Topological Structure of Moist Equatorial Waves

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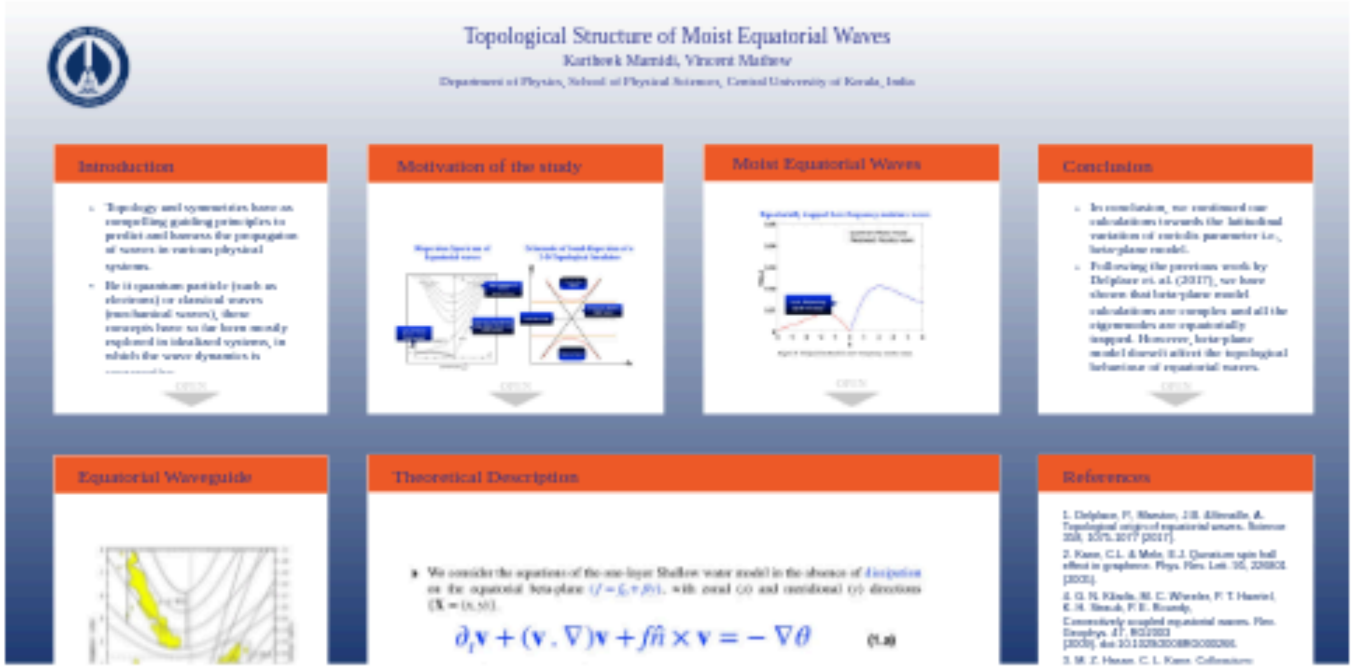
Abstract

Symmetries and topology have been actively introduced currently to characterize the mode structure of waves in various systems physics, giving rise to the concepts of topological insulators, topological superconductors and topological photonics, to name a few. Very recently, the equatorial wave systems have been described from a topological point of view by Delplace et. al. (Science 358, 1075-1077 (2017)). It was shown that the emergence of unidirectional edge waves (Yanai and Kelvin waves) can be attributed to the topological bound states. An f-plane model is used to connect the topological invariants, the Chern numbers, to the existence of these modes. We have extended this analysis by incorporating a beta plane model thereby including the Earth's sphericity from beginning. Equatorial beta plane model renders the Poincare and Rossby waves also equatorially trapped. Further, the effect of moisture balance on the topology of the equatorial waveguide is examined. It is shown that the presence of a new eastward propagating mode within a low-frequency regime is similar to the observed MJO mode. We explained how moisture localizes these low-frequency unidirectional oscillations. The topological origin of moist waves is emphasized by relating their topological invariants, or Chern numbers. From this perspective, equatorial moist waves also show the strong similarities with bulk-edge correspondence encountered in quantum valley Hall effect and its classical analogues. Our study shows that the topological origin of MJO-like mode and its localization due to low-level moisture encode essential information of the tropical climatic systems.

