ is said to be uniformly continuous, if for $\varepsilon > 0$, $\exists\ \delta > 0$ such that*

$$|Fv - Fv^*| < \varepsilon\ \forall\ \ |v - v^*| < \delta.$$

Definition 0.13. *A family T in $C(I, R)$ is called uniformly bounded, if $\exists$ a constant, where $|f(t)| < k \quad \forall\ t \in P$ and $f \in T$.*
A family T is equi-continuous, if

$$|f(v) - f(v^*)| < \varepsilon \quad forall \quad v, v^* \in J,$$

with

$$|v - v^*| < \delta.$$

Theorem 0.14. *If a family T=$(f(v))$ in $C(I, R)$ is uniformly bounded and equi-continuous on P, then F has a uniformly convergent sub-equence $(f_n(v))$=1. Thus a subset T in $C(I, R)$ is relatively compact, iff T equi-continuous and uniformly bounded on J.*
Theorem 0.15. *Let $\mathbb{X}$ be a Banach space and $T : \mathbb{X} \to \mathbb{X}$ is function, which is completely continuous, then either*
(1) v=λTv has a solation, if λ=1. OR
(2) $\{$ $v \in \mathbb{X}$: v=λTv, for $\lambda \in (0, 1)$) $\}$ has a solution.
Definition 0.16. *The solution of FDEs is Hyers-Ulam stable, if $\exists\ \mathbb{K}_f > 0$ and we can find $\mathbb{L}_f > 0$, such that for each solution $v(t)$ to the system there exists a another solution $v^*(t)$, such that*

$$|v(t) - v^*(t)| \le \mathbb{K}_f\ \mathbb{L}_f.$$

Existence and Stability Analysis of FDEs

The concerned section, is devoted to the investigation of solutions and stability results of the following PVBs for FDEs with conditions contains the conventional derivatives is given by

(-1)

where $3 < \varsigma \leq 4$ and $\forall\ s,t \in AC^4[0,1]$, $f : P \times \mathbb{R} \to \mathbb{R}$ is continues and $h(1) = v$ is non-local function. In the following theorems, we provides the integral representation and the Green function for the concerned problem (**??**).
Theorem 0.17. *If $3 < \varsigma \leq 4$ and $\forall\ \sigma, t \in [0,1]$, then the solution to fractional differential equation subject to the condition involving ordinary derivatives*

(-1)

is Given by,

$$v(t) \quad = \quad t^3 h(v) + \int_0^1 \mathbb{H}(t,\sigma)\omega(\sigma)d\sigma,$$

(-1)

where $\mathbb{H}(t,\sigma)$ is represents the Green's function and given by ,

$$\mathbb{H}(t,\sigma) = \frac{1}{\Gamma(\varsigma)}\{$$

$$\begin{array}{ll} (t-\sigma)^{\varsigma-1} - t^3(1-\sigma)^{\varsigma-1}, & 0 \leq \sigma \leq t \leq 1, \\ -t^3(1-\sigma)^{\varsigma-1}, & 0 \leq t \leq \sigma \leq 1. \end{array}$$

Proof. Consider $f(t, v(t))$=$\omega(t)$, then (**??**) become,

(-3)

Then in veiw of Lemma

$$v(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + I^{\varsigma}\omega(t), \qquad \text{(-3)}$$

By using the boundary conditions $v(0) = 0$ in (, we get

$$\gamma_0 = 0.$$

Therefore equation (

$$v(t) = \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + I^{\varsigma}\omega(t). \quad \text{(-3)}$$

Now differentiating equation (

$$\gamma_1 = 0.$$

Therefore equation(

$$D^1 v(t) = \gamma_1 + 2\gamma_2 t + 3\gamma_3 t^2 + I^{\varsigma-1}\omega(t). \quad \text{(-3)}$$

Again differentiating equation (

$$\gamma_2 = 0.$$

Now using the boundary conditions $v(1) = h(v)$ and putting the values of $\gamma_1, \gamma_2, \gamma_3$ in equation (

$$\gamma_3 = h(u) - \frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\sigma)^{\varsigma-1}\omega(\sigma)d\sigma,$$

(-4)

Putting these values in equation (

$$v(t) = \quad t^3 h(v) + \tfrac{1}{\Gamma(\varsigma)} \int_0^1 \mathbb{H}(t,\sigma)\omega(\sigma)d\sigma,$$

