Conservation of total wave action and the one dimensional evolution of simple waves in the solar wind

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Abstract

We investigate the evolution of outwardly propagating simple MHD waves in a model of the expanding solar wind using MHD simulations. In order to understand the different evolution of slow, Alfvén and fast modes, the question of wave-action conservation is re-examined theoretically. Using the fluctuation averaged Lagrangian, we discuss the conservation of total wave action and Equi-partition of wave energy for MHD waves. Results show that, even though the wave action for a simple monochromatic wave is subject to loss under resonance/degeneracy condition - conditions that can occur in the expanding solar wind in the regions where plasma ß crosses one, the total wave action possessed by all modes remains conserved, representing a wave action exchange between different degrees of freedom. The Expanding Box simulations demonstrate the results of the theoretical modeling, and reveal further details about mode-mixing, Alfvén resonance and wave steepening. All of these may help to understand the evolution of fluctuations from the inner heliosphere out to Earth orbit and beyond.

Conservation of total wave action and the evolution of simple waves in the solar wind

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ABSTRACT

The question of wave-action conservation for outwardly propagating simple MHD waves in the expanding solar wind is re-examined. Using the fluctuation averaged Lagrangian, we discuss the conservation of total wave action and Equi-partition of wave energy for MHD waves. It is shown that, even though the wave action for a simple monochromatic wave is subject to loss under resonance/degeneracy condition - conditions that can occur in the expanding solar wind in the regions where plasma β crosses one, the total wave action possessed by all modes remains conserved, representing a wave action exchange between different degrees of freedom. Expanding Box simulations demonstrate the results of the theoretical modeling, and reveal further details about mode-mixing, Alfvén resonance and wave steepening. All of these may help to understand the evolution of fluctuations from the inner heliosphere out to Earth orbit and beyond.

INTRODUCTION

Basic to the understanding of the wave evolution in the highly structured solar wind, is the comprehension of the simpler, isotropic case, i.e. that of evolution in a olain, isotropic radial wind. This apparently simple problem is not well-known yet. In the linear case, only the evolution of Alfvén waves is well understood: the WKB approximation predicts a 1/R decrease of the specific energy. However, the WKB approximation (as well as the finite frequency approximations, are not able to cope with the mode mixing introduced by the expansion, and the mode conversion caused by resonance introduced by singular point in MHD eigen-modes, especially for the magnetosonic modes. Early studies on such topic mainly focused on its WKB solution. However, our simulation with Expanding Box Model [4,5] reveals that even in the WKB limit, the evolution of monochromatic magnetosonic waves would deviate significantly from the WKB prediction due to mode conversion, provided that resonance conditions are partially satisfied *en route* in their propagation.

In this study, we are trying to extend the theory of conservation of wave action [1,2,3] to explain the deviation from WKB theory introduced by MHD mode conversion. <u>We are proposing that the mode conversion we found in the simulation is a direct result of MHD resonance/degeneracy, which would lead to a conservative exchange of wave action between different eigen modes (degrees of freedom of this system), and hence the total of the wave action remains unchanged.</u>