Topological Structure of Moist Equatorial Waves



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PRESENTED AT:

INTRODUCTION

- Topology and symmetries have as compelling guiding principles to predict and harness the propagaton of waves in various physical systems.

- Be it quantum particle (such as electrons) or classical waves (mechanical waves), these concepts have so far been mostly explored in idealized systems, in which the wave dynamics is conservative.

- Very recently, the equatorial wave systems have been described from a topological point of view by Delplace et. al (2017).
- It was shown that the emergence of unidirectional edge waves (Yanai and Kelvin waves) can be attributed to the topological bound states.
- An f-plane model is used to connect the topological invariants, the Chern numbers, to the existence of these modes
- Topology guarantees the existence of equatorial Yanai and Kelvin waves, obviating the need to carry out the classic but more complex calculation on the equatorial beta-plane.
- We have extended this analysis by incorporating a beta plane model thereby including the Earth's sphericity from beginning.
- Equatorial beta plane model renders the Poincare and Rossby waves also equatorially trapped. Further, the effect of moisture balance on the topology of the equatorial waveguide is examined.

Equatorial Waveguide

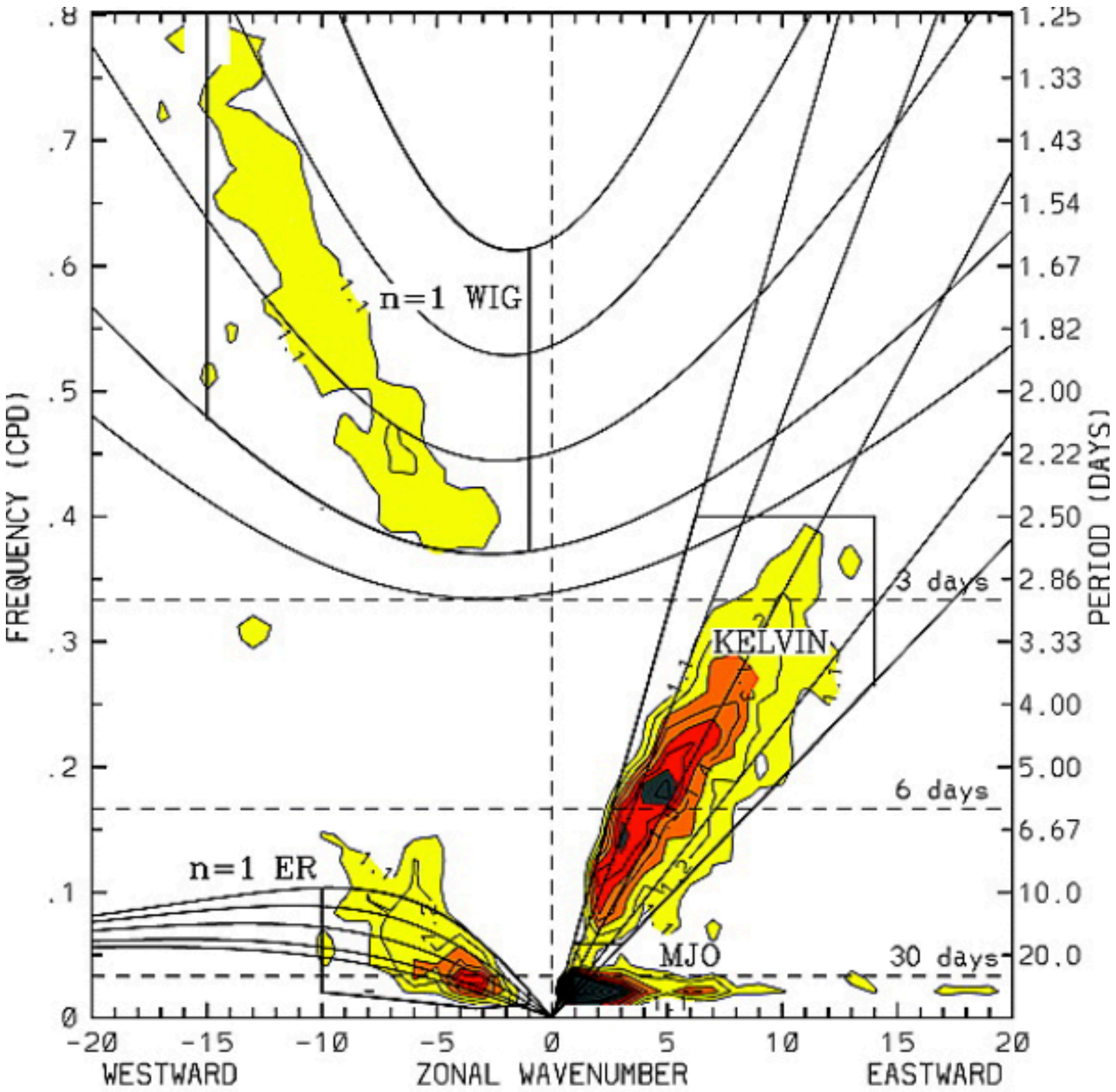
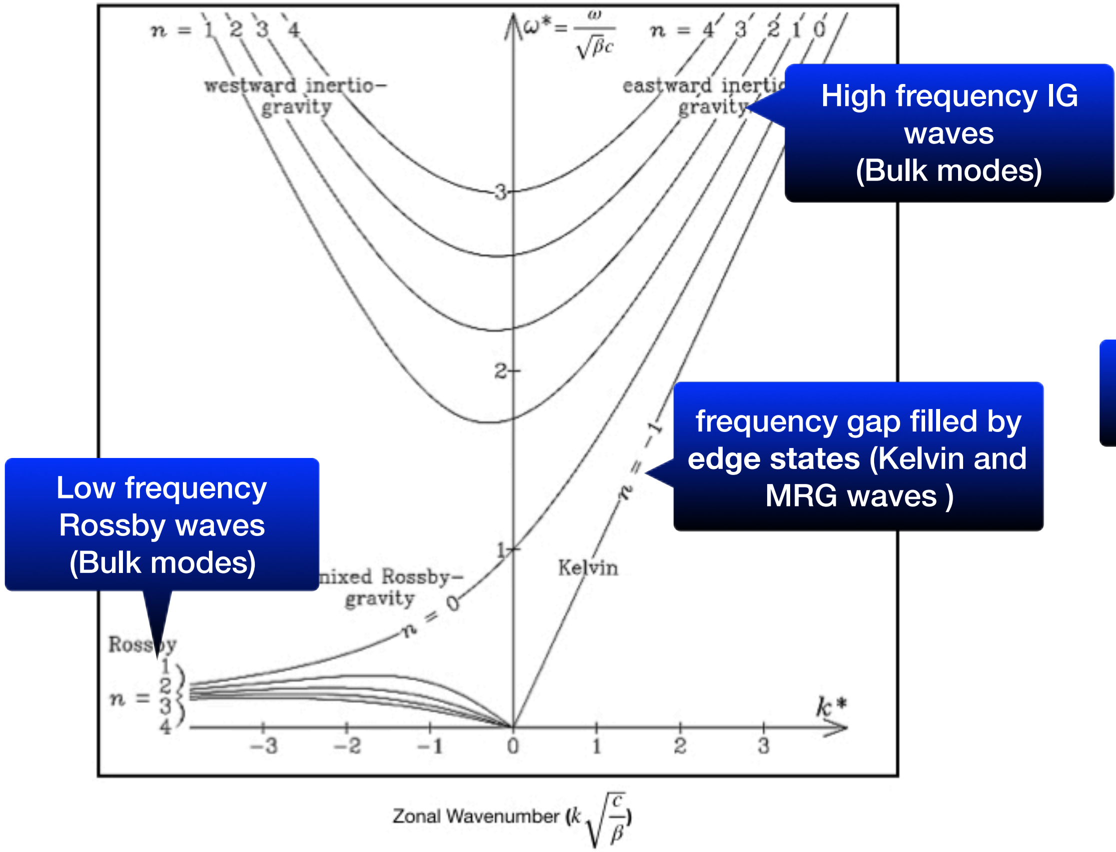


Figure 1 : Observed wavenumber–frequency spectrum of the equatorial symmetric component of brightness temperature[4]

- The flow in the tropical atmosphere is characterized by a perturbation in rainfall and cloud distribution.
- These perturbations are the manifestations of equatorial waves.
- Several authors suggested that these equatorial waves are the significant physical processes within the tropics, which play a crucial role in the large-scale dynamics.