(-4)

where

$$\mathbb{H}(t,\sigma) = \frac{1}{\Gamma(\varsigma)} \{$$

$$
\begin{array}{ll}
(t-\sigma)^{\varsigma-1} - t^3(1-\sigma)^{\varsigma-1}, & 0 \leq \sigma \leq t \leq 1, \\
-t^3(1-\sigma)^{\varsigma-1}, & 0 \leq t \leq \sigma \leq 1.
\end{array}
$$

In view of the established results for linear BVP (**??**), which is equivalent to the following integral equation as

$$
v(t) = t^3 h(v) + \int\limits_0^1 \mathbb{H}(t,\sigma) f(\sigma, v(\sigma)) d\sigma.
$$

(-4)

The equation (is integral representation of our proposed problem (**??**).

□

Lemma 0.17.1. *The function* $\mathbb{H}(t,\sigma)$ *satisfies the following properties*
(i) $\mathbb{H}(t,\sigma)$ *is continuous* $\forall\ s,t \in [0,1]$,
(ii) $\max\limits_{t,s\in[0,1]} \mathbb{H}(t,\sigma) \leq \frac{6\Gamma(\varsigma)}{\Gamma(4+\varsigma)}$.

$$
\max_{t,s\in[0,1]} \mathbb{H}(t,\sigma) = \mathbb{H}(\sigma,\sigma)
$$

$$
\max_{t,s\in[0,1]} \int\limits_0^1 \mathbb{H}(t,\sigma) d\sigma = \max_{t,s\in[0,1]} \frac{1}{\Gamma(\varsigma)} \int\limits_0^1 \sigma^3 (1-\sigma)^{\varsigma-1} d\sigma,
$$

$$
= \frac{6\Gamma(\varsigma)}{\Gamma(4+\varsigma)}.
$$

(-4)

Equation (, is deserted value of constructed Green function.

Existence, uniqueness and Data Dependence Results

In this section, we produced some results for existence, uniqueness and data dependence. We also provides the following assumption must holds, which are need for further investigation in this work.

(H_1) *For arbitrary* $v, u \in X$, $\exists$ *a constant* $\mathbb{K}_h \in [0, 1)$, *such that*

$$|h(v) - h(u)| \leq \mathbb{K}_h \parallel v - u \parallel;$$

(H_2) *For arbitrary* $v \in X$, *there exist* $\mathbb{C}_h, \mathbb{M}_h > 0, \quad b_1 \in [0, 1)$ *such that*

$$|h(v)| \leq \mathbb{C}_h \parallel v \parallel^{b_1} + \mathbb{M}_h;$$

(H_3) *For arbitrary* $(t, v) \in X$, $\exists$ $\mathbb{C}_f, \mathbb{M}_f > 0, \quad b_2 \in [0, 1)$, *such that*

$$|f(t, v)| \leq \mathbb{C}_f |v|^{b_2} + \mathbb{M}_f.$$

(H_4) *To derive uniqueness of solution the following assumption holds true for.* $\mathbb{L}_f > 0$, *such that is* $|f(t, v) - f(t, v^*)| \leq \mathbb{L}_f |v - v^*|$.

Operator equations

In this section, we convert our obtained integral equation into operator equation. For which we define $T : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$,

$$T(v) = v, \qquad v \in X,$$

where $Tv = Fv + Gv$, $F : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$ *and* $G : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$. *Where*

$$Fv = t^3 h(v),$$

$$Gv = \int_0^1 \mathbb{H}(t, \sigma) f(\sigma, v(\sigma)) d\sigma.$$

Hence the proposed problem gained the of a operator equation $Tv = Fv + Gv = v$. *The fixed points of the constructed operator equation are the deserted solutions of concerned BVP* (**??**).

Theorem 0.18. *The "operator $F : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$ is Lipschitz with constant $\mathbb{C}_h < 1$ and satisfies the condition"*

$$\|Fv\| \leq \mathbb{C}_h \|v\|^{b_1} + \mathbb{M}_h.$$

Proof. We defined $F : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$ is given by

$$Fv = t^3 h(v),$$

to prove F is Lipschitz, we have

$$\|Fv - Fu\| = \max_{t \in [0,1]} |t^3 h(v) - t^3 h(u)|,$$

using assumption of (H_1), we have

$$\|Fv - Fu\| \leq \|k\| \|v - u\|. \qquad \text{where} \quad k = \mathbb{K}_h < 1,$$

For growth condition we consider

$$\|Fv\| = \max_{t \in [0,1]} |t^3 h(v)|,$$

using assumption of (H_2), we get

$$\|Fv\| \leq \mathbb{C}_h \|v\|^{b_1} + \mathbb{M}_h.$$

The above result shows that F satisfies the Lipschitz condition with constant $\mathbb{C}_h$.