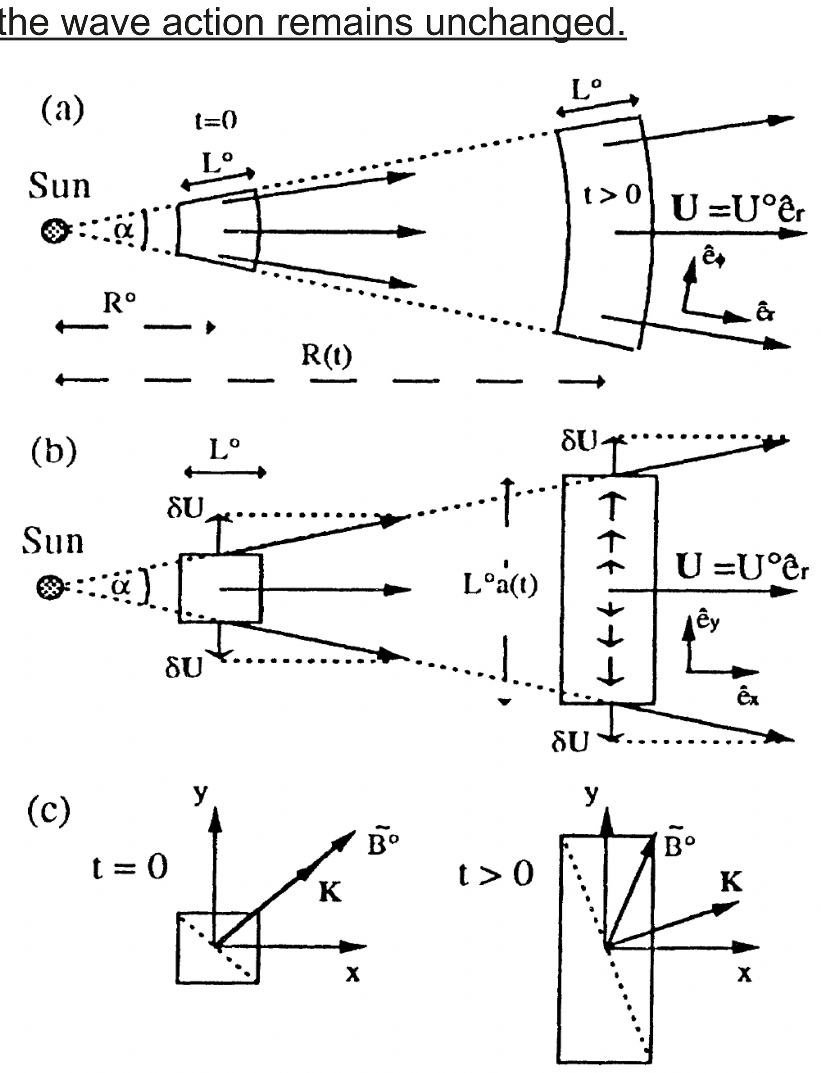


Fig 1: Sketch of the evolution of a plasma volume advected by a spherical wind with constant speed. (a) Exact evolution, (b) approximated evolution in the limit of small angular size, and (c) transformation of a parallel wave (**k** || $\mathbf{B}_0$) into an oblique wave. (Fig 1 from [4])

WAVE ACTION

$$\mathcal{L} = \frac{1}{2}\rho U^2 - \frac{P}{\gamma - 1} - \frac{B^2}{2\mu_0}$$

$$\downarrow$$

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_2 + o(\delta^2)$$

$$\downarrow$$

$$\langle \mathscr{L}_2 \rangle = L = \frac{1}{2}\rho_0 \delta\vec{u}^2 - \frac{1}{2}\rho_0 c_s^2 \left(\frac{\delta\rho}{\rho_0}\right)^2 - \frac{1}{2}\rho_0 v_a^2 \left(\frac{\delta\vec{B}}{B_0}\right)^2$$

$$\downarrow$$

$\delta\vec{u}, \delta\rho, \delta\vec{B} \rightarrow \delta\vec{u}_M, \delta\rho_M, \delta\vec{B}_M$ M can be Alfven/Slow/Fast

$$\downarrow$$

Conservation

$$\boxed{\frac{\partial}{\partial t}\left(\frac{E_M}{\omega_M}\right) + \nabla \cdot \left(\boldsymbol{c}_M \frac{E_M}{\omega_M}\right) = 0}$$

Wave Action Density: $\frac{E_M}{\omega_M} = \frac{\partial L_M}{\partial \omega_M}, E_M = \omega_M \frac{\partial L_M}{\partial \omega_M} - L_M$

CONSERVATION OF TOTAL WAVE ACTION

$$\boxed{\frac{E_{M_1}}{\omega_r}} \xrightarrow{\text{Degeneracy/Resonance}} \boxed{\frac{E'_{M_1}}{\omega_r}} + \boxed{\frac{E'_{M_2}}{\omega_r}} + \boxed{\frac{E'_{M_3}}{\omega_r}}$$