Motivation of the study

Dispersion Spectrum of Equatorial waves



Schematic of band dispersion of a 2-D Topological Insulator

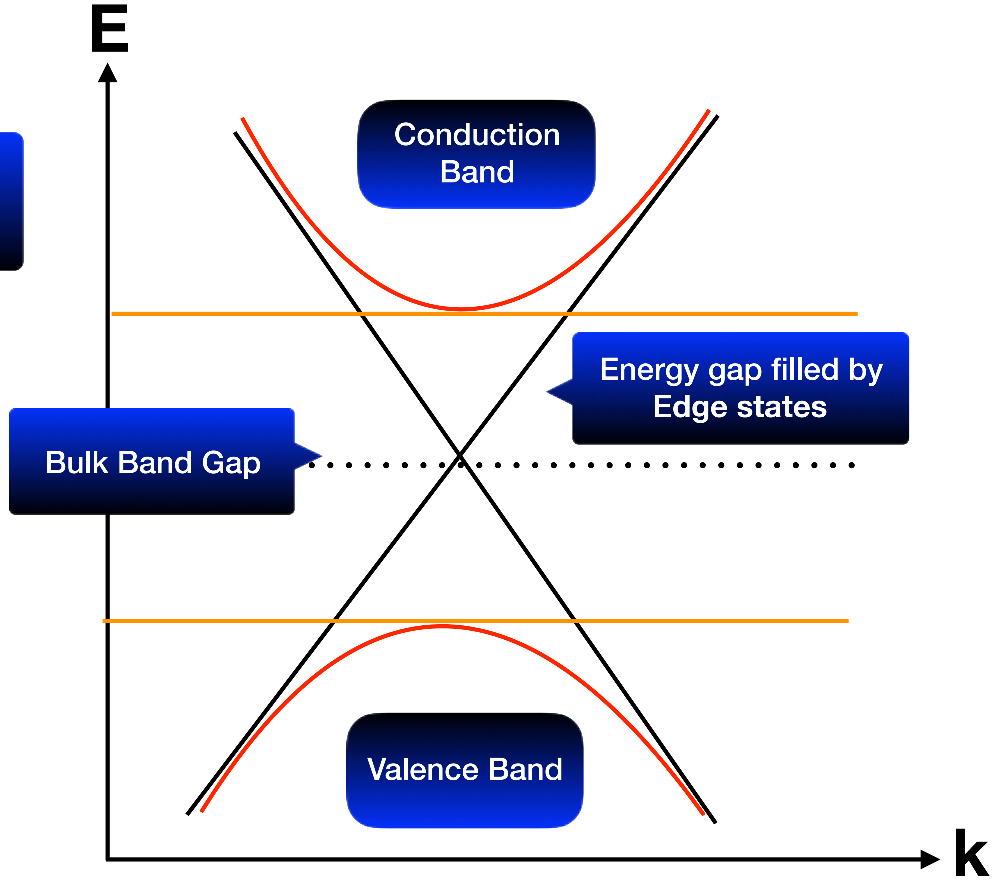
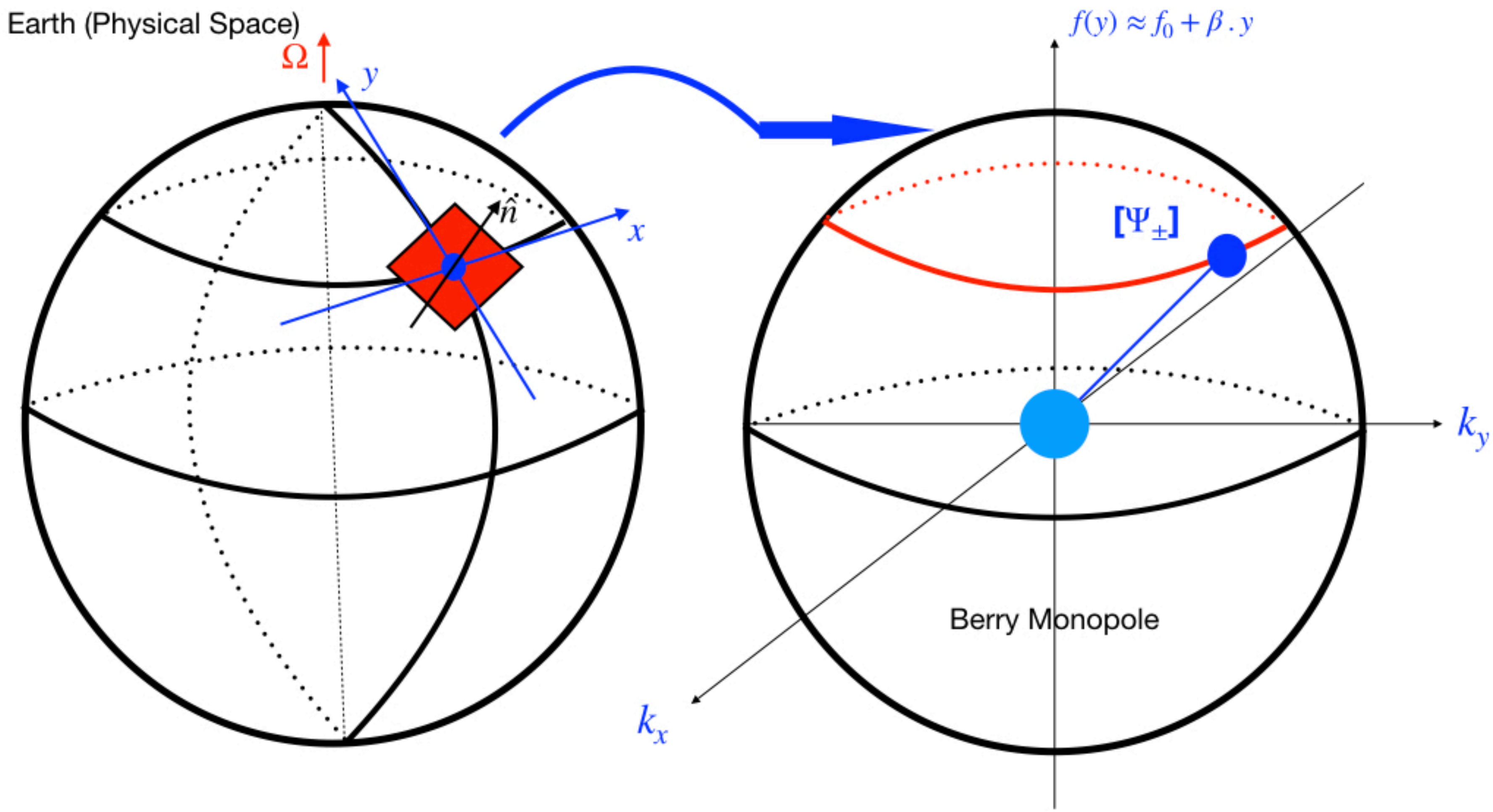