□

Theorem 0.19. *The operators $G : C(P \times R, R) \to C(P \times R, R) \to$ is continuous and satisfies the following*

$$\|Gv\| \leq 2 \left(\frac{\mathbb{C}_f \|v\|^{b_2} + \mathbb{M}_f}{\Gamma(\varsigma + 1)} \right), \quad \text{for every} \quad v \in AC^4[0,1].$$

Proof. As $G : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$ is given as

$$Gv = \int_0^1 \mathbb{H}(t,\sigma) f(\sigma, v(\sigma)) d\sigma,$$

To prove that G is continuous. We will prove as,

$$\|Gv_n - Gv\| \to 0 \quad \text{as } n \to \infty$$

Let $\{v_n\}$ is a sequence in bounded set such that

$$B_K = \{\|v\| \leq K : v \in X)$$

Now as f is continuous, so $\quad f(\sigma, v_n(\sigma)) \to f(\sigma, v(\sigma))$ as $n \to \infty$.

$$t^3(1-\sigma)^{\varsigma-1}|f(\sigma, v_n) - f(\sigma, u)| \leq t^3(1-\sigma)^{\varsigma-1}\{2\mathbb{C}_f + \mathbb{M}_f\},$$

is also integrable for all s $\in [0,1]$.
By convergent theorem (Lebesgue), we have

$$\int_0^t (t-\sigma)^{\varsigma-1}[f(\sigma, v_n(\sigma)) - f(\sigma, v(\sigma))]d\sigma \to 0,$$

and

$$t^3 \int_0^1 (1-\sigma)^{\varsigma-1}[f(\sigma, v_n(\sigma)) - f(\sigma, v(\sigma))]d\sigma \to 0 \quad \text{as} \quad n \to \infty,$$

so

$$\|Gv_n - Gv\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence G is continuous.
Now to derive growth condition, we do the following

$$||Gv|| = \max_{t\in[0,1]} \left| \frac{1}{\Gamma(\varsigma)} \int_0^t (t-\sigma)^{\varsigma-1} f(\sigma, v(\sigma)) d\sigma - \frac{t^3}{\Gamma(\varsigma)} \int_0^1 (1-\sigma)^{\varsigma-1} f(\sigma, v(\sigma)) d\sigma \right|,$$

Hence by assumption (H_3) we get

$$||Gv|| \leq \frac{\mathbb{C}_f \|v\|^{b_2} + \mathbb{M}_f}{\Gamma(\alpha+1)}(1+1), \|Gv\| \leq 2\left(\frac{\mathbb{C}_f \|v\|^{b_3} + \mathbb{M}_f}{\Gamma(\varsigma+1)}\right).$$

Thus G is satisfy the define growth condition.

$\square$

Theorem 0.20. *The operator $G : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$ is Compact and α-Lipschitz with constant zero.*

Proof. As $G : C(P \times \mathbb{R}, \mathbb{R}) \to C(P \times \mathbb{R}, \mathbb{R})$, is given by

$$Gv = \frac{1}{\Gamma(\varsigma)} \left(\int_0^t (t-\sigma)^{\varsigma-1} f(\sigma, v_n(\sigma)) d\sigma + t^3 \int_0^1 (1-\sigma)^{\varsigma-1} f(\sigma.v_n(\sigma)) d\sigma \right),$$

in order to prove G Compact. We have to show that G is both equi-continuous and uniform bounded.
Let us consider $D \subseteq B_k \subseteq X$, for this is sufficient to show that $G(D)$ is relatively compact in X, Let v_n in $D \subseteq B_k$, $\quad \forall \ v_n \in D$,
in the light of Theorem

$$Gv_n(\tau) = \frac{1}{\Gamma(\varsigma)} \left(\int_0^\tau (\tau-\sigma)^{\varsigma-1} f(\sigma, v_n(\sigma)) d\sigma + \tau^3 \int_0^1 (1-\sigma)^{\varsigma-1} f(\sigma, v_n(\sigma)) d\sigma \right),$$

Now by using assumption (H_3), we have

As $t \to \tau$, the R.H.S in above relation tends to 0, that is

$$|Gu_n(\tau) - Gu_n(t)| \to 0 \quad \text{as} \quad t \to \tau.$$

Thus Gu_n is uniformly continuous.