$$E_{M_1} = E'_{M_1} + E'_{M_2} + E'_{M_3}$$

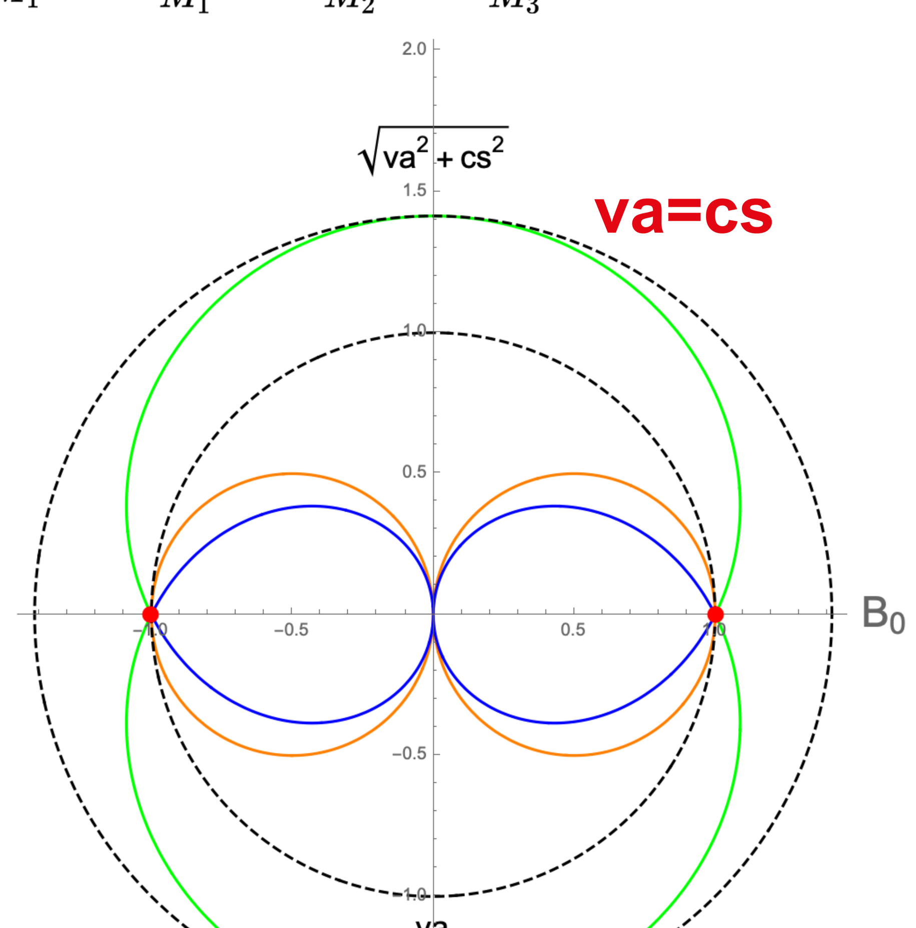


Fig 2: Phase velocity diagram of MHD eigen modes when Alfven speed (va) is equal to sound speed (cs). The singular point at parallel propagation (**k** || $\mathbf{B}_0$) facilitate the mode conversion between Alfven/Slow/Fast mode. (Alfven mode resonates, Slow/Fast modes degenerate)

SIMULATION RESULTS

Initialize a monochromatic wave (Alfven/Slow/Fast)

$$\downarrow$$

At time=t, calculate:
Wave Energy Density

$$\mathscr{E}_w = \frac{1}{2}\langle\rho\rangle\langle\boldsymbol{\delta u}^2\rangle + \frac{\langle\delta p^2\rangle}{2\langle\rho\rangle c_s^2} + \frac{\langle\boldsymbol{\delta B}^2\rangle}{2\mu_0}$$

Calculate $E_w = E_k + E_p + E_m$

$$\downarrow$$

Calculate the wave vector k

$$\boldsymbol{k}(t) = (k_x^0, k_y^0/a(t), 0)$$

$$a(t) = \frac{R(t)}{R_0} = 1 + \frac{U_0}{R_0} \cdot t$$

and eigen polarizations:

$$\delta\vec{u}_M, \delta\rho_M, \delta\vec{B}_M$$

$$\downarrow$$

Wave energy for each mode:

$$\mathscr{E}_{w,(A,S,F)} = \mathscr{E}_{k,(A,S,F)}/\mathscr{E}_k * \mathscr{E}_w$$