Figure 2 : Comparison of Equatorial waves (Classical Waves) with the Band Structure (Quantum Waves) of the Topological Insulator



(a) beta-plane geometry on a rotating planet **(b) Corresponding Parameter space** (k_x, k_y, f)

Theoretical Description

- We consider the equations of the one-layer Shallow water model in the absence of dissipation on the equatorial beta-plane $(f = f_0 + \beta y)$, with zonal (x) and meridional (y) directions $\{\mathbf{X} = (x, y)\}$.

$$\partial_t \mathbf{v} + (\mathbf{v} \,.\, \nabla)\mathbf{v} + f\hat{n} \times \mathbf{v} = -\nabla\theta \qquad \textbf{(1.a}$$

$$\partial_t \theta + \nabla \,.\, (\mathbf{v}\theta) = 0 \qquad \textbf{(1.b}$$

- The velocity filed is described by $\mathbf{V}(\mathbf{x}, t) = (u(\mathbf{x}, t), v(\mathbf{x}, t))$, and here $\nabla = (\partial/\partial x, \partial/\partial y)$, and $\hat{n}$ is the unit vector in the vertical direction, and θ is the potential temperature perturbation.
- We non-dimensionalised and linearised the above equation of motion at ($u = 0$ and $\theta = \hat{\theta}$) and We look for the solution (Eigen modes) in the form of a wave (Planar wave), with amplitudes of the fields ($\zeta(x, y, t) = \zeta(y)e^{i(\omega t - k_x x - k_y y)}$).