Thus $\{Gu_n\}$ is equi -continuous. Hence $G(D) \subset G(D)$. Thus by Arzela Ascoli theorem $G(D)$ is relatively compact in X. Further G is $\alpha-$ Lipschitz with constant zero.

$\square$

0.1 Uniqueness of solutions of BVP (??) Of PFDE

In this section, we developed the condition for uniqueness and boundness of BVP (??).

Theorem 0.21. *The consider BVP (??) has at least one solution and the set of the solutions is bounded.*

Proof. As the operators $F, G, T : C(P \times R, R) \to C(P \times R, R)$ are continuous and bounded. Further F, G are α-Lipschitz with constant K and 0, and T is α Lipschitz with Lipshitz constant K. Since $\mathbb{K}_h < 1$ so T is a contraction mapping.
Consider the set of solution

$$S = \{v \in X : \quad 0 \leq \lambda \leq 1, \quad v = \lambda T v\} \ \ \textit{is bounded.}$$

For boundness, consider

$$\|v\| = \max_{t\in[0,1]} |\lambda T v| \leq \max_{t\in[0,1]} \lambda |Tv| \leq \max_{t\in[0,1]} \lambda |Fv + Gv|,$$

$$\|v\| \leq \lambda \left(\mathbb{C}_h \|v\|^{b_1} + \mathbb{M}_h + 2 \left(\frac{\mathbb{C}_f \|v\|^{b_2} + \mathbb{M}_f}{\Gamma(\varsigma + 1)} \right) \right)$$

From above it is clear that S is bounded. If not, let $\varsigma = \|v\| \to \infty$ as $0 < b_1, b_2 < 1$,

$$\|v\| = \|\lambda T v\| \leq \lambda \|Tv\| \leq \lambda \|Fv + Gv\|$$

$$1 \leq \frac{\lambda}{\varsigma} \left(\mathbb{C}_h \varsigma^{b_1} + 2 \left(\frac{\mathbb{C}_f \varsigma^{b_2} + \mathbb{M}_f}{\Gamma(\varsigma + 1)} \right) \right),$$

as $\varsigma \to \infty$, which means $1 \leq 0$ is not possible. Hence a set S is bounded. □

Theorem 0.22. *If $\delta = K + \frac{2\mathbb{L}_f}{\Gamma(\varsigma+1)}$ and less then 1, then our proposed BVP* **??** *has a unique solution.*

Proof. Consider $u, v \in X$, with Contraction principle

using assumption (H_4), we have

$$|Tv(t) - Tu(t)| \leq \Big(K + \frac{2\mathbb{L}_f}{\Gamma(\varsigma + 1)} \Big) |v - u|,$$

$$|Tv(t) - Tu(t)| = \Delta|v - u| \leq \delta.$$

where

$$\Delta = K + \frac{2\mathbb{L}_f}{\Gamma(\varsigma + 1)},$$

there exist unique solution to BVP (**??**).

□

Stability analysis of BVP of FDEs

In the concerned section of this work, we established condition for the Hyers-Ulam stability for the BVP of FDE (**??**).

Theorem 0.23. *If the assumption* (H_1)*–*(H_4) *holds, then the solution is Hyers-Ulam stable.*

Proof. Let v *and* $v^* \in C^4(I, R)$ be any two solution of BVPs **??**. For stability

(-7)

$$v(t) = t^3 h(v) + \int_0^1 \mathbb{H}(t, \sigma) f(\sigma, v(\sigma)) d\sigma,$$

$$v^*(t) = t^3 h(v^*) + \int_0^1 \mathbb{H}(t, \sigma) f(\sigma, v^*(\sigma)) d\sigma.$$