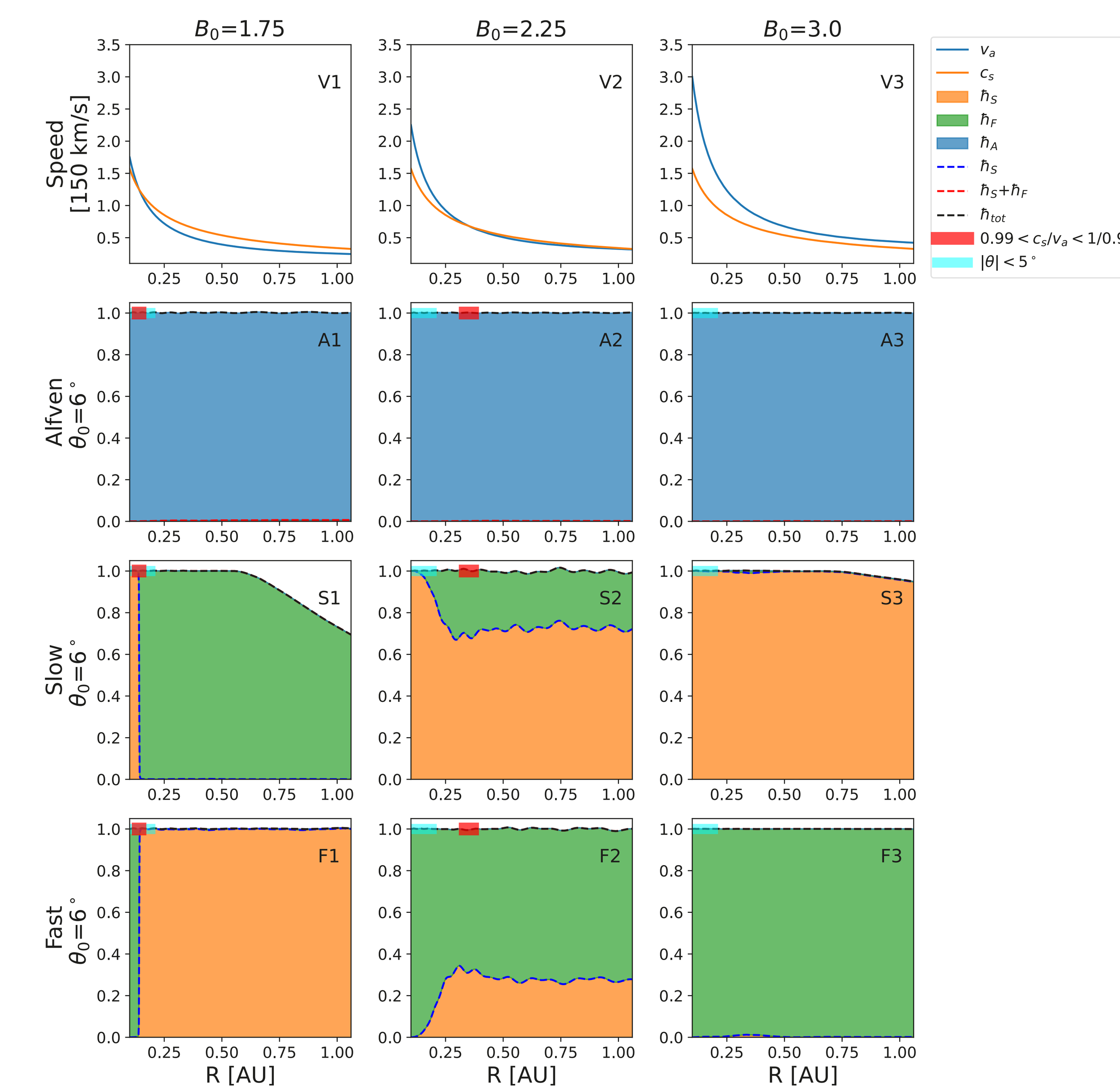
Wave action for each mode:

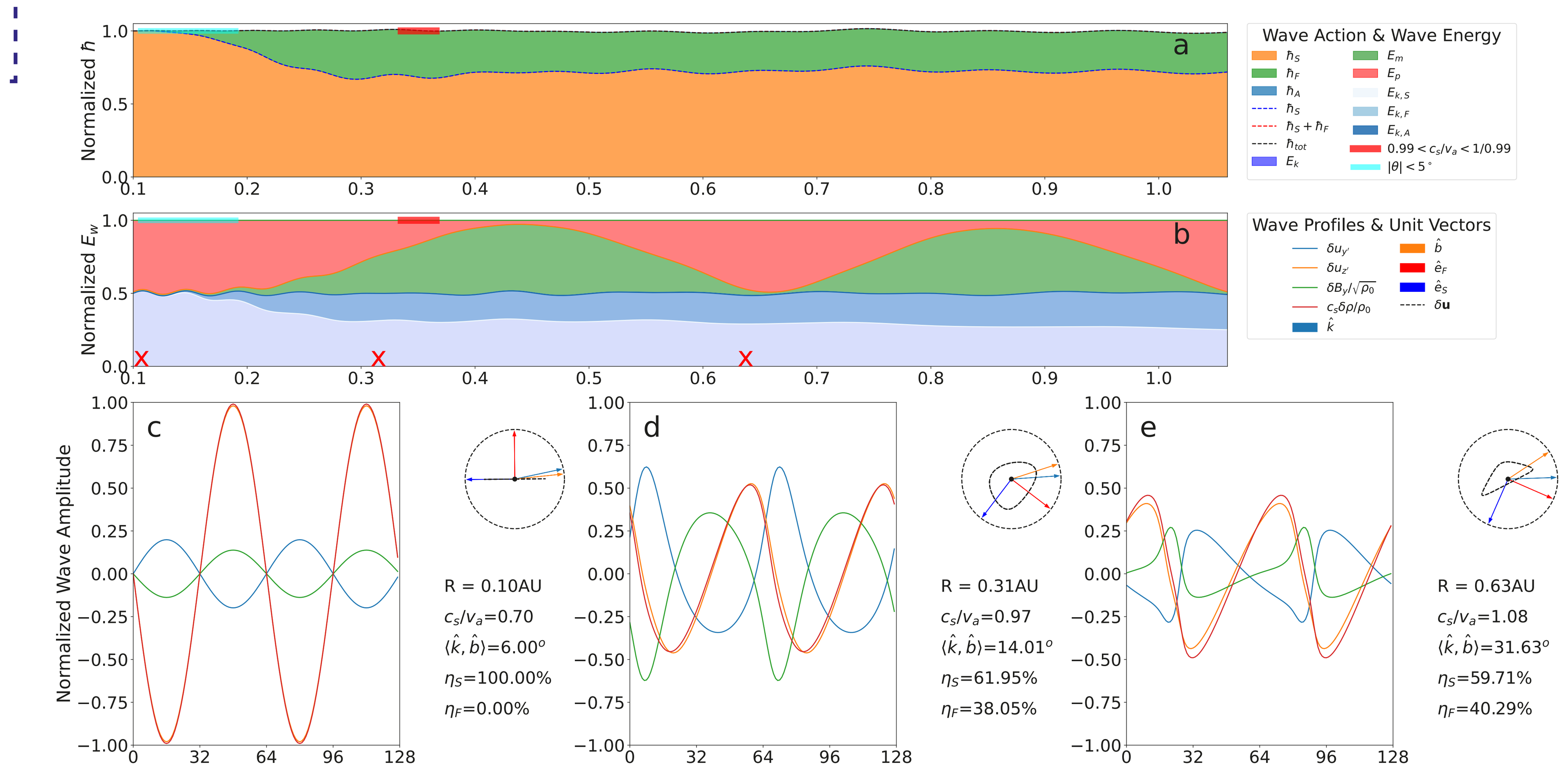
$$\hbar_{A,S,F} = \int_{V(t)} \frac{\mathscr{E}_{w,(A,S,F)}}{\omega_{A,S,F}} d\nu = \frac{E_{w,(A,S,F)}}{\omega_{A,S,F}}$$

$$\downarrow$$

Total Wave Action:

$$\hbar_{tot} = \hbar_A + \hbar_S + \hbar_F = \text{const}$$





Conclusion & Discussion:

In this study, we reviewed the definition of wave action developed by [1,2,3] and showed that the conservation law could be violated if the system passes through the resonance singular point of MHD system (i.e. parallel propagation & va=cs). We then propose that under such conditions, the total of the wave action is a conserved quantity. To prove our conjecture, we carried out simulations with the Expanding Box Model (EBM) [4,5] to study the nonlinear evolution of monochromatic MHD waves in the radial solar wind. In each simulation runs we initialize the box with monochromatic MHD waves and found that **the total of wave action possessed by three MHD eigen modes is an adiabatic conserved quantity**, instead of the wave action possessed by single modes (especially for the magnetosonic modes).

The simulations further reveal that:
1. Alfven waves are extraordinarily stable, inducing near no magnetosonic fluctuations.
2. The mixed Slow/Fast modes form a beat, causing fluctuations of magnetic/elastic energy ratio.
3. The mixed modes seem to delay the formation of shock.

Future Applications:
Our findings can be readily applied as diagnostic tool for simulation study. For example, the total of wave action can be used as a proxy for nonlinearity of the system, to quantify the ratio nonlinear/linear effects in the system. It is hence very interesting to apply our finding to the nonlinear evolution of other coherent structures (e.g. solitons, vortexes) in the solar wind.

Reference:

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