Resultant eigenvalue equation

$$\frac{\partial u}{\partial t} + fv + \frac{\partial \theta}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} - fu + \frac{\partial \theta}{\partial y} = 0$$

$$\frac{\partial \theta}{\partial t} + (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

$$\omega \begin{pmatrix} \hat{\theta} \\ \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 0 & k_x & k_y \\ k_x & 0 & -if \\ k_y & if & 0 \end{pmatrix} \begin{pmatrix} \hat{\theta} \\ \hat{u} \\ \hat{v} \end{pmatrix}$$

Where $(f = f_0 + \beta y)$

By rearranging the above linear eigenvalue equation
The dynamical system becomes

$$i\partial_t \Psi = \mathbf{H}\Psi \qquad \textbf{(2)}$$

Where $\Psi = (\mathbf{v}, \theta)$ and $\mathbf{H}$ is the Hermitian operator

In a Quantum mechanical context it can be referred as a Hermitian system and can easily applied to the Conservative systems (idealized physical systems in which the wave amplitude is neither attenuated nor amplified)

The time reversal symmetry is broken due to non-zero Coriolis parameter and this broken symmetry generates the gaps in the wave spectrum[1]

$$t \to -t, x' \to x, \theta \to \theta, u \to -u$$

Diagonalization of equation (2) leads to three eigenmodes $\omega_0 = 0$, $\omega_\pm = \pm\sqrt{k^2 + f^2}$, where $k^2 = k_x^2 + k_y^2$ and corresponding **three** eigenvectors (wave bands)

$$\Psi_+(k_x, k_y, f) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{k}{\sqrt{k^2+f^2}} \\ \frac{k_x}{k} - i\frac{fk_y}{k\sqrt{k^2+f^2}} \\ \frac{k_y}{k} + i\frac{fk_x}{k\sqrt{k^2+f^2}} \end{pmatrix} \qquad \Psi_-(k_x, k_y, f) = \Psi_+(-k_x, -k_y, f) \qquad \Psi_0(k_x, k_y, f) = \frac{1}{\sqrt{k^2+f^2}} \begin{pmatrix} f \\ ik_y \\ -ik_x \end{pmatrix}$$

- We now consider an atmospheric component of the model that describes the dynamics of anomaly circulation of the lower troposphere, represented by the Linear shallow water equations describing the horizontal structure of the first baroclinic wave motion on an equatorial β-plane, along with vertically integrated moisture equation.
- We non-dimensionalize the governing equations using the spatial, temporal and potential temperature scales defined by

$$[L_0] = \{\frac{C_0}{\beta}\}^{1/2} \quad [T_0] = \{\frac{1}{C_0\beta}\}^{1/2} \quad \text{and} \quad [\theta_0] = H\frac{d\bar{\theta}}{dz}$$

$$\frac{\partial u}{\partial t} + \epsilon^* u + fv + \gamma\frac{\partial \theta}{\partial x} = 0 \qquad \textbf{(3.a)}$$

$$\frac{\partial v}{\partial t} + \epsilon^* v - fu + \gamma\frac{\partial \theta}{\partial y} = 0 \qquad \textbf{(3.b)}$$

$$\frac{\partial \theta}{\partial t} + \gamma[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] - \eta B s = 0 \qquad \textbf{(3.c)}$$

$$\frac{\partial s}{\partial t} + \Gamma_q[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] + \Lambda u + Bs = 0 \qquad \textbf{(3.d)}$$

where, $\Gamma_q = \frac{H}{\bar{q}}\frac{d\bar{q}}{dz}$

Resultant eigenvalue equation

$$\omega\begin{pmatrix}\hat{\theta}\\ \hat{u}\\ \hat{v}\\ \hat{s}\end{pmatrix} = \begin{pmatrix} 0 & \gamma k_x & \gamma k_y & i\eta B \\ \gamma k_x & i\epsilon^* & -if & 0 \\ \gamma k_y & if & i\epsilon^* & 0 \\ 0 & \Gamma_q k_x + i\Lambda & \Gamma_q k_y & iB \end{pmatrix}\begin{pmatrix}\hat{u}\\ \hat{v}\\ \hat{\theta}\\ \hat{s}\end{pmatrix}$$

Again by rearranging the above linear eigenvalue equation
The dynamical system becomes

$$\boxed{i\partial_t \Psi = \hat{H}\Psi}$$

Where $\Psi = (\mathbf{v}, \theta, s)$ and $\hat{H}$ is the Non-Hermitian operator

The parameters Λ: the non-dimensional values of evaporation- wind feedback component. Γ_q represents the strength of moisture convergence. B and γ represents, the moisture relaxation timescale and temperature scale respectively. ε^* represents the Rayleigh damping parameter.