Consider

$$|u(t)-v(t)| = \max_{t\in[0,1]} |t^3h(v)+\int_0^1 \mathbb{H}(t,\sigma)f(\sigma,v(\sigma))d\sigma]-t^3h(v^*)+\int_0^1 \mathbb{H}(t,\sigma)f(\sigma,v^*(\sigma))d\sigma|,$$

$$\leq \max_{t\in[0,1]} |t^3h(v)-t^3h(v^*)|+\max_{t\in[0,1]} |\int_0^1 \mathbb{H}(t,\sigma)[f(\sigma,v(\sigma)-f(\sigma,v^*(\sigma))]d\sigma|,$$

using assumption (H_1) and (H_4), we have

$$\leq \mathbb{K}_h \parallel v-v^* \parallel + \max_{t\in[0,1]} \int_0^1 \mathbb{H}(t,\sigma)d\sigma\mathbb{L}_f \parallel v-v^* \parallel,$$

using maximum value of green function

$$\parallel v(t)-v^*(t) \parallel \leq \parallel v-v^* \parallel [\mathbb{K}_h + \frac{6\Gamma(\varsigma)}{\Gamma(4+\varsigma)}\mathbb{L}_f],$$

$$\parallel v(t)-v^*(t) \parallel \leq K_1\ K_2,$$

(-9)

where $K_1 = \parallel v-v^* \parallel$ and $K_2 = [\mathbb{K}_h + \frac{6\Gamma(\varsigma)}{\Gamma(4+\varsigma)}\mathbb{L}_f]$.
Hence the solution of BVP **??**, is Hyers-Ulam stable.

□

Examples

In this section, we provide some examples which illustrate the our proposed problem of BVP of FDEs.

Example 1. *Consider the following BVP for FDEs*

(-9)

where $\varsigma = 7/2$, *Now where* $\mathbb{H}(t,\sigma)$ *is,*

$$\mathbb{H}(t,\sigma) = \frac{1}{\Gamma(7/2)}\{$$

$$\begin{array}{ll} (t-\sigma)^{5/2} - t^3(1-\sigma)^{5/2}, & 0 \le \sigma \le t \le 1, \\ -t^3(1-\sigma)^{5/2}, & 0 \le t \le \sigma \le 1. \end{array}$$

Now

$$|f(t,v)| \le \mathbb{C}_f |v|^{b_2} + \mathbb{M}_f.$$

where

$$C_f = 1/40, \quad M_f = 1/40, \quad b_2 = 1,$$

$$|f(t,v)| \le 0.05,$$

Now calculating

$$\Delta = K + \frac{2\mathbb{L}_f}{\Gamma(\varsigma+1)},$$

where

$$K = 1/30, L_f = 1/40, \varsigma = 7/2,$$

$$|Tv(t) - Tu(t)| \le 0.0376 < 1.$$

Assumption $(H_1)-(H_2)$, holds, therefore solution of concerned problem has at least one solution.
For the stability of BVP

$$|v(t)-v^*(t)| \leq \| v-v^* \| [\mathbb{K}_h + \frac{6\Gamma(\varsigma)}{\Gamma(4+\varsigma)}\mathbb{L}_f].$$

Let

$$K_h = 1/30, L_f = 1/40 and \varsigma = 7/2,$$

$$|v(t)-v^*(t)| \leq 0.0045 < 1.$$

Hence the solution BVP

Example 2. *Consider the BVP for FDEs*

(-10)

where $\varsigma = 5/2$,
where $\mathbb{H}(t,\sigma)$ *is*

$$\mathbb{H}(t,\sigma) = \frac{1}{\Gamma(5/2)} \{$$

$$\begin{array}{ll} (t-\sigma)^{3/2} - t^3(1-\sigma)^{3/2}, & 0 \leq \sigma \leq t \leq 1, \\ -t^3(1-\sigma)^{3/2}, & 0 \leq t \leq \sigma \leq 1. \end{array}$$ *Now*

(-11)

Now calculating

$$\Delta = K + \frac{2\mathbb{L}_f}{\Gamma(\varsigma+1)},$$

where

$$K = 1/40, L_f = 1/50, \varsigma = 5/2,$$

$$|Tv(t) - Tu(t)| \leq 0.03074 < 1.$$

The assumption $(H_1) - (H_2)$ *holds. Hence the proposed BVP is at least one solution.*
For the stability of BVP **??** *of FDE,*

$$|v(t) - v^*(t)| \leq \parallel v - v^* \parallel [\mathbb{K}_h + \frac{6\Gamma(\varsigma)}{\Gamma(4+\varsigma)} \mathbb{L}_f],$$

Let

$$K_h = 1/40, L_f = 1/50 and \varsigma = 5/2,$$

$$|v(t) - v^*(t)| \leq 0.0011097 < 1.$$

Hence the solution BVP **??** *has a unique solution and the solution has stable.*

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