Equatorially trapped Low-frequency moisture waves

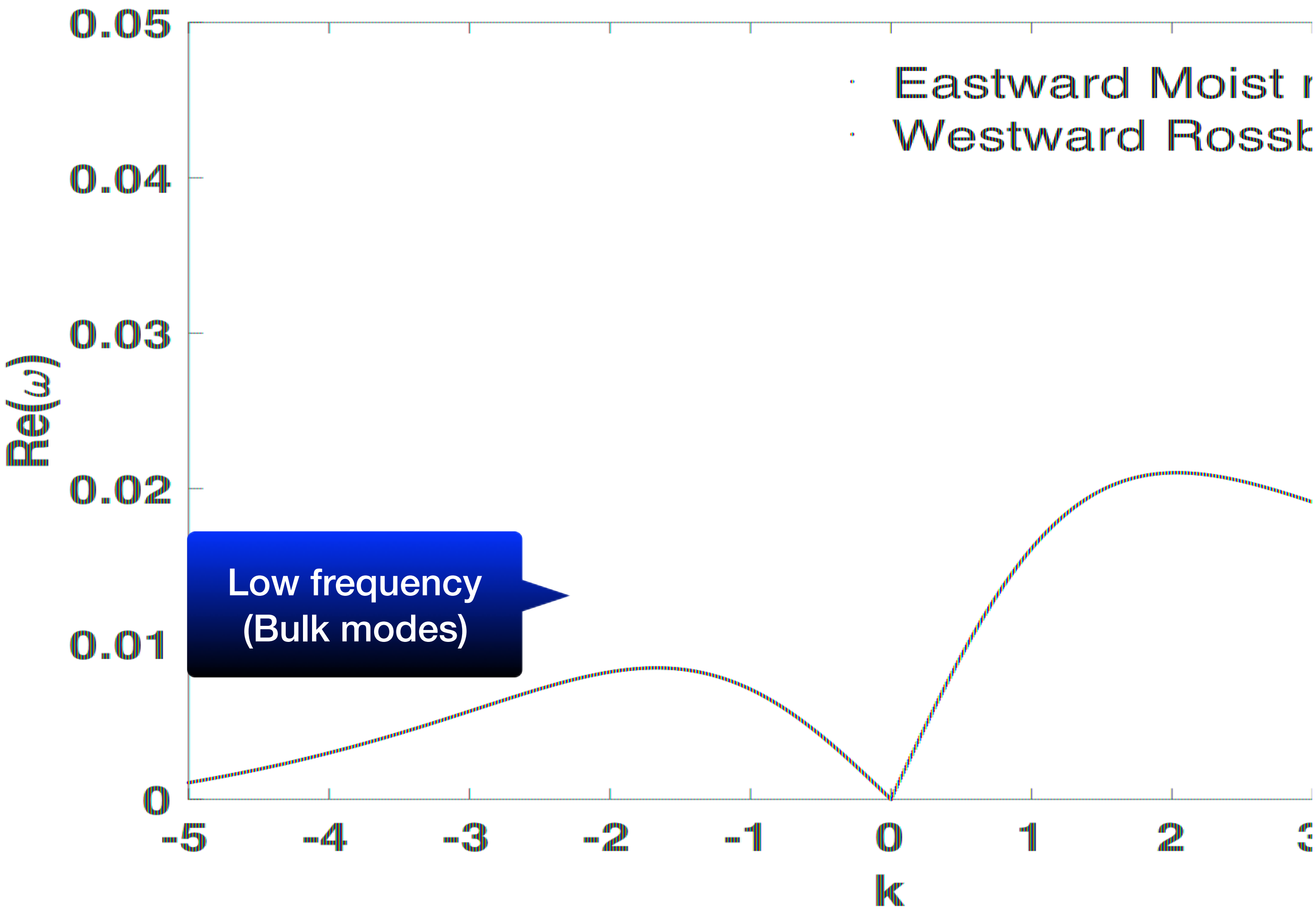


Figure 4 : Dispersion Relation (low frequency modes only)

CONCLUSION

- In conclusion, we continued our calculations towards the latitudinal variation of coriolis parameter i.e., beta-plane model.
- Following the previous work by Delplace et. al. (2017), we have shown that beta-plane model calculations are complex and all the eigenmodes are equatorially trapped. However, beta-plane model doesn't affect the topological behaviour of equatorial waves.
- We further incorporated vertically integrated moisture equation along with the momentum damping into the shallow water system, and we have seen that a new eastward propgating low-frequency mode, identified to be MJO emerges.
- However, in contrast with the dry shallow water system, incorporation of moisture and damping makes the system non-Hermitian.
- Understanding the topological properties of non-Hermitian Hamiltonian in continuum system is still an open area.